Using the assumption $Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$ for solving some kinds of linear third order P.D.Es .

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ABSTRACT:
The main aim of this research is to find the complete solution of some kinds of linear partial differential equations of third order with constant coefficients which have the general form

$$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$$

By using the assumption

$$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$$

This assumption will transform the above equation to the non-linear second order ordinary differential equation with two independent functions which have the general form .
Note :- we used u instead of u(x) also v instead of v(y).

$$A(u''+3uu'+u^3)+B(v''+3vv'+v^3)+C(vu'+vu^2)+D(uv'+uv^2)+$$
$$E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0.$$

keywords: linear third order P.D.Es, independent functions, non-linear second order ordinary differential equation

الملخص :
الهدف الرئيسي لهذا البحث هو إيجاد الحل التام لبعض أصناف المعادلات التفاضلية الجزئية الخطية من الرتبة الثالثة ذات المعاملات الثابتة و التي صيغتها العامة

$$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$$

باستخدام الفرضية :-

$$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$$

هذه الفرضية سوف تحول المعادلة أعلاه إلى معادلة تفاضلية اعتيادية لا خطية من الرتبة الثانية بدالتين مستقلتين و التي صيغتها العامة

$$A(u''+3uu'+u^3)+B(v''+3vv'+v^3)+C(vu'+vu^2)+D(uv'+uv^2)$$
$$+E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0.$$

1- **INTRODUCTION**

Many of researches tried to find new methods for solving (P.D.Es).
The researcher Kudaer [5],2006 studied the linear second order (O.D.Es) ,which have the form $y'' + P(x)\, y' + Q(x)\, y = 0$

and used the assumption $y(x) = e^{\int Z(x)\, dx}$ to find the general solution of it , and the solution depends on the forms of P(x) and Q(x) .

The researcher Abd Al-Sada [1], 2006 studied the linear second order (P.D.Es) with constant coefficients and which have the form
$AZ_{xx}+BZ_{xy}+CZ_{yy} + DZ_x+EZ_y+FZ=0$.
Where A , B , C , D , E and F are arbitrary constants , and used the assumption $Z(x,y) = e^{\int u(x)\, dx + \int v(y)\, dy}$ to find the complete solution of it, and the solution dependeds on the values of A , B , C , D , E and F .

The researcher Hani [3],2008, studied the linear second order (P.D.Es) ,which have three independent variables , and it s general form
$AZ_{xx}+BZ_{xy}+CZ_{xt} +DZ_{yy}+EZ_{yt}+FZ_{tt} +GZ_x+HZ_y+IZ_t+JZ=0$
Where A,B,C,…,I and J are arbitrary constants , and used the assumption $Z(x,y,t) = e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$ to find the complete solution of it . The solution depended on the values of A,B,C,…,I and J.

The researcher Hanon [4] , 2009, studied the linear second order (P.D.Es) , with variable coefficients which had the form
$A(x,y)Z_{xx}+B(x,y)Z_{xy}+C(x,y)Z_{yy}+ D(x,y)Z_x +E(x,y)Z_y +F(x,y)Z=0$
Where some of A(x,y),B(x,y),C(x,y),D (x,y),E(x,y) and F(x,y) , are functions of x or y or both x and y . To solve this kind of equations , she used the assumptions

$$Z(x,y) = e^{\int \frac{u(x)}{x}\, dx + \int v(y)\, dy} \quad , \quad Z(x,y) = e^{\int u(x)\, dx + \int \frac{v(y)}{y}\, dy} \quad \text{and}$$

$$Z(x,y) = e^{\int \frac{u(x)}{x}\, dx + \int \frac{v(y)}{y}\, dy}$$

These assumptions give the complete solution of the above equation and the solution depends on the forms of A(x,y),B(x,y),C(x,y),D(x,y), E(x,y) and F(x,y) .
Finally , the researcher Mohsin [6] , 2010 , studied the nonlinear second order (P.D.Es) ,of homogeneous degree which have the general form

$$\begin{aligned} &A(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{xx} + B(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{xy} + \\ &C(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_{yy} + D(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_x + \\ &E(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z_y + F(x,y,Z,Z_x,Z_y,Z_{xx},Z_{xy},Z_{yy})Z = 0 \end{aligned}$$

Where A,B,C,D,E and F are linear functions of dependent variable Z and partial derivatives of dependent variable with respect to the independent variables x and y , by using the following assumptions

$$Z(x,y) = e^{\int u(x)dx+\int v(y)dy} \quad , \quad Z(x,y) = e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$$

$$Z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy} \quad \text{and} \quad Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int \frac{v(y)}{y}dy},$$

to find the complete solutions of the above kind of equations
In our work we search functions u(x)and v(y) , such that the assumption

$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$, gives the complete solutions of linear third order partial differential equations, which have the general form given by

$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$

and it depends on the values of A,…,I and J
This assumption transforms the above (P.D.Es) to the nonlinear second order ordinary differential equation with two independent functions u(x) and v(y) .

