On edge- addition problem

Suaad A. A. Suady
Alaa A. Najim
Department ofmathematics
College of science – University of Basrah

Abstract:

For given positive integers t and d, "Edge-addition problem" can stated as: Given a graph G with diameter d and a positive integer $d'(d'<d)$ to obtain a graph H from G, how many edges must be added to G such that the resulting H has diameter of at most d'. Let $F(t,d)$ denoted the minimum diameter of an altered graph obtained by adding $t-$extra edges to a graph with diameter d. And let $p(t,d)$ (res. cycle) denoted the minimum diameter of graph obtained by adding $t-$extra edges to a path (res. cycle) with diameter d. Clearly that $F(t,d)=p(t,d)$.Let $T_p(P,d)$ denoted the minimum number of edges that have to be add to a path of length d in order to obtain a graph of diameter at most P. In this paper we find exact value to $p(t,d)$, $T_p(P,d)$ for some t and d (res. P and d). Also we prove $C(t,d) \geq C(t',d')$ if ($d \geq d'$and $t = t'$) or ($d = d'$and $t \leq t'$).

Keywords : Diameter , Altered graph , Edge addition .

الملخص:

افترض t و d اعداد صحيحه موجبه فأن مسئلة اضافة حافه يمكن ان نعرفها بما يلي افترض البيان G وبقطر d ويكن لدينا عدد صحيح موجب $d'(d'<d)$ للحصول على بيان H من G ما هو عدد الحواف التي يمكن اضافتها الى G بحيث يكون البيان H له قطر على الاكثر d' . لتكن $F(t,d)$ ترمز الى اقل قطر للبيان الاخر الذي نحصل عليه باضافة t من الحافات الى البيان بقطر d وليكن $p(t,d)$ (حلقه) ترمز الى اقل قطر لبيان نحصل عليه بأضافة t من الحافات الى مسار (حلقه) بقطر d . من الواضح ان $F(t,d)=p(t,d)$ ليكن $T_p(P,d)$ يرمز الى عدد الحافات الذي ينبغي اضافتها الى المسار بطول d للحصول على بيان بقطر على الاكثر P . في هذا البحث سنجد القيمه الدقيقه الى $p(t,d)$, $T_p(P,d)$ لبعض قيم t و d (P و d) . كذالك نبرهن $C(t,d) \geq C(t',d')$ اذا كان $(d = d',\ t \leq t')$ او $(d \geq d',\ t = t')$

1. Introduction

In this note, a graph $G=(V,E)$ always means a simple undirected (without loops and multiple edges)with vertex –set V and edge-set E .(In general, we follow the graph-theoretic terminology of [2])G. The distance $d_G(u,v)$ between two vertices $u,v \in V(G)$ is the length (number of edges)of a shortest path in G joining u and v ; if no such path exists , we set $d_G(u,v)=\infty$. The diameter $D(G)$ is the maximum value of $d_G(u,v)$ taken over all pairs of vertices $u,v \in V(G)$, thus a graph has finite diameter if and only if it is connected. It is well – known that when the underlying topology of an interconnection network of a system is modeled by a graph G, the Diameter of G is an important measure for communication efficiency and message delay of the System [13]. The edge – addition and deletion problems are well-known [3] .

Let $F(t,d)$ denoted the minimum diameter of an altered graph obtained by adding $t-$ extra edges to a graph with diameter d . Determining the exact value of $F(t,d)$ is fairly difficult in general since it has been proved by Schoone , Bodlander and Van LeuWeen [13] . For given integers t,d and a connected graph G ,constructing an altered graph G' of G by adding $t-extra$ edges to G such G' has diameter of at most d in NP-complete. Thus, the problem of determining sharp upper bounds of $F(t,d)$ is of interesting.

Let $p(t,d)$ (res. $C(t,d)$) denoted the minimum diameter of a graph obtained by adding $t-extra$ edges to a path (res.cycle) with diameter d . In [4]Deng and Xu show that $F(t,d)=p(t,d)$ for any integer t and d.

