Quasi-Extending Modules and Related Concepts

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Abstract

Let R be a commutative ring with identity, and M be a unitary R-module. In this paper the notation of Quasi-Extending module is introduced and studied as a stronger form of extending module, and gives examples, characterization and some basic properties of this concept. In other hand, we introduce QS-prime module, as a link between extending and Quasi-extending.

Keywords: Quasi-Extending Modules, unitary R-module, QS-prime module

الملخص:

لتكن R حلقه ابداليه مع عنصر محايد ولتكن M مقاس وحدوي. في هذا البحث تم تقديم مقاس شبه موسع ودراسته كصيغه قويه المقاسات الموسعه واعطاء بعض الامثله والخواص والصفات الاساسيه لهذا المفهوم. من جانب اخر قدمنا مفهوم المقاس الموسع الاولي كحلقة وصل بين المقاس الموسع ومقاس شبه الموسع .

Introduction

Let R be a commutative ring with identity, and M be a unitary R-module. An R-module M is called extending if every submodule of M is essential in a direct summand of M[1]. Where a non-zero submodule N of M is essential in M if $N \cap K \neq (0)$ for every non-zero submodule K of M[3]. A submodule Q of an R-module is called a quasi-prime if $r_1 r_2\ m \in N$ for $r_1, r_2 \in R$ and $m \in M$, then either $r_1\ m \in N$ or $r_2\ m \in N$ [4]. An R-module M is called quasi-prime iff <0> is a quasi-prime submodule of M [4]. An R-module M is uniform, if every non-zero submodule of M is essential in M [3]. A submodule of an R-module M is called closed if it has no proper essential extending [2].

$1:Quasi-Extending modules

In this section, we introduce the definition of quasi-extending module, and give some basic properties, examples, and characterizations of this concepts .

Definition (1.1)

An R-module M is called Quasi-extending module if every non-zero submodule of M is essential in a quasi-prime direct summand of M.

Examples and Remarks (1.2)

1. Every Quasi-extending module is extending module, but the converse is not true in general as the following examples shows: The Z-module Z is extending but not Quasi-extending because the submodule $< 5 >$ is essential in Z, but Z is not quasi-prime submodule.

2. Z_{p^∞} as Z-module is extending but not Quasi-extending, becauseZ_{p^∞} has no quasi-prime submodule.
3. Z_6 as Z-module is Quasi-extending, because the submodule $\{\overline{0},\overline{2},\overline{4}\}$, and $\{\overline{0},\overline{3}\}$ are the only non-zero quasi-prime direct summand of Z_6.
$\{\overline{0},\overline{3}\}$ is essential in $\{\overline{0},\overline{3}\}$ which is a direct summand of Z_6 and $\{\overline{0},\overline{2},\overline{4}\}$ is essential in $\{\overline{0},\overline{2},\overline{4}\}$ which is a direct summand of Z_6
4.Uniform module M is not Quasi-extending because the only direct summand of M is (0) and M. Hence there is no quasi-prime submodule Q of M such that, the non-zero submodule N is essential in Q which is a direct summand of M.
Now, we introduce the first characterization.

Theorem (1.3)

An R-module M is Quasi-extending if and only if every closed submodule of M is quasi-prime direct summand.

Proof

Suppose that M is a Quasi-extending module, and Let N be a closed submodule of M. Then there exists a quasi-prime direct summand of M say K such that N is essential in K. But N is closed submodule of M .Hence N=K . Hence N is a quasi-prime direct summand.

Conversely: suppose that every closed submodule of M is a quasi-prime direct summand. Then by [1] M is extending module. Now, let L be a non-zero proper submodule of M .Then L is essential in V, where V is a direct summand of M. Hence V is closed, and then by hypothesis V is a quasi-prime direct summand of M. Thus L is essential in a quasi-prime direct summand of M .Thus, M is Quasi-extending module.

In the following theorem, we give many characterizations of Quasi–extending module.