2- Solving Special Types of the Linear Third Order Partial Differential Equations with Constant Coefficients :

The general form of these equations are given by:

$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$

To obtaine the complete solution of these equations we classify the above equation to the following cases :

Case (1) :

$CZ_{xxy}+ EZ_{xx}+FZ_{yy}+GZ_{xy}=0$ (i.e. A=B=D=H= … =J=0) s.t. C,E,F and G are not identically zero .

Case (2) :

$BZ_{yyy}+DZ_{xyy}+ EZ_{xx}+FZ_{yy}+GZ_{xy}=0$ (i.e. A=C=H= … =J=0) s.t. B,D,E,F and G are not identically zero .

Case (3) :

$DZ_{xyy}+ HZ_x+IZ_y+JZ=0$. (i.e. A=B=C= … =G=0) s.t. D,H,I and J are not identically zero .

Case (4):

$AZ_{xxx} +DZ_{xyy}+HZ_x+IZ_y+JZ=0$.(i.e. B=C= … =G=0) s.t. A,D,H,I and J are not identically zero .

Case (5):

$CZ_{xxy}+ EZ_{xx}+FZ_{yy}+GZ_{xy} +HZ_x+IZ_y+JZ=0$ (i.e. A=B=0) s.t. C,D, E,F,G,H,I and J are not identically zero .

Case (6):

$AZ_{xxx}+CZ_{xxy}+DZ_{xyy}+ EZ_{xx}+FZ_{yy}+GZ_{xy} + HZ_x+IZ_y+JZ =0$ (i.e. B=0) s.t. A,C,D,E,F,G,H,I and J are not identically zero .

3. Description of the suggested method :

Let us consider the linear third order partial differential equations which have the general form

$$AZ_{xxx}+BZ_{yyy}+CZ_{xxy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0 \qquad \ldots (1)$$

In order to find the complete solution of equation (1) , we search two independent functions u(x) and v(y) , such that the assumption

$$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}, \qquad \ldots (2)$$

given the complete solution of it, this assumption will transform equation (1) to non-linear second order ordinary differential equation. By finding Z_x, Z_{xx}, Z_{xxx}, Z_{xy}, Z_{xxy}, Z_{xyy}, Z_y, Z_{yy} and Z_{yyy} from the equation (2), we get

$$Z_x = u(x)\ e^{\int u(x)\ dx + \int v(y)\ dy} \qquad ,$$

$$Z_{xx} = [u'(x) + u^2(x)]e^{\int u(x)\ dx + \int v(y)\ dy}$$

$$Z_y = v(y)\ e^{\int u(x)\ dx + \int v(y)\ dy} \qquad ,$$

$$Z_{yy} = [v'(y) + v^2(y)]e^{\int u(x)\ dx + \int v(y)\ dy}$$

$$Z_{xy} = u(x)\,v(y)\,e^{\int u(x)\ dx + \int v(y)\ dy} \qquad ,$$

$$Z_{xxx} = [u''(x) + 3u(x)u'(x) + u^3(x)]e^{\int u(x)dx + \int v(y)dy}$$

$$Z_{xxy} = [v(y)(u'(x) + u^2(x))]e^{\int u(x)dx + \int v(y)dy}$$

$$Z_{xyy} = [u(x)(v'(y) + v^2(y))]e^{\int u(x)dx + \int v(y)dy}$$

$$Z_{yyy} = [v''(y) + 3v(y)v'(y) + v^3(y)]e^{\int u(x)dx + \int v(y)dy},$$

and by substituting Z ,Z_x, Z_{xx} , Z_{xxx} , Z_{xy} , Z_{xxy} , Z_{xyy} , Z_y , Z_{yy} and Z_{yyy} into the equation (1) , we get

$$\{A(u'' + 3uu' + u^3) + B(v'' + 3vv' + v^3) + C(vu' + vu^2) + D(uv' + uv^2) + E(u' + u^2) + F(v' + v^2) + Guv + Hu + Iv + J\}e^{\int u(x)dx + \int v(y)dy} = 0$$