For some small t 's and special d 's,the exact value of $p(t,d)$ have been determined. for example, $p(1,d)=\left\lfloor \frac{d+1}{2} \right\rfloor$ for $d\geq 2$, $p(2,d)=\left\lceil \frac{d+1}{3} \right\rceil$ for $d\geq 3$, $p(3,d)=\left\lceil \frac{d+2}{4} \right\rceil$ for $d\geq 5$, determined by Schoone el at [3]. $\left\lceil \frac{d}{t+1} \right\rceil \leq p(t,d) \leq \left\lceil \frac{d}{t+1} \right\rceil +1$ for $t=4,5$ and $d\geq 4$ and $p(t,(2k-1)(t+1))+1=2k$ for any positive integer k determined by Deng and Xu in [4]. In 2006 A.A.Najim and Xu, J. [6] established the best bound for $p(t,d)$ which is $\left\lceil \frac{d-2}{t+1} \right\rceil \leq p(t,d) \leq \left\lceil \frac{d-2}{t+1} \right\rceil +1$, for $t\geq 6$ and $d\geq 3$.

Let $T_P(p,d)$ be minimum number of edges that have to be add to a path of length d to transform it to a graph of diameter at most p . Schoone in [11] proved that it is NP-complete to determine the $T_P(p,d)$. Alon [1] determined $T_P(2,d)=d-2$ for $d\geq 273$ and in general, $T_P(p,d)\leq (d+1)/\left\lfloor \frac{p}{2} \right\rfloor$.

The results are:

1. $P(t,d)=2$ for $t=4,5$ and $4\leq d\leq t+1$.

2. $p(t,d)=4$ for $t\geq 6$ and $d=3t+4,3t+5$.

3. $p(t,d)=k+2$ for $t\geq 2$, $k\geq 1$ and $d=k(t+1)+5$.

4. $C(t,d')\geq C(t,d)$ if $d'\geq d$.

5. $C(t,d)\geq C(t',d)$ if $t\leq t'$.

6. $T_P(p,d)=4$ For $p\geq 7$ and $d=5(p-1)$.

7. $T_P(p,d)=k+m-1$ Where $p\geq 12$, $0\leq r\leq 3$ and

$$(m,i,j)=\begin{cases}(7,0,-8) & where \quad k=0,1 \quad ....,(1)\\ (7,1,-5) & where \quad 0\le k\le p-7 \quad ....(2)\\ (5,0,-8) & where \quad 0\le k\le 3 \quad ....(3)\\ (5,3,8) & where \quad 0\le k\le 4 \quad ....(4)\end{cases}$$

Lemma 2.1 :- $P(t,d)=\left\lceil\frac{d-2}{t+1}\right\rceil+1=2$ for $t=4,5$ and $4\le d\le t+1$.

proof :- Let $p=x_0x_1...x_d$ be an (x_0-x_d) - path and let G be an altered graph

obtained from p by adding $t-extra$ edges and having diameter, $d(G)=p(t,d)$

Where $4\le d\le t+1$ and $t=4,5$.

From Deng Z.G. [4], we have $1\le P(t,d)\le 2$.
We show that $p(t,d)\ne 1$. Suppose that $p(t,d)=1$.
Thus G is complete undirected graph .
From lemma A. A. Najim [8] the number of the extra edges is equal to $t_d\ge\frac{d(d-1)}{2}$
$t\ge 6$ contradiction since $t=4,5$. Thus $p(t,d)=2$.■
Lemma 2.2 :- $p(t,d)=4$ for $t\ge 6$ and $d=3t+4,\ 3t+5$.