Theorem (1.4)

Let M be an R-module .Then the following statements are equivalent:
1. M is Quasi-extending R-module.
2. Every closed submodule of M is a quasi-prime direct summand of M.
3. If K is a direct summand of E(M), then K∩M is a quasi-prime direct summand of M.

Proof

(1)→(2) Follows by theorem(1.3)
(2)→(3) Suppose that every closed submodule of M is a quasi-prime direct summand of M. Let K be a direct summand of E(M), then E(M)=K⊕ L , for some submodule L of E(M) .We claim that K∩M is closed submodule of M .Suppose that K∩M is an essential in N, where N is a submodule of M, and let n∈N .Thus n=k+ℓ, where k ∈K and ℓ∈L .Now consider that n∉K ,so ℓ≠0 . But M is essential in E (M) and 0≠ℓ∈L submodule of E (M).Therefore , there exists r∈R such that 0≠r ℓ∈M, r n= r k+ r ℓ and r k = r n–r ℓ∈K∩M submodule of N .Thus r ℓ= r n-r k ∈L∩N. But K∩M is essential in N, then (0)= (K∩M) ∩L is essential in N∩L and hence N∩L=(0) .Then r ℓ= 0 which is a contradiction. Therefore K∩M is closed in M, and hence by hypothesis K∩M is a quasi-prime direct summand of M.

(3)→(1) Let K be a submodule of M. Hence by [3] K⊕C is essential in M, where C is the relative complement of K. But M is essential in E(M), then by [3] K⊕C is essential in E(M). Therefore E(M)=E(K⊕C)=E(K) ⊕E(C).That is E(K) is a summand of E(M), then by hypothesis E(K)∩M is a quasi-prime direct summand of M .Now, K=K∩M is essential in E(K)∩M, where K is essential in E(K).Thus K is essential in a quasi-prime direct summand of M. So, M is a Quasi-extending module.

The following theorem is another characterization of Quasi-extending modules.

Theorem (1.5)

An R-module M is Quasi-extending module if and only if for each submodule N of M, there is a direct decomposition M= M_1 ⊕M_2 such that N submodule of M_1,where M_1 is a quasi-prime submodule of M and N⊕ M_2 is essential submodule of M.

Proof

Suppose that M is a Quasi-extending module, and let N be a submodule of M. Then N is essential in a quasi-prime direct summand K of M. Thus M = K ⊕ K_1 where $K_1$is a submodule of M. Since N is essential in K and K_1 is essential in K_1 , then N+K_1 is essential in K+K_1=M [3].Thus N+K_1 is essential submodule of M .

Conversely: Let N be a submodule of M .Then by hypothesis, there is a direct decomposition M= M_1 ⊕M_2 such that N submodule of M_1, where M_1 is a quasi-prime submodule of M, and N + M_2 is essential in M. We claim that N is essential in M_1 .Let K be a non-zero submodule of M_1, then K is a submodule of M, and since N + M_2 is essential in M, then (N + M_2) ∩K≠0. Let 0≠k =n + m_2,where k ∈K, n∈N and m_2 ∈ M_2. Hence m_2 = k-n∈ M_1 ∩M_2 = (0) .Therefore k = n ∈K ∩N ≠(0). That is N is essential in M_1. Hence M is a Quasi-extending module.

Proposition (1.6)

Let M be a Quasi-extending module, and N is a direct summand of M. Then N is a uniform submodule.

Proof

Let V be a non-zero proper submodule of a direct summand N of M. Since M is a Quasi-extending, then there exists a quasi-prime direct summand Q of M such that V essential in Q. But N is a direct summand of M, and Q is a direct summand of M, then M=N⊕W and M=Q⊕L for some submodule W, L of M. But M=M∩M. Hence (N⊕W) ∩(Q⊕L)=M. Hence M= (Q∩N) ⊕ (L∩W). Thus, let x∈N then x∈M .That is x = y_1 + y_2 ,where y_1 ∈Q∩N, y_2 ∈L∩W. But x-y_1 =y_2 .Hence x-y_1 ∈N∩L∩W=(0). Thus x=y_1 ∈Q∩N .Therefore N is a submodule of Q∩N. That is N=N∩Q .This gives N is a submodule of Q . Now, we prove that V essential in N. Let K be a non-zero submodule of N, then K is a submodule of Q. Hence V∩K≠(0). Thus, sincs V essential in Q, we have V essential in N. That is N is a uniform.