Since

$$e^{\int u(x)dx + \int v(y)dy} \neq 0 \quad , so ,$$

$$A(u''(x) + 3u(x)u'(x) + u^3(x)) + B(v''(y) + 3v(y)v'(y) + v^3(y)) + C(v(y)u'(x) + v(y)u^2(x)) + D(u(x)v'(y) + u(x)v^2(y)) + E(u'(x) + u^2(x)) + F(v'(y) + v^2(y)) + Gu(x)v(y) + Hu(x) + Iv(y) + J = 0 \ldots (3)$$

Equation(3)is non-linear second order ordinary differential equation and contains two independent functions u(x) and v(y) .

4. The Complete Solution of the Linear P.D.Es of third order :

To find the complete solution of the linear partial differential equation of third order with constant coefficients we go back to the previous cases on the beginning of this paper :-

Case (1)):- If A=B=D=H= … =J=0 ,so, the P.D.E is given by :-

CZ_{xxy}+ EZ_{xx}+FZ_{yy}+GZ_{xy}=0 , then the equation(3) becomes

$$C(vu'+vu^2)+E(u'+u^2)+F(v'+v^2)+Guv=0$$

And the complete solution is given by :-

i) $$Z(x,y)=e^{\lambda x-\frac{G\lambda+C\lambda^2}{2F}y}[d_1\cos\sqrt{\frac{E\lambda^2}{F}-(\frac{G\lambda+C\lambda^2}{2F})^2}\,y+ d_2\sin\sqrt{\frac{E\lambda^2}{F}-(\frac{G\lambda+C\lambda^2}{2F})^2}\,y]$$

; $d_1= A_3\cos f_1$, $d_2=A_3\sin f_1$

If $\frac{E\lambda^2}{F}\neq(\frac{G\lambda+C\lambda^2}{2F})^2$

Where d_1,d_2 and λ are arbitrary constants

ii) $Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{G\lambda+C\lambda^2}{2F}y}$; $A_3=e^g$

If $\frac{E\lambda^2}{F}=(\frac{G\lambda+C\lambda^2}{2F})^2$

Where A_3 and λ are arbitrary constants .

Proof : Since

$$C(vu'+vu^2)+E(u'+u^2)+F(v'+v^2)+Guv=0$$

Here also we cann´t separate the variables in this equation , so we suppose that $u=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$C\lambda^2v+E\lambda^2+Fv'+Fv^2+G\lambda v=0\Rightarrow v'+v^2+\frac{G\lambda+C\lambda^2}{F}v+\frac{E\lambda^2}{F}=0$$

i) If $\frac{E\lambda^2}{F}\neq(\frac{G\lambda+C\lambda^2}{2F})^2$, we get :

$$\frac{dv}{[v+\frac{C\lambda^2+G\lambda}{2F}]^2+l_1^2}+dy=0\ ;\ l_1^2=\frac{E\lambda^2}{F}-(\frac{G\lambda+C\lambda^2}{2F})^2$$

$$\Rightarrow\frac{1}{l_1}\tan^{-1}[\frac{v+\frac{C\lambda^2+G\lambda}{2F}}{l_1}]=c_1-y\ \Rightarrow v=l_1\tan(f_1-l_1y)-\frac{C\lambda^2+G\lambda}{2F}$$; $f_1=l_1c_1$,so,

$$Z(x,y)=e^{\int\lambda dx+\int(l_1\tan(f_1-l_1y)-\frac{C\lambda^2+G\lambda}{2F})dy}=e^{\lambda x+\ln|\cos(f_1-l_1y)|-\frac{C\lambda^2+G\lambda}{2F}y+g}$$

$$\Rightarrow Z(x,y)=A_3\cos(f_1-l_1y)e^{\lambda x-\frac{C\lambda^2+G\lambda}{2F}y}\qquad ;A_3=e^g$$

And the complete solution is given by :-

i) $$Z(x,y)=e^{\lambda x-\frac{G\lambda+C\lambda^2}{2F}y}[d_1\cos\sqrt{\frac{E\lambda^2}{F}-(\frac{G\lambda+C\lambda^2}{2F})^2}\,y+d_2\sin\sqrt{\frac{E\lambda^2}{F}-(\frac{G\lambda+C\lambda^2}{2F})^2}\,y]$$

