Numerical Solution of the TwoPoint Boundary Value Problems by Using Non-polynomialsSpline Method

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Abstract

In this paper, a numerical schemeis proposed for the numerical solutions of the second order two point boundary value problems using non-polynomial spline method. The numerical results are obtained for different values of (n). Three test problems have been considered to test the accuracy of the proposed method , and to compare the compute results with exact solutions and other known methods.
Keywords: non-polynomial spline,finite difference,tow-point boundary value problem, exact solution.

الملخص:

في هذا البحث نقترح تطبيقلطريقةالسبلاين لغير متعددات الحدود على مسائل قيم ذات نقطتين حدودية من الرتبة الثانية.النتائج العددية التي حصلنا عليها كانت لقيم (n) المختلفة, ثم حل ثلاث مسائل لاختبار دقة الطريقة المقترحة, فضلا عن مقارنتها مع الحلول الدقيقة (المضبوطة)وطرائق اخرى معروفة.

1. Introduction

Solutions of boundary value problems(BVPs) can sufficiently closely be approximated by simple and efficient numerical method . Among these numerical methods are finite difference method, standard 5-point formula, iteration method , relaxation method ect. But here the non-polynomials spline method(AbdEl-Salamand Zaki(2010),Caglaret al.(2010) and El-Danaf (2008))with finite difference will be considered. Boundary value problems arise in several branches of physical differential equation will have them. Problems involving the wave equation, such us the determination of normal modes, are often stated as boundary value problems(Sebestyen (2011)). The analysis of these problems involves the eigen functions of a differential operator.

Consider the second-order of linear BVP (Gebreslassie et al.(2012) , Keenan (1992) and Sebestyen (2011))

$$y''+p(x)y'+q(x)y=r(x) \quad (1)$$

with boundary conditions:

$$y(x_0)=\gamma_1,\ y(x_n)=\gamma_2 \quad (2)$$

where γ_1 and γ_2 are real constantsand $p(x), q(x)$ and $r(x)$ are continuous functions defined on the interval $[x_0, x_n]$.

In this paper , we solved this problem numerically with non-polynomials spline method and comparing itwith analytic(exact)solution. The paper is organized as follows: In sections 2, we introduce derivation of our method .Analysis of the method is presented in section 3.In section 4, non-polynomial spline solutions are displayed. The convergence analysis is shown in section 5, In section 6, numerical results are included to show the applications and advantages of our method.

2. Derivation of the Non-Polynomial Spline Schemes

We divide the interval $[a,b]$ into $n+1$ equal subintervals using the grid points $x_i = a+ih$, $i=1,2,\ldots,n+1$ with $x_0 = a$, $x_{n+1} = b$, $h=\dfrac{b-a}{n+1}$ where n is arbitrary positive integer.Let $y(x)$ be the exact solution of the problem(1-2) and y_i be an approximation to $y(x_i)$ obtained by the segment $P_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}) .Each non-polynomialspline segment $P_i(x)$ has the form(Abd El-Salam, F.A. et al.(2010) , Caglar, H. andCaglar, N. (2012) andEl-Danaf (2008)).

$P_i(x) = a_i \sin k(x-x_i) + b_i \cos k(x-x_i) + c_i (x-x_i)^2 + d_i (x-x_i) + e_i$ (3)

for $i=0,1,\ldots,n$,

where a_i, b_i, c_i, d_i and e_i are arbitrary non zero constant and k is frequency of the trigonometric functions, which will be used to raise the accuracy of the method and Eq.(3)reduce to quarticpolynomial spline function in $[a,b]$ as $k \to 0$.

To derive the expression for the coefficients of Eq.(3) in term of y_{i+1} , y_i , S_i , S_{i+1} and M_i we first define:

$$\left.\begin{aligned} &P_i(x_i) = y_i,\ P_i(x_{i+1}) = y_{i+1} \\ &P_i''(x_i) = S_i,\ P_i''(x_{i+1}) = S_{i+1} \\ &\quad P_i'''(x_i) = M_i \end{aligned}\right\} (4)$$

After some algebraic manipulation of the Eq. (1) and using the notations defined in Eq. (4), the following expressions follow

$$a_i = \frac{h^2\left(\theta(S_i - S_{i+1}) + hM_i \sin(\theta)\right)}{\theta^3\left(\cos(\theta)-1\right)}\ ,\ b_i = \frac{-h^3 M_i}{\theta^3}$$

$$c_i = \frac{\theta\left(S_i \cos(\theta) - S_{i+1}\right) + hM_i \sin(\theta)}{2\theta\left(\cos(\theta)-1\right)}$$

$$d_i = -\frac{1}{h}(y_{i+1} - y_i) + \frac{h}{\theta^2}(S_i - S_{i+1}) + \frac{h^2 M_i \sin(\theta) + h\theta\left(S_i \cos(\theta) - S_{i+1}\right)}{2\theta\left(\cos(\theta)-1\right)}$$

$$e_i = y_i + \frac{h^2\left(\theta(S_{i+1} - S_i) - hM_i \sin(\theta)\right)}{\theta^3\left(\cos(\theta)-1\right)}$$

where $\theta = kh$ and $i=1,2,\ldots n$.