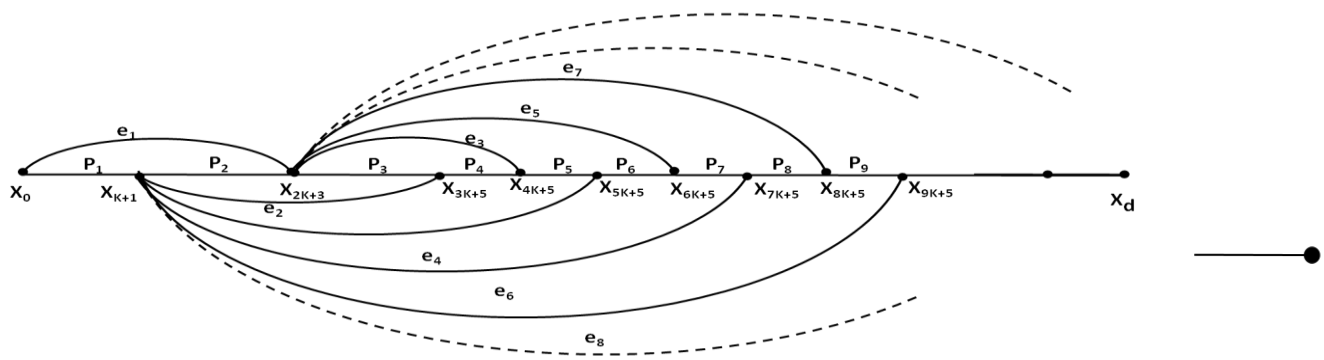
proof:- At first we prove that $p(t,d)\ge 4$ where $d=3t+4,\ 3t+5$.
From A.A.Najim. [9] $p(t,d)=4$ where $t\ge 4,\ 3t+1\le d\le 3t+3$ and

From S. A. AL-Bachary [12] $p(t,d)\ge p(t,d')$ if $d\ge d'$; then we get $p(t,d)\ge 4$.
From A.A.Najim. [10] $p(t,d)\le\left\lceil\frac{d-2}{t+1}\right\rceil+1$, for $t\ge 6$; we get
$p(t,d)\le 4$.So, $p(t,d)=4$.■

Theorem 2.1:-

$p(t,d)=k+2$ for $t\ge 2$, $k\ge 1$ and $d=k(t+1)+5$.

proof :-
Let G be an altered graph construct from a single path $P=x_0x_1...x_d$ plus $t-$extra edges
such that the diameter of G is $p(t,d)$.



Construction of theorem 2.1

Let $d = k(t+1)+5$. we add $t-$ extra edges:

$e_1 = x_0 x_{2k+3}$.

$e_r = x_h x_{k(r+1)+5}$ where $r = 2,3,...,t$ and $h = \begin{cases} k+1 & , if \;\; r \;\; is \, even \\ 2k+3 & , if \;\; r \;\; is \, odd \end{cases}$

Now the end – vertices of these edges divide P into $t+1$ segments :

$P_1 = (x_0, x_{k+1})$

$P_2 = (x_{k+1}, x_{2k+3})$

$P_3 = (x_{2k+3}, x_{3k+5})$

$P_i = (x_{(i-1)k+5}, x_{ik+5})$ where $t \geq 3$ and $i = 4,5,...,t+1$

It is easy to see that :

$\varepsilon(p_1) = k+1$

$\varepsilon(p_i) = k+2$ where $i = 2,3.$

$\varepsilon(p_i) = k$ where $i = 4,5,...,t+1$.

We will prove the distance between any two vertices x and y of $V(G)$, is less than or equal $k+2$.

Now we define $\frac{t(t+1)}{2}$ cycles , $c^1, c^2, ..., c^{\frac{t(t+1)}{2}}$ as :

$$\left.\begin{array}{l} c_t^{\,1} = p_1 \cup p_2 + e_1 \\ c_t^{\,2} = p_1 \cup p_3 + e_1 + e_2 \end{array}\right\} \quad t \geq 2$$

$$\left.\begin{array}{l} C_t^{\,3} = p_2 \cup p_3 + e_2 \\ C_t^{\,4} = p_1 \cup p_4 + e_2 + e_3 + e_4 \\ C_t^{\,5} = p_2 \cup p_4 + e_2 + e_3 \\ C_t^{\,6} = p_3 \cup p_4 + e_3 \end{array}\right\} \quad t \geq 3$$