; $d_1=A_3\cos f_1$, $d_2=A_3\sin f_1$

Where d_1,d_2 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

ii) If $\frac{E\lambda^2}{F}=(\frac{G\lambda+C\lambda^2}{2F})^2$, we get : $v(y)=\frac{1}{y-c_1}-\frac{G\lambda+C\lambda^2}{2F}$, therefore

$$Z(x,y)=e^{\int\lambda dx+\int[\frac{1}{y-c_1}-\frac{G\lambda+C\lambda^2}{F}]dy}=e^{\lambda x+\ln|y-c_1|-\frac{G\lambda+C\lambda^2}{F}y+g}$$

And the complete solution is given by :-

$$Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{G\lambda+C\lambda^2}{2F}y} \quad ; \quad A_3=e^g$$

Where A_3 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

<u>Example</u> : To solve the P.D.E :

$3Z_{xxy}+2Z_{xx}+\frac{1}{4}Z_{yy}-4Z_{xy}=0$, here $C=3, E=2, F=\frac{1}{4}, G=-4$

Since $\frac{E\lambda^2}{F}\neq(\frac{G\lambda+C\lambda^2}{2F})^2$, then by using the formula (as in case-1- i)

We get the complete solution which is form

$$Z(x,y)=e^{\lambda x-(8\lambda+6\lambda^2)y}[d_1\cos\sqrt{36\lambda^4+96\lambda^3+56\lambda^2}\,y+d_2\sin\sqrt{36\lambda^4+96\lambda^3+56\lambda^2}\,y]$$

Where d_1 , d_2 and λ are arbitrary constants

<u>Case (2)):-</u> If A=C=H= … =J=0,so, the P.D.E is given by :-

$BZ_{yyy}+DZ_{xyy}+EZ_{xx}+FZ_{yy}+GZ_{xy}=0$, then the equation(3) becomes

$$B(v''+3vv'+v^3)+D(uv'+uv^2)+E(u'+u^2)+F(v'+v^2)+Guv=0$$

And the complete solution is given by :-

i) $$Z(x,y)=e^{\lambda y-\frac{G\lambda+D\lambda^2}{2E}x}[d_1\cos\sqrt{\frac{B\lambda^3+F\lambda^2}{E}-(\frac{G\lambda+D\lambda^2}{2E})^2}\,x+d_2\sin\sqrt{\frac{B\lambda^3+F\lambda^2}{E}-(\frac{G\lambda+D\lambda^2}{2E})^2}\,x]$$

; $d_1=A_3\cos f_1$, $d_2=A_3\sin f_1$

If $\frac{B\lambda^3+F\lambda^2}{E}\neq(\frac{G\lambda+D\lambda^2}{2E})^2$.

Where d_1,d_2 and λ are arbitrary constants

ii) $Z(x,y)=A_3(x-c_1)e^{\lambda y-\frac{G\lambda+D\lambda^2}{2E}x}$; $A_3=e^g$

If $\frac{B\lambda^3+F\lambda^2}{E}=(\frac{G\lambda+D\lambda^2}{2E})^2$.

Where A_3 and λ are arbitrary constants .

<u>Proof :</u> Since

$$B(v''+3vv'+v^3)+D(uv'+uv^2)+E(u'+u^2)+F(v'+v^2)+Guv=0$$

Here also we cann´t separate the variables in this equation ,so we suppose that v = λ where λ is an arbitrary constant, then the last equation becomes

$$B\lambda^3+D\lambda^2u+F\lambda^2+Eu'+Eu^2+G\lambda u=0\Rightarrow u'+u^2+\frac{G\lambda+D\lambda^2}{E}u+\frac{F\lambda^2+B\lambda^3}{E}=0$$

i) If $\frac{F\lambda^2+B\lambda^3}{E}\neq(\frac{G\lambda+D\lambda^2}{2E})^2$, we get :

$u=b_1\tan(f_1-b_1x)-\frac{G\lambda+D\lambda^2}{2E}$; $f_1=b_1c_1$, $b_1^2=\frac{F\lambda^2+B\lambda^3}{E}-(\frac{G\lambda+D\lambda^2}{2E})^2$,so,

$$Z(x,y)=e^{\int(b_1\tan(f_1-b_1x)-\frac{G\lambda+D\lambda^2}{2E})dx+\int\lambda dy}=e^{\lambda y+\ln|\cos(f_1-b_1x)|-\frac{G\lambda+D\lambda^2}{2E}x+g}\Rightarrow$$

$$Z(x,y)=A_3\cos(f_1-b_1x)e^{\lambda y-\frac{G\lambda+D\lambda^2}{2E}x}\quad ;A_3=e^g$$