Applying the first and third derivative continuities at (x_i, y_i) in Eq.(3),i.e $P_{i-1}^{(\mu)}(x_i) = P_i^{(\mu)}(x_i)$ where $\mu = 1,3$ obtain the following relations

$$\left(\frac{h^2\left(\theta\sin\theta+2\cos\theta-2\right)}{2\theta^2\left(\cos\theta-1\right)}\right)\left(M_{i-1}+M_i\right)=\frac{\left(y_{i+1}-2y_i+y_{i-1}\right)}{h}+$$
$$\frac{h\left(2\theta\sin\theta+2\cos\theta-\theta^2\cos\theta-2\right)}{2\theta^2\left(\cos\theta-1\right)}S_{i-1}-\frac{h\left(2\theta\sin\theta+4\cos\theta+\theta^2\cos\theta-\theta^2-4\right)}{2\theta^2\left(\cos\theta-1\right)}S_i \qquad (5)$$
$$+\frac{h\left(2\cos\theta+\theta^2-2\right)}{2\theta^2\left(\cos\theta-1\right)}S_{i+1}$$
$$M_{i-1}+M_i=\frac{\theta\sin\theta}{h\left(\cos\theta-1\right)}\left(S_{i-1}-S_i\right) (6)$$

For $i=1,2,\ldots,n$.

Subtracting Eq .(6) from Eq. (5) yield the relations

$$y_{i-1}-2y_i+y_{i+1}=h^2\left(S_{i-1}\left(\frac{1}{2(1-\cos\theta)}-\frac{1}{\theta^2}\right)+2S_i\left(\frac{1}{\theta^2}-\frac{\cos\theta}{2(1-\cos\theta)}\right)+S_{i+1}\left(\frac{1}{2(1-\cos\theta)}-\frac{1}{\theta^2}\right)\right)$$

which can be written as ,

$$\frac{1}{h^2}(y_{i-1}-2y_i+y_{i+1})=\alpha S_{i-1}+2\beta S_i+\alpha S_{i+1} (7)$$

where

$$\alpha=\left(\frac{1}{2(1-\cos\theta)}-\frac{1}{\theta^2}\right) \text{and } \beta=\left(\frac{1}{\theta^2}-\frac{\cos\theta}{2(1-\cos\theta)}\right),$$

where $k\to 0$,then $\alpha=\frac{1}{12}$ and $\beta=\frac{5}{12}$ therefore the Eq.(7) we get

$$\frac{12}{h^2}(y_{i-1}-2y_i+y_{i+1})=S_{i-1}+10S_i+S_{i+1}.$$

the local truncation errors t_i , $i=1,2,...,n-1$, in equation (7) can beobtained as follows:
First we re-write the equation.(7) in the form ,

$$y_{i-1}-2y_i+y_{i+1}=h^2\left(\alpha y_{i-1}^{(2)}+2\beta y_i^{(2)}+\alpha y_{i+1}^{(2)}\right) \qquad (8)$$

the terms $y_{i-1}^{(2)}, y_i^{(2)}$ and $y_{i+1}^{(2)}$ in equation (8) are expanded around the point x_i using Taylor series and the expressions for t_i , $i=1,2,...,n-1$.can be obtained,

$$t_i=h^2\left(1-2\alpha-2\beta\right)y_i''+\frac{h^4}{12}\left(1-12\alpha\right)y_i^{(4)}+\frac{h^6}{360}\left(1-30\alpha\right)y_i^{(6)}+O\left(h^8\right) (9)$$

Now the (9) gives rise to the class of methods of different orders as follow:

Second order method:

For any choice of arbitrary α and β with $\alpha=\frac{1}{6},\beta=\frac{1}{3}$ and $\alpha+\beta=\frac{1}{2}$,then local truncation error is

$$t_i=\frac{-h^4}{12}y_i^{(4)}+O\left(h^6\right), \qquad \text{for } i=1,2,\ldots,n, \qquad (10)$$

Fourth order method:

If $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$, $T_i = O(h^6)$ then the resulting method is fourth order method. Then the local truncation error is :

$$t_i = \frac{-h^6}{240} y_i^{(6)} + O(h^8) , \qquad \text{for } i = 1, 2, \ldots, n , \quad (11)$$

3. Analysis of the method

To illustrate the application of the spline method developed in the previous section, we consider the linear of second–order BVP that is given in Eq. (1). At the grid point (x_i, y_i) ,the proposed problem in Eq.(1) may be discretized by

$$S_i + p(x)y' + q(x)y = r(x) \quad (12)$$

Solving equation(12) for S_i , we get

$S_i = -p(x)y' - q(x)y + r(x)$ (13) and approximate first derivative by using finite-difference .