Note

By proposition(1.6) and remarks and examples(1.2)(4), we see that a direct summand of a Quasi-extending module is not Quasi-extending.

Before we give the next result we introduce the following definition .

Definition(1.7)

A ring R is fully quasi-prime if and only if every R-module is a quasi -prime.

Proposition (1.8)

Let M be an R-module over a fully quasi-prime ring. Then every submodule N of M is a quasi-prime submodule.

Proof

Let N be a submodule of M, since R is a fully quasi-prime ring, then $\frac{M}{N}$ is a quasi-prime R-module. Hence by[4, coro. 2.2.2] N is a quasi-prime submodule of M.

Proposition (1.9)

Let M be a semi-simple R-module over a fully quasi-prime ring R. Then M is Quasi-extending module.

Proof

Let M be a semi-simple R-module over a fully quasi-prime ring. Let N be a submodule of M .Then N is a direct summand of M [1]. But R is a fully quasi-prime ring, then by proposition(1.8) N is a quasi-prime submodule. Then N is essential in a quasi-prime direct summand of M. Hence M is a Quasi-extending module.

It is well Known in our work, that every Quasi-extending module is extending but not conversely.

The following proposition gives the sufficient condition under which the converse is true. And this condition will by study in the next section

Proposition (1.10)

Let M be an R module such that every direct summand of M is a quasi-prime. Then M is Quasi-extending

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Proof:

Follows directly.

$2:QS-prime module

In this section we introduce the concepts of QS-Prime module as a link between extending module and Quasi-extending module.

Definition (2.1)

An R- module M is called QS-prime module if every proper direct summand of M is a quasi-prime. A ring R is QS-prime ring if R is QS-prime module as R-module.

Examples (2.2)

1. Z_5 as a Z-module is QS-prime, where Z_5 has no non-zero proper direct summand N.
2. Z_{12} as a Z-module is not QS-prime module.

Proposition (2.3)

Let M be an R- module, over a fully quasi-prime ring, then every module is QS-prime module.

Proof

Let M be an R-module over a fully quasi-prime ring R, then by proposition(1.8) every submodule of M is a quasi-prime. Hence M is QS-prime.

Proposition(2.4)

Let M be QS-prime module .Then M is Quasi-extending module iff M is extending.

Proof

Let M be extending module, and N be a non-zero proper submodule of M. Then N is essential in a direct summand K of M .Since M is QS-prime, then K is quasi-prime direct summand. Therefore M is Quasi-extending module.

The converse is obvious.

Proposition(2.5)

An R-module M is a Quasi-extending if and only if M is extending and QS-prime module.

Proof

Follow directly.

Proposition (2.6)

Let M be a semi-simple R-module. Then M is a Quasi-extending module if and only if M is QS-prime module.

Proof

suppose that M is Quasi-extending module, and let N be a proper direct summand of M .Then N is closed submodule of M. Now, since M is a Quasi-extending, then by proposition(1.4) N is quasi-prime direct summand. That is M is QS-prime module.

Conversely: Suppose that M is QS-prime module, and let N be a proper submodule of M. Then N is a direct summand of M. But M is QS-prime module, then N is quasi-prime in M .Thus N is a quasi-prime direct summand of M, and N is essential in N. Therefore M is a Quasi-extending module.

We end this section with following proposition

Preposition (2.7)

Every direct summand of QS-prime Module is QS-prime module.

Proof

Let M be QS-prime module, and N be a summand of M. Then N is quasi prime direct summand . let K be a direct summand of N ,then by [4,Prop. 2.2.13] K is a quasi-prime. Therefore N is QS-prime module.

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