$$\left.\begin{array}{l} C_t^{\,7+q} = p_1 \cup p_{i+1} + e_1 + e_{i-1} + e_i \\ C_t^{\,8+q} = p_2 \cup p_{i+1} + e_{i-1} + e_i \\ C_t^{\,9+q} = p_3 \cup p_{i+1} + e_2 + e_{i-1} + e_i \\ C_t^{\frac{i(i+1)}{2}} = p_i \cup p_{i+1} + e_{i-2} + e_i \end{array}\right\} \text{where } t \geq 4,\ q = \sum_{r=4}^{i} (r-1)m\ ,\ i = 4,5,\ldots,t\ ,\ \ m = \begin{cases} 0 & if \quad r = 4 \\ 1 & if \quad r \succ 4 \end{cases} .$$

$$C_t^{\,10+i+q'} = p_{4+i} \cup p_{t+1} + e_{2+i} + e_{3+i} + e_j + e_{j+1} \text{where } t \geq 5,\ q' = \sum_{r=4}^{j} r\ , i = 0,1,\ldots, j-4\ , j = 4,5,\ldots,t-1\ .$$

Their lengths are :

$\varepsilon(C_t^{\,i}) = 2k + 4$ where $i = 1,4,5$.

$\varepsilon(C_t^{\,i}) = 2k + 5$ where $i = 2,3$.

$\varepsilon(C_t^{\,6}) = 2k + 3$.

$$\left.\begin{array}{l} \varepsilon(C_t^{\,9+q}) = 2k + 5 \\ \varepsilon(C_t^{\,S+q}) = 2k + 4 \end{array}\right\} \text{where }, S = 7,8, q = \sum_{r=4}^{i} (r-1)m \ \text{ and } \ m = \begin{cases} 0 & if \quad r = 4 \\ 1 & if \quad r \succ 4 \end{cases} .$$

$\varepsilon(C_t^{\,10+i+q'}) = 2k + 4$ where $t \geq 5$, $q' = \sum_{r=4}^{j} r\ , i = 0,1,\ldots, j-4\ , j = 4,5,\ldots,t-1$.

$\varepsilon(C_t^{\frac{i(i+1)}{2}}) = 2k + 2$ where $i = 4,5,\ldots,t$.

It is easy to see that any two vertices x and y of $V(G)$, are contained in cycle

$c^{\frac{t(t+1)}{2}}$ define above the fact :

$^{z}): Z = 1,2,\ldots,\frac{t(t+1)}{2} \} \le \left\lfloor \frac{2k+5}{2} \right\rfloor$

t $p(t, k(t+1)+5) \le k+2$ . ...(1)

omKerjouas[7] we have

$1)+5 \ge \frac{k(t+1)+5+t-3}{t+1}$

$\frac{1}{+1}$

$k(t+1)+5) \ge k+2$ ....(2)

tions (1) and (2) we get

$1)+5) = k+2$. ■

NowLet $C = (x_0 x_1 \ldots x_d)$ be a cycle of length d and let $C(t,d)$ be a minimum diameter of C after adding t – extra edges . Suppose G isthe new graph after addition .

Lemma 2.3 :- $C(t,d') \ge C(t,d)$ if $d' \ge d$.

proof :- Let d and d' be two positive integer such that $d' \ge d$.
Let $C(t,d)$ and $C(t,d')$ be minimum diameter of altered graph G and G' obtained from a single paths $P = (x_0, x_1, \ldots x_d)$ and $P' = (x_0, x_1, \ldots x_{d'})$ respectively plus t -extra edges.
Then there are $x, y \in V(G)$ and $x', y' \in V(G')$ such that

$$\left.\begin{aligned} C(t,d) &= d_G(x,y) \\ C(t,d') &= d_{G'}(x',y') \end{aligned}\right\} \ldots\ldots\ldots\ldots(1)$$

Suppose that $C(t,d) > C(t,d')$.........(2)