The following approximation for the first-order derivative of y in Eq. (13) can be used (Caglar et al (2010) and Rashidinia et al.(2008)).

$$y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}$$

So Eq. (13) becomes

$$\left.\begin{aligned} S_{i-1} &= -\frac{p(x_{i-1})(-y_{i+1} + 4y_i - 3y_{i-1})}{2h} - q(x_{i-1})y_{i-1} + r(x_{i-1}) \\ S_i &= -\frac{p(x_i)(y_{i+1} - y_{i-1})}{2h} - q(x_i)y_i + r(x_i) \\ S_{i+1} &= -\frac{p(x_{i+1})(3y_{i+1} - 4y_i + y_{i-1})}{2h} - q(x_{i+1})y_{i+1} + r(x_{i+1}) \end{aligned}\right\} (14)$$

Substituting Eq. (14)in Eq. (7),we get the following equation:

$$\begin{aligned} &\left[-1 + h\alpha\left(\frac{3p(x_{i-1})}{2} - \frac{p(x_{i+1})}{2}\right) + h\beta p(x_i) - h^2\alpha q(x_{i-1})\right] y_{i-1} \\ &+ \left[2 + 2h\alpha\left(-p(x_{i-1}) + p(x_{i+1})\right) - 2h^2\beta q(x_i)\right] y_i \\ &+ \left[-1 + h\alpha\left(\frac{p(x_{i-1})}{2} - \frac{3p(x_{i+1})}{2}\right) - h\beta p(x_i) - h^2\alpha q(x_{i+1})\right] y_{i+1} \\ &\quad = -h^2\alpha r(x_{i-1}) - 2h^2\beta r(x_i) - h^2\alpha r(x_{i+1}) \end{aligned} \quad (15)$$

4. Non-polynomial Spline Solutions.

The tri-diagonal linear system (15) can be written in the following matrix form,
$AY + h^2DR = G$ (16)
where $A = N + hBQ - h^2Bq$ and A is tri-diagonal and diagonally matrix of order $(n-1)$
Here $N = (L_{ij})$ is a tri-diagonal matrix defined by

$$L_{ij} = \begin{cases} 2 & i = j = 1,2,...,n-1, \\ -1 & |i-j| = 1, \\ 0 & otherwise. \end{cases}$$

and $BQ = Z_{ij}$, $Bq = U_{ij}$ are tri-diagonal matrices defined by

$$Z_{ij} = \begin{cases} 2\alpha\left(-p(x_0) + p(x_2)\right), & i = j = 1, \\ \alpha\left(\frac{3p(x_{i-1})}{2} - \frac{p(x_{i+1})}{2}\right) - h\beta p(x_i), & i > j, \\ 2\alpha\left(-p(x_{i-1}) + p(x_{i+1})\right), & i = j, \\ \alpha\left(\frac{p(x_{i-1})}{2} - \frac{3p(x_{i+1})}{2}\right) - h\beta p(x_i), & i < j, \\ 2\alpha\left(-p(x_{n-2}) + p(x_n)\right), & i = j = n-1, \end{cases}$$

and

$$U_{ij} = \begin{cases} 2\beta q(x_i), & i = j = 1,2,...,n-1. \\ \alpha q(x_{i-1}), & i > j \\ \alpha q(x_{i+1}), & i < j \end{cases}$$

$R = \left(r(x_1), r(x_2), ..., r(x_{n-2}), r(x_{n-1})\right)^T$, $Y = \left(y_1, y_2, ..., y_{n-2}, y_{n-1}\right)^T$
The tri-diagonal matrix D is defined by

$$\begin{bmatrix} 2\beta & \alpha & & & & \\ \alpha & 2\beta & \alpha & & & \\ 0 & \alpha & 2\beta & \alpha & & \\ & \ddots & & \ddots & \ddots & \\ & & \alpha & 2\beta & \alpha & \\ & & & \alpha & 2\beta & \end{bmatrix}$$

$G = \left(g_1, 0, 0, 0, ..., 0, g_{n-1}\right)^T$
Where

$$g_1 = -h^2\alpha r(x_0) - \left(-1 + h\alpha\left(\frac{3p(x_0)}{2} - \frac{p(x_2)}{2}\right) + h\beta p(x_1) - h^2\alpha q(x_0)\right)\gamma_1$$

$g_i = 0$, for $i = 2,3,...,n-2,$

$$g_{n-1} = -h^2\alpha r(x_n) - \left(-1 + h\alpha\left(\frac{p(x_{n-2})}{2} - \frac{3p(x_n)}{2}\right) - h\beta p(x_{n-1}) - h^2\alpha q(x_{n-1})\right)\gamma_n$$