And the complete solution is given by :-

$$Z(x,y)=e^{\lambda y-\frac{G\lambda+D\lambda^2}{2E}x}[d_1\cos\sqrt{\frac{B\lambda^3+F\lambda^2}{E}-(\frac{G\lambda+D\lambda^2}{2E})^2}\,x+$$

$$d_2\sin\sqrt{\frac{B\lambda^3+F\lambda^2}{E}-(\frac{G\lambda+D\lambda^2}{2E})^2}\,x]$$

; $d_1=A_3\cos f_1$, $d_2=A_3\sin f_1$

Where d_1,d_2 and λ are arbitrary constants

Domain : $-\infty<x<\infty$, $-\infty<y<\infty$

ii) If $\frac{F\lambda^2+B\lambda^3}{E}=(\frac{G\lambda+D\lambda^2}{2E})^2$, we get : $u=\frac{1}{x-c_1}-\frac{G\lambda+D\lambda^2}{2E}$, therefore

$$Z(x,y)=e^{\int\lambda dy+\int[\frac{1}{x-c_1}-\frac{G\lambda+D\lambda^2}{2E}]dx}=e^{\lambda y+\ln|x-c_1|-\frac{G\lambda+D\lambda^2}{2E}x+g}$$

And the complete solution is given by :-

$$Z(x,y)=A_3(x-c_1)e^{\lambda y-\frac{G\lambda+D\lambda^2}{2E}x}\quad ;\quad A_3=e^g$$

Where A_3 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

Example : To solve the P.D.E :

$$Z_{yyy} + \frac{1}{2}Z_{xyy} - \frac{1}{3}Z_{xx} - 4Z_{yy} + 2Z_{xy} = 0$$, here

$$B = 1, D = \frac{1}{2}, E = -\frac{1}{3}, F = -4, G = 2$$

Since $\frac{F\lambda^2 + B\lambda^3}{E} \neq (\frac{G\lambda + D\lambda^2}{2E})^2$, then by using the formula (as in case-2- i)

We get the complete solution which is form

$$Z(x,y) = e^{\lambda y + (3\lambda + \frac{3}{2}\lambda^2)x}[d_1 \cos\sqrt{3\lambda^2 - 12\lambda^3 - \frac{9}{4}\lambda^4}\,x + d_2 \sin\sqrt{3\lambda^2 - 12\lambda^3 - \frac{9}{4}\lambda^4}\,x]$$

Where d_1 , d_2 and λ are arbitrary constants

Case (3): If A=B=C= … =G=0 , so the P.D.E is given by :

$DZ_{xyy}+ HZ_x+IZ_y+JZ=0$, then the equation (3) becomes

$$D(u(x)v'(y) + u(x)v^2(y)) + Hu(x) + Iv(y) + J = 0$$

And the complete solution is given by

i)
$$Z(x,y) = e^{-\lambda^2 x - \frac{-I}{2D\lambda^2}y}[d_1 \cos\sqrt{\frac{H}{D} - \frac{J}{D\lambda^2} - (\frac{-I}{2D\lambda^2})^2}\,y + d_2 \sin\sqrt{\frac{H}{D} - \frac{J}{D\lambda^2} - (\frac{-I}{2D\lambda^2})^2}\,y]$$

; $d_1= A_3 \cos f_1$, $d_2=A_3 \sin f_1$

If $\frac{H}{D} - \frac{J}{D\lambda^2} \neq (\frac{-I}{2D\lambda^2})^2, \lambda \neq 0$

Where d_1,d_2 and λ are arbitrary constants

ii) $Z(x,y) = A_3(y - c_1)e^{-\lambda^2 x + \frac{I}{2D\lambda^2}y}$; $A_3 = e^g$

If $\frac{H}{D} - \frac{J}{D\lambda^2} = (\frac{-I}{2D\lambda^2})^2, \lambda \neq 0$

Where A_3 and λ are arbitrary constants .