Since $C(t,d) = d(G)$ and $C(t,d') = d(G')$

Then $C(t,d') \ge d_{G'}(u',v')$, $\forall$ $u',v' \in V(G')$

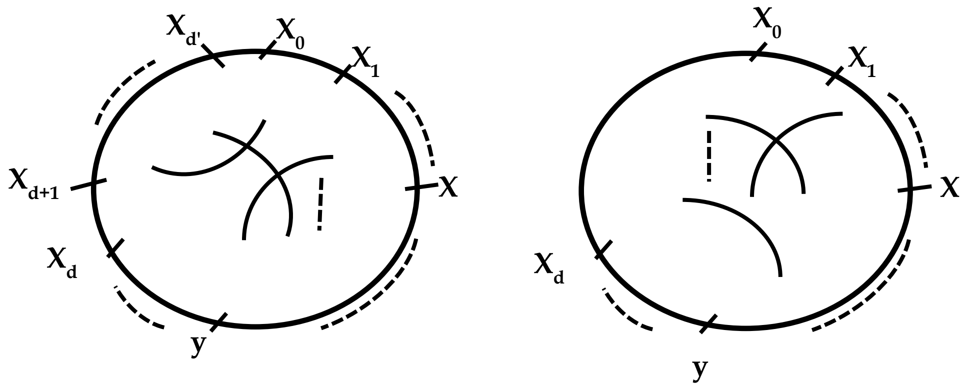
$\Rightarrow$ $C(t,d') \ge d_{G'}(u,v)$, $\forall$ $u,v \in \{x_0, x_1, \ldots, x_d\}$

$\Rightarrow$ $C(t,d') \ge d_{G'}(x,y)$

From (1) and (2) above we get

$\Rightarrow$ $d_G(x,y) > d_{G'}(x,y)$ but this contradiction

Then $C(t,d') \geq C(t,d)$ ■



$G' \quad G$

Lemma (2.4) :- $C(t,d) \geq C(t',d)$ if $t \leq t'$.

proof :- Let t and t' be two positive integer such that $t \leq t'$

Let $C(t,d)$ and $C(t',d)$ be minimum diameter of altered graph G and G' obtained from a

single path $P=(x_0,x_1,\ldots x_d)$ plus t -extra edges and t' -extra edges respectively.

This mean $V(G)=V(G')$ there are $x,y \in V(G)$ and $x',y' \in V(G')$ such that

$C(t,d)=d_G(x,y)$, and $C(t',d)=d_{G'}(x',y')$

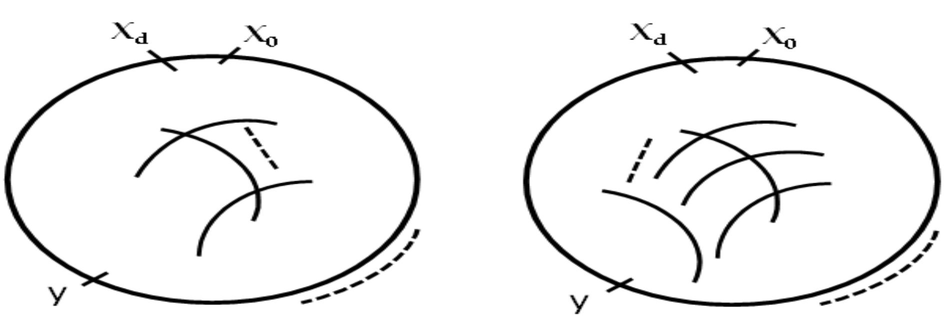
Since $d_G(x,y) \geq d_G(u,v)$, $\forall u,v \in V(G)$

$\Rightarrow d_G(x,y) \geq d_G(x',y')$

Since $d_G(x,y) \geq d_G(x',y')$

$\Rightarrow d_G(x,y) \geq d_{G'}(x',y')$

Then $C(t,d) \geq C(t',d)$. ■



$G \quad G'$

Definition 2.1:-The regular segment is a sub paths of the path p_n containing in its interior endpoint of an edge in subgraph H of G .