We assume that ,

$\overline{Y} = \left(y(x_1), y(x_2), ..., y(x_{n-1})\right)^T$

Be the exact solution of the given boundary value problem (1) at nodal point x_i , for $i = 0,1,...,n-1$. and then we have

$$A\overline{Y} + h^2DR = T(h) + G \, , \tag{17}$$

If we subtract equation (16) from equation (17) we get the following

$$A\left(\overline{Y} - Y\right) = AE = T(h) \tag{18}$$

5. Convergence Analysis

Our main purpose now is to derive a bound $\|E\|_\infty$ We now turn back to the error equation in (18) and rewrite it in the form ,

$$E = A^{-1}T = \left[N + hBQ - h^2Bq\right]^{-1}T = \left[I + N^{-1}\left(hBQ - h^2Bq\right)\right]^{-1}N^{-1}T$$

$$\|E\|_\infty \le \left\|\left[I + N^{-1}\left(hBQ - h^2Bq\right)\right]^{-1}\right\|_\infty \left\|N^{-1}\right\|_\infty \|T\|_\infty \tag{19}$$

In order to derive the bound on $\|E\|_\infty$, the following two lemmas are needed.

Lemma 1(Quarteroni el at (2000)**):**If G is a square matrix of order n and $\|G\| < 1$, then the $(I+G)^{-1}$ exists and $\left\|(I+G)^{-1}\right\| \le (1-\|G\|)^{-1}$.

Lemma 2:The matrix $\left(N + hBQ - h^2BR\right)$ is nonsingular if $\|p\|_\infty < \dfrac{8h\varepsilon}{(a-b)^2(8\alpha + 2\beta)}$ and $\|q\|_\infty < \dfrac{8(1-\varepsilon)}{(a-b)^2}$ where $0 < \varepsilon < 1$.

Proof:

Since, $A = \left(N + hBQ - h^2BR\right) = \left[I + N^{-1}\left(hBQ - h^2Bq\right)\right]N$ and the matrix N is nonsingular, so to prove A nonsingular it is sufficient to show $\left[I + N^{-1}\left(hBQ - h^2Bq\right)\right]$ nonsingular.

Since

$$\left\|N^{-1}\left(hBQ - h^2Bq\right)\right\|_\infty \le \left\|N^{-1}\right\|_\infty \left(\left\|hBQ - h^2Bq\right\|_\infty\right) \le \left\|N^{-1}\right\|_\infty \left(\|hBQ\|_\infty + \left\|h^2Bq\right\|_\infty\right) \tag{20}$$

Moreover,

$\left\|N^{-1}\right\|_\infty \le \dfrac{(b-a)^2}{8h^2}$ (Rashidinia et al.(2008)).

, $\|hBQ\|_\infty \le h(8\alpha + 2\beta)\|p\|_\infty$ and $\|hBq\|_\infty \le h^2\|q\|_\infty$

Where

$\|p\|_\infty = \max_{a \le x_i \le b}|p(x_i)|$ and $\|q\|_\infty = \max_{a \le x_i \le b}|q(x_i)|$

There for ,substituting $\|N\|_\infty, \|hBQ\|_\infty, \left\|h^2Bq\right\|_\infty$ in equation (20) we get ,

$$\left\|N^{-1}\left(hBQ - h^2Bq\right)\right\|_\infty \le \frac{(b-a)^2}{8h}(8\alpha + 2\beta)\|p\|_\infty + \frac{(b-a)^2}{8}\|q\|_\infty . \tag{21}$$

Since ,

$$\begin{cases} \|p\|_\infty < \dfrac{8h\varepsilon}{(a-b)^2(8\alpha+2\beta)} \\ \|q\|_\infty < \dfrac{8(1-\varepsilon)}{(a-b)^2} \end{cases} \quad (22)$$

There for equations (21) and (22) leads to $\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le 1$.From lemma (1) , it show that the matrix A is nonsingular. Since $\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le 1$ so using lemma(1) and equation (19) follow that

$$\|E\|_\infty \le \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1-\|N^{-1}\|_\infty \left\|\left(hBQ-h^2Bq\right)\right\|_\infty}.$$

From equation(10) we have

$$\|T_i\|_\infty = \frac{1}{12}h^4M_4 \quad , M_4 = \max_{a\le x\le b}\left|y^{(4)}(x)\right|$$

then,

$$\|E\|_\infty \le \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1-\|N^{-1}\|_\infty \left\|\left(hBQ-h^2Bq\right)\right\|_\infty} \cong \mathrm{O}\left(h^2\right). \quad (23)$$

Also from equation (11) we have

$$\|T_i\|_\infty = \frac{1}{240}h^6M_6 \quad , M_6 = \max_{a\le x\le b}\left|y^{(6)}(x)\right|$$

Then,

$$\|E\|_\infty \le \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1-\|N^{-1}\|_\infty \left\|\left(hBQ-h^2Bq\right)\right\|_\infty} \cong \mathrm{O}\left(h^4\right). \quad (24)$$

Theorem 1

Let $y(x)$ is the exact solution of the continuous BVP (1) with the boundary condition (2) and let $y(x_i)$,for $i=1,2,...,n-1$, satisfies the discrete BVP (16). Further, if $e_i = y(x_i)-y_i$ then

1- $\|E\|_\infty \cong \mathrm{O}\left(h^2\right)$ for second order convergent method.