Lemma 2.5 :- The number of regular segments in any simple graph contains Hamilton path p_n is $2(\varepsilon - n) - 1$.

proof:- Let G be simple connected graph have Hamilton path $P_n = x_0 x_1 ... x_n$. Suppose $t = \varepsilon - n$ and $E_1 = \{e_1, e_2, ..., e_t\}$ is a set of extra edges adding to p_n .

Let $H = [E_1]$ denoted the subgraph of G induced by the set of edges E_1 . And let $V(H)$ be the set of the end points of extra edges. Then

$|V(H)| \le 2t$, since each segment is a maximal sub path of p_n and every regular segment includes only two vertices from $V(H)$.

Then the number of regular segments is $|V(H)| - 1$.

Then the number of regular segments are $2(\varepsilon - n) - 1$. ■

Theorem 2.2 :- $T_p(P,d) = k + m - 1$

where $p \ge 12$, $d = (k+m)(p-i) + r + j$, $0 \le r \le 3$ and

$$(m,i,j) = \begin{cases} (7,0,-8) & \text{where } k = 0,1 \text{ ....(1)} \\ (7,1,-5) & \text{where } 0 \le k \le p-7 \text{ ....(2)} \\ (5,0,-8) & \text{where } 0 \le k \le 3 \text{ ....(3)} \\ (5,3,8) & \text{where } 0 \le k \le 4 \text{ ....(4)} \end{cases}$$

proof :- Let $d = (k+m)(p-i) + r + j$

from equations (1) , (3) and from A.A.Najim[10]we get

$$T_p(P,d) \ge \left\lceil \frac{d}{p} \right\rceil - 1 = \left\lceil \frac{(k+m)(p-i)+r+j}{p} \right\rceil - 1$$

$$T_p(P,d) \ge \left\lceil \frac{(k+m)(p-i)+r+j}{p} \right\rceil - 1 = k + m - 1 + \left\lceil \frac{-i(k+m)+r+j}{p} \right\rceil$$

$=k+m-1+\left\lceil \frac{r-8}{p} \right\rceil$ since $i=0$ and $j=-8$

$=k+m-1$;since $\left\lceil \frac{r-8}{p} \right\rceil = 0$. So ,$T_p(P,d) \geq k+m-1$

from equations (1) , (2) , (3) and from A.A.Najim[10] we get

$$T_p(P,d) \leq \left\lfloor \frac{d-2}{p-2} \right\rfloor -1 = \left\lfloor \frac{(k+m)(p-i)+r+j-2}{p-2} \right\rfloor -1$$

$$= \left\lfloor \frac{(k+m)(p-i-s)+s(k+m)+r+j-2}{p-2} \right\rfloor -1 \quad \text{where } S = \begin{cases} 2 & , if \quad i=0 \\ 1 & , if \quad i=1 \end{cases}$$

$$=k+m-1+\left\lfloor \frac{s(k+m)+r+j-2}{p-2} \right\rfloor \quad . \quad \text{Since} \quad \left\lfloor \frac{s(k+m)+r+j-2}{p-2} \right\rfloor = 0$$

So $T_p(P,d) \leq k+m-1$

from equations (2) , (4) and from A.A.Najim [10] we get

$$T_p(P,d) \geq \left\lceil \frac{d-2}{p-1} \right\rceil -1 = \left\lceil \frac{(k+m)(p-i)+r+j-2}{p-1} \right\rceil -1$$

$$T_p(P,d) \geq \left\lceil \frac{(k+m)(p-i-s)+s(k+m)+r+j-2}{p-1} \right\rceil -1 \quad \text{where } S = \begin{cases} 0 & , if \quad i=1 \\ -2 & , if \quad i=3 \end{cases}$$

$$=k+m-1+\left\lceil \frac{s(k+m)+r+j-2}{p-1} \right\rceil ; \quad \text{Since} \quad \left\lceil \frac{s(k+m)+r+j-2}{p-1} \right\rceil = 0$$