2- $\|E\|_\infty \cong \mathrm{O}\left(h^4\right)$ for fourth order convergent method.

which are given by (23) and (24), neglecting all errors due to round off.

6. Numerical Examples

In this section we illustrate the numerical technique discussed in the previous sections by the following two pointboundary value problems of system(1-2), in order to illustrate the comparative performance of our method(15) over other existing methods . All calculations are implemented by Maple 13.

Example1 : Consider the linear boundary value problem of the form(Kalyani andRama Chandra Rao (2013)):

$y''-2y'-2y=-2 \quad 0\le x\le 1$

$y(0)=0, y(1)=0,$

with exact solution

$$y=\frac{(e^{(1-\sqrt{3})}-1)e^{(1+\sqrt{3})x}}{e^{(1+\sqrt{3})}-e^{(1-\sqrt{3})}}+\frac{(1-e^{(1+\sqrt{3})x})}{e^{(1+\sqrt{3})}-e^{(1-\sqrt{3})}}+1 .$$

The numerical solutions of the example (1) are present in the Table 1 contains results for our method for different values of n , Table 2 shows compared of our method with B-spline method (Cagar el at.(2006)) , Finite differencemethod (Fang et at.(2002)) ,Non-polynomial spline(Kalyani andRama Chandra Rao(2013)) and exact solution. Figure 1 shows the exact and numerical solution for $h=0.1$.

Example2 : Consider the linear boundary value problem of the form(Islam (2005)):

$$y''+\frac{2}{x}y'-\frac{2}{x^2}y=\frac{\sin(lnx)}{x^2}\quad 1\le x\le 2$$

$y(1)=1, y(2)=2$.

with exact solution is given by

$$y=c_1x+\frac{c_2}{x^2}-\frac{3}{10}\sin(lnx)-\frac{1}{10}\cos(lnx)$$

where $c_1=1.1392070132$ and $c_2=-0.0392070132$.

The numerical solutions of the example(2) are presented in Table 3 for different values of h and subintervals n .Table 4 shows compared of our method withNon-polynomial (Islam(2005)) and exact solution.The Figure2 shows the comparison of the exact and numerical solutions (for choosing $h=\frac{1}{32}$.

Example 3 : Consider the linear boundary value problem of the form (Rashidiniaet al.(2008)):

$$-\frac{d}{dx}\left(e^{1-x}\frac{dy}{dx}\right)=1+e^{1-x}\quad 0\le x\le 1,$$

$y(0)=0, y(1)=0$.

with exact solution,

$$y(x)=x\left(1-e^{x-1}\right) .$$

The numerical solutions of the example(3) are presented in Table 5 for different values of subintervals n .Table 6shows the exact solution and numerical solution for subintervals n ,Table5shows compared of our method withCubic spline method(Rashidinia et al.(2008))and exact solution for example(3).

x	Numerical solution				Exact Solution
	$h=\frac{1}{10}$	$h=\frac{1}{20}$	$h=\frac{1}{40}$	$h=\frac{1}{80}$	
0.1	0.05724756655	0.05718993120	0.05717560293	0.05717201498	0.0571708338
0.2	0.1062514945	0.1061303070	0.1061001848	0.1060926428	0.1060901588
0.3	0.1462674618	0.1460785360	0.1460315870	0.1460198343	0.1460159614
0.4	0.1761853324	0.1759277761	0.1758637871	0.1758477733	0.1758424928
0.5	0.1944229959	0.1941016040	0.1940217769	0.1940018051	0.1939952135
0.6	0.1987860689	0.1984149125	0.1983227516	0.1982997012	0.1982920867
0.7	0.1862827254	0.1858904522	0.1857930781	0.1857687320	0.1857606816
0.8	0.1528795521	0.1525171170	0.1524271796	0.1524047004	0.1523972596
0.9	0.09317985516	0.09293168746	0.09287012672	0.09285474541	0.0928496490

Table 1. The numerical solution and exact solution of example (1) at different values of subintervals**.**

x	Finite difference method (Fang et at.(2002))	B-spline method (Cagar el at.(2006))	Non-polynomial spline(KalyaniandRama Chandra Rao (2013))	Our method	Exact solution
0.1	0.0399	0.05657	0.05730	0.05724756655	0.0571708338
0.2	0.0897	0.104297	0.106357	0.1062514945	0.1060901588
0.3	0.1302	0.1464167	0.146421	0.1462674618	0.1460159614
0.4	0.1604	0.1763667	0.176383	0.1761853324	0.1758424928
0.5	0.1787	0.193999	0.194657	0.1944229959	0.1939952135
0.6	0.1827	0.1982966	0.199044	0.1987860689	0.1982920867
0.7	0.1695	0.18655	0.186544	0.1862827254	0.1857606816
0.8	0.1350	0.153771	0.15311	0.1528795521	0.1523972596
0.9	0.0735	0.909366	0.093335	0.09317985516	0.0928496490

Table 2. Comparison of our method with other methods for example(1),when $h=\frac{1}{10}$.

x	Numerical solution				Exact Solution
	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$	
$\frac{17}{16}$	1.057684044	1.057684614	1.057684773	1.057684829	1.057684826
$\frac{18}{16}$	1.116067219	1.116068588	1.116068955	1.116069075	1.116069082
$\frac{19}{16}$	1.175173380	1.175175547	1.175176120	1.175176299	1.175176319
$\frac{20}{16}$	1.235002859	1.235005712	1.235006459	1.235006685	1.235006720
$\frac{21}{16}$	1.295541307	1.295544686	1.295545563	1.295545825	1.295545868
$\frac{22}{16}$	1.356765435	1.356769160	1.356770124	1.356770405	1.356770456
$\frac{23}{16}$	1.418646779	1.418650674	1.418651677	1.418651959	1.418652026
$\frac{24}{16}$	1.481154166	1.481158064	1.481159066	1.481159333	1.481159417
$\frac{25}{16}$	1.544255337	1.544259093	1.544260055	1.544260294	1.544260395
$\frac{26}{16}$	1.607918004	1.607921491	1.607922378	1.607922598	1.607922696
$\frac{27}{16}$	1.672110531	1.672113637	1.672114420	1.672114619	1.672114704
$\frac{28}{16}$	1.736802353	1.736804980	1.736805636	1.736805819	1.736805879
$\frac{29}{16}$	1.801964231	1.801966295	1.801966811	1.801966974	1.801967000
$\frac{30}{16}$	1.867568386	1.867569818	1.867570175	1.867570296	1.867570305
$\frac{31}{16}$	1.933588556	1.933589297	1.933589481	1.933589562	1.933589548

Table 3. The numerical solution and exact solution of example (2) at different values of subintervals .

X	**Numerical solution**				
	Non-polynomial (Islam (2005))	Our method	Absolute error Our method	Absolute error Non-polynomial ((Islam (2005))	Exact solution
$\frac{17}{16}$	1.057689584	1.057684044	0.000004758	0.000006438	1.057684826
$\frac{18}{16}$	1.116075308	1.116067219	0.000006226	0.000009312	1.116069082
$\frac{19}{16}$	1.175182204	1.175173380	0.000005885	0.000010111	1.175176319
$\frac{20}{16}$	1.235011351	1.235002859	0.000004631	0.000009738	1.235006720
$\frac{21}{16}$	1.295548878	1.295541307	0.000003010	0.000008748	1.295545868
$\frac{22}{16}$	1.356771804	1.356765435	0.000001348	0.000007482	1.356770456
$\frac{23}{16}$	1.418651862	1.418646779	0.0000001641	0.000006140	1.418652026
$\frac{24}{16}$	1.481158003	1.481154166	0.000001414	0.000004852	1.481159417
$\frac{25}{16}$	1.544258044	1.544255337	0.000002351	0.000003681	1.544260395
$\frac{26}{16}$	1.607919746	1.607918004	0.000002950	0.000002662	1.607922696
$\frac{27}{16}$	1.672111494	1.672110531	0.000003210	0.000001816	1.672114704
$\frac{28}{16}$	1.736802734	1.736802353	0003145	0.000001140	1.736805879
$\frac{29}{16}$	1.801964230	1.801964231	0.000002770	0.0000006511	1.801967000
$\frac{30}{16}$	1.867568200	1.867568386	0.000002105	0.0000000311	1.867570305
$\frac{31}{16}$	1.933588374	1.933588556	0.000001174	0.000000041	1.933589548

Table 4 . Comparison of our method with other method[9] for example (2) , when $h = \frac{1}{16}$.