So ,$T_p(P,d) \geq k+m-1$

from equations (4) and from A.A.Najim[10] we get

$$T_p(P,d) \leq \left\lfloor \frac{d-7}{p-3} \right\rfloor -1 = \left\lfloor \frac{(k+m)(p-i)+r+j-7}{p-3} \right\rfloor -1 = k+m-1+\left\lfloor \frac{r+j-7}{p-3} \right\rfloor$$

Since $\left\lfloor \frac{r+j-7}{p-3} \right\rfloor = 0$ .So $T_p(P,d) \leq k+m-1$

From above result we get $T_p(P,d) = k+m-1$. ■

Lemma 2.6 :- $T_p(P,d)=4$ For $p \geq 7$ and $d=5(p-1)$.

proof :-From Kerjouas [7] we have

$$p(t,d) \geq \left\lceil \frac{d}{t+1} \right\rceil \quad , \quad \text{put} \quad P=p(t,d)$$

$$P \geq \left\lceil \frac{d}{t+1} \right\rceil = \left\lceil \frac{5(p-1)}{t+1} \right\rceil \Rightarrow P \geq \frac{5p-5}{t+1}$$

$$t \geq 4-\left\lfloor \frac{5}{P} \right\rfloor = 4. \quad \text{since } p \geq 7 \text{ .So} \quad T_p(P,d) \geq 4.$$

From A.A.Najim[10] we have

$$T_p(P,d) \leq \left\lfloor \frac{d-7}{p-3} \right\rfloor -1 = \left\lfloor \frac{5(p-1)-7}{p-3} \right\rfloor -1$$

$$T_p(P,d) \leq 5+\left\lfloor \frac{3}{p-3} \right\rfloor -1 \Rightarrow T_p(P,d) \leq 4 \text{ ,since } p \geq 7 \text{ .Thus} \quad T_p(P,d)=4. \blacksquare$$

References

[1] Alon , N. , Gyarfas, A. and Ruszinko , M. ," Decreasing the diameter of bounded degree graphs",J. Graph Theory 35(2000), 161-172 .

[2] Bondary , J.A. and Murty U. S. R. "Graph Theory with Application",Macamillan Press, London, 1976 .

[3] Chun , F. S. R., Garey, M. R., "Diameter bounds for altered graph" . J. graph Theory.1984,8(4):511-534 .

[4] Deng , Z.G. and Xu , J. –M., "On diameters of altered graph" . J. Mathematical study. 37(1)(2004), 35-41 .

[5] H.-X. Ye, C. Yang, J.-M.Xu ,"Diameter Vulnerability of graphs by edge deletion", Journal of discrete Mathematics, Vol. 309, pp. 1001-1006, 2009 .

[6] J. Bond and C.peyarat," Diameter Vulnerability in network " Graph Theory with Application to Algorithm and Computer Science, John Wiley and sons Inc. , pp.123-149, 1985 .

[7] Kerjouas, Arete – Vulerabitite,"du Diameter dan les Resaux d' Interconnection" , LRI Research Report no. 261(1986), University de paris-Sud ,Orsay, France.

[8] A.A. Najim. , J.-M. Xu."On edge addition of altered graphs",Journal of University

of Science and Technology of China , Vol.35,No.6, PP.725-731, 2005 .
[9]A.A. Najim , J.-M. Xu "On Edge addition and Edge deletion ", Journal of University of Science and Technology of China, Vol.36, No.9, PP.951-955, 2006.
[10] A.A. Najim , J.-M. Xu"On Edge addition and Edge deletion of Graphs ",Ph.D. thesis, Journal of University of Science and Technology of China,2006 .
[11] Schoone , A. A., Bodlaender, H. L.and Van Lee ween, J.,"Diameter increase caused by edge deletion". J. Graph Theory 11(1987), 409-427 .
[12] S. A. AL-Bachary"On Edge addition and Edge deletion problems of Graphs", M.Sc. thesis, university of Basrah , 2009.
[13] XuJuming, Topological Structure and Analysis of Interconnection Network, Dordrecht/Boston/ London,2001.