X	Numerical solution				Exact Solution
	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$	
$\frac{1}{16}$	0.03803901942	0.03802823942	0.03802554599	0.03802487270	0.03802464833
$\frac{2}{16}$	0.07292034196	0.07289926781	0.07289400240	0.07289268622	0.07289224754
$\frac{3}{16}$	0.1043383717	0.1043076226	0.1042999399	0.1042980194	0.1042973794
$\frac{4}{16}$	0.1319612268	0.1319215717	0.1319116640	0.1319091874	0.1319083618
$\frac{5}{16}$	0.1554286194	0.1553809962	0.1553690977	0.1553661229	0.1553651319
$\frac{6}{16}$	0.1743495725	0.1742951074	0.1742814996	0.1742780977	0.1742769643
$\frac{7}{16}$	0.1882999590	0.1882399903	0.1882250075	0.1882212625	0.1882200142
$\frac{8}{16}$	0.1968198529	0.1967559551	0.1967399910	0.1967360004	0.1967346702
$\frac{9}{16}$	0.1994106748	0.1993446855	0.1993281985	0.1993240776	0.1993227039
$\frac{10}{16}$	0.1955321191	0.1954661690	0.1954496922	0.1954455733	0.1954442008
$\frac{11}{16}$	0.1845988460	0.1845353918	0.1845195387	0.1845155759	0.1845142551
$\frac{12}{16}$	0.1659769186	0.1659187788	0.1659042537	0.1659006232	0.1658994127
$\frac{13}{16}$	0.1389799701	0.1389303649	0.1389179716	0.1389148741	0.1389138415
$\frac{14}{16}$	0.1028650771	0.1028276703	0.1028183249	0.1028159888	0.1028152102
$\frac{15}{16}$	0.05682831897	0.05680726615	0.05680200654	0.05680069185	0.0568002536

Table 5. The numerical solution and exact solution of example (3) at different values of subintervals**.**

X	Exact solution	Our method	Cubic spline method (Rashidinia et al.(2008))	Absolute error Our method	Absolute error Cubic spline method (Rashidinia et al.(2008))
$\frac{1}{16}$	0.03802464833	0.03803901942	0.03804327288	0.00001437109	0.00001862455
$\frac{2}{16}$	0.07289224754	0.07292034196	0.07292860910	0.00002809442	0.00003636156
$\frac{3}{16}$	0.1042973794	0.1043383717	0.1043503642	0.0000409923	0.0000529848
$\frac{4}{16}$	0.1319083618	0.1319612268	0.1319766022	0.0000528650	0.0000682404
$\frac{5}{16}$	0.1553651319	0.1554286194	0.1554469757	0.0000634875	0.0000818438
$\frac{6}{16}$	0.1742769643	0.1743495725	0.1743704418	0.0000726082	0.0000934775
$\frac{7}{16}$	0.1882200142	0.1882999590	0.1883228008	0.0000799448	0.0001027866
$\frac{8}{16}$	0.1967346702	0.1968198529	0.1968440464	0.0000851827	0.0001093762
$\frac{9}{16}$	0.1993227039	0.1994106748	0.1994355106	0.0000879709	0.0001128067
$\frac{10}{16}$	0.1954442008	0.1955321191	0.1955567909	0.0000879183	0.0001125901
$\frac{11}{16}$	0.1845142551	0.1845988460	0.1846224409	0.0000845909	0.0001081858
$\frac{12}{16}$	0.1658994127	0.1659769186	0.1659984064	0.0000775059	0.0000989937
$\frac{13}{16}$	0.1389138415	0.1389799701	0.1389981924	0.0000661286	0.0000843509
$\frac{14}{16}$	0.1028152102	0.1028650771	0.1028787347	0.0000498669	0.0000635245
$\frac{15}{16}$	0.0568002536	0.05682831897	0.05683595867	0.00002806537	0.00003570507

Table 6. Compared of our method with Cubic spline method [13] and exact solution when $k \to 0$ and $h = \frac{1}{16}$.

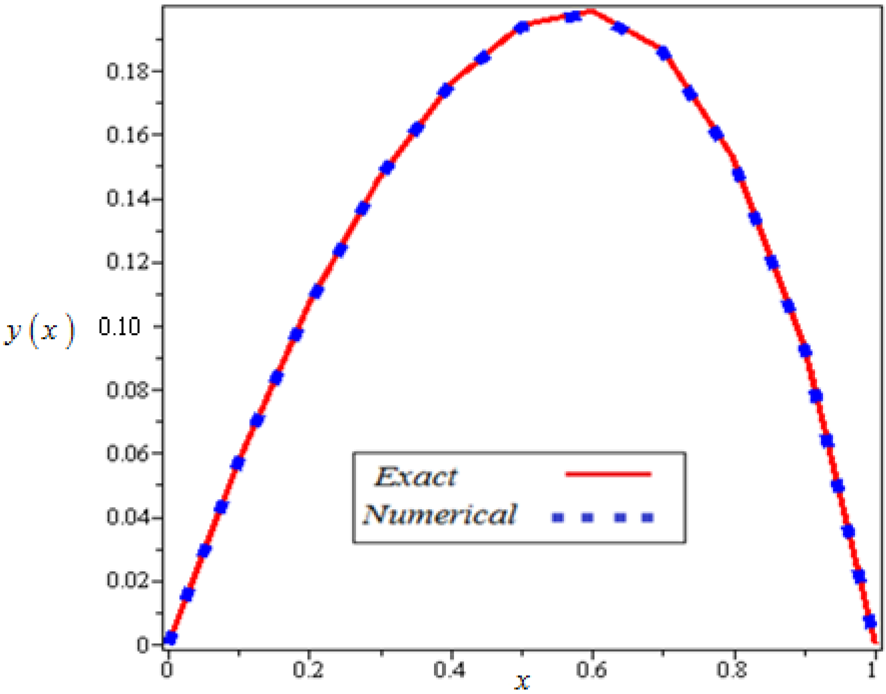

Figure1. Comparison of exact and numerical solutions of example(1) for $h = 0.1$.

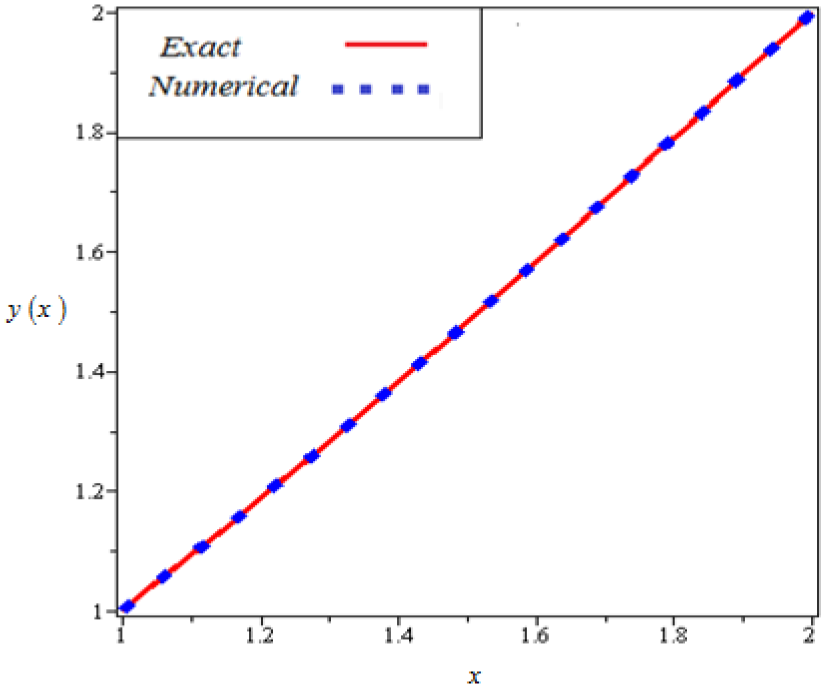

Figure 2. Comparison of exact and numerical solutions of example (2) for $h = \frac{1}{32}$.

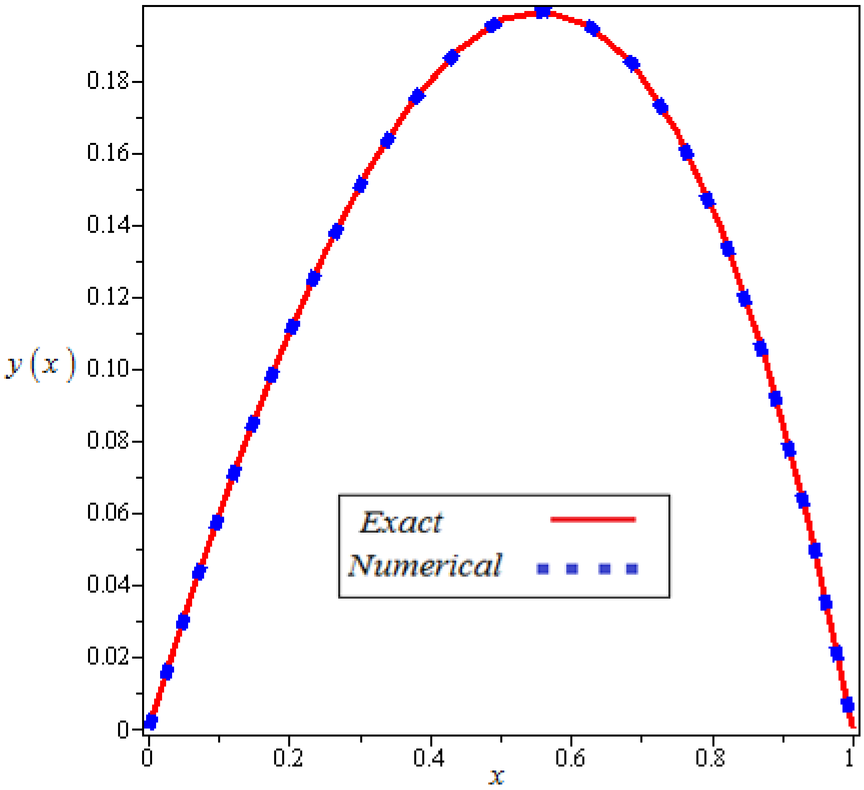

Figure3. Comparison of exact and numerical solutions of example(3) for $h = \frac{1}{16}$.

7. Conclusion

In this paper, non-polynomial spline functions are used to develop a class of numerical methods for solving two point boundary value problems. The numerical results obtained by using the method described in this study give acceptable results. We have concluded that numerical results converge to the exact solution when h goes to zero.

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