Proof : Since

$$D(uv' + uv^2) + Hu + Iv + J = 0 \Rightarrow u(Dv' + Dv^2 + H) + Iv + J = 0$$

Here we can separable the variables [2], and we can write $u = \frac{-Iv - J}{Dv' + Dv^2 + H} = -\lambda^2$, so , u= $-\lambda^2$, and

$$Iv + J - (Dv' + Dv^2 + H)\lambda^2 = 0 \Rightarrow v' + v^2 - \frac{I}{D\lambda^2}v - \frac{J}{D\lambda^2} + \frac{H}{D} = 0$$

$$\Rightarrow v' + v^2 + \frac{-I}{D\lambda^2}v + \frac{H}{D} - \frac{J}{D\lambda^2} = 0$$

i) If $\frac{H}{D}-\frac{J}{D\lambda^2}\neq(\frac{-I}{2D\lambda^2})^2, \lambda\neq 0$, we get :

$v=k_1\tan(f_1-k_1y)-\frac{-I}{2D\lambda^2}$; $f_1=k_1c_1$, $k_1^2=\frac{H}{D}-\frac{J}{D\lambda^2}-(\frac{-I}{2D\lambda^2})^2$, so,

$$Z(x,y)=e^{\int -\lambda^2 dx+\int (k_1\tan(f_1-k_1y)-\frac{-I}{2D\lambda^2})dy}=e^{-\lambda^2 x+\ln|\cos(f_1-k_1y)|-\frac{-I}{2D\lambda^2}y+g}\Rightarrow$$

$$Z(x,y)=A_3\cos(f_1-k_1y)\,e^{-\lambda^2 x-\frac{-I}{2D\lambda^2}y} \qquad ; A_3=e^g$$

And the complete solution is given by :-

$$Z(x,y)=e^{-\lambda^2 x-\frac{-I}{2D\lambda^2}y}[d_1\cos\sqrt{\frac{H}{D}-\frac{J}{D\lambda^2}-(\frac{-I}{2D\lambda^2})^2}\,y+d_2\sin\sqrt{\frac{H}{D}-\frac{J}{D\lambda^2}-(\frac{-I}{2D\lambda^2})^2}\,y]$$

; $d_1=A_3\cos f_1$, $d_2=A_3\sin f_1$, where d_1,d_2 and λ are arbitrary constants
Domain: $-\infty < x < \infty$, $-\infty < y < \infty$

ii) If $\frac{H}{D}-\frac{J}{D\lambda^2}=(\frac{-I}{2D\lambda^2})^2$,we get : $v=\frac{1}{y-c_1}-\frac{-I}{2D\lambda^2}$, therefore

$$Z(x,y)=e^{\int -\lambda^2 dx+\int [\frac{1}{y-c_1}-\frac{-I}{2D\lambda^2}]dy}=e^{-\lambda^2 x+\ln|y-c_1|-\frac{-I}{2D\lambda^2}y+g}$$

And the complete solution is given by :-

$$Z(x,y)=A_3(y-c_1)\,e^{-\lambda^2 x-\frac{-I}{2D\lambda^2}y} \quad ; \quad A_3=e^g$$

Where A_3 and λ are arbitrary constants
Domain : $-\infty < x < \infty$, $-\infty < y < \infty$.

<u>Example</u> : To solve the P.D.E :
$-3Z_{xyy}-Z_x+Z_y-3Z=0$, here $D=-3, H=-1, I=1, J=-3$

Since $\frac{H}{D}-\frac{J}{D\lambda^2}\neq(\frac{-I}{2D\lambda^2})^2, \lambda\neq 0$, then by using the formula (as in case-3- i)

We get the complete solution which is form

$$Z(x,y)=e^{-\lambda^2 x-\frac{1}{6\lambda^2}y}[d_1\cos\frac{\sqrt{12\lambda^4-36\lambda^2-1}}{6\lambda^2}y+d_2\sin\frac{\sqrt{12\lambda^4-36\lambda^2-1}}{6\lambda^2}y]$$

; $\lambda\neq 0$

Where d_1 , d_2 and λ are arbitrary constants

<u>Case (4) :-</u> If B=C= … =G=0 , so the P.D.E is given by : $AZ_{xxx}+DZ_{xyy}+HZ_x+IZ_y+JZ=0$, then the equation (3) becomes

$$A(u''+3uu'+u^3)+D(uv'+uv^2)+Hu+Iv+J=0$$

And the complete solution is given by :-

i) $Z(x,y)=e^{\lambda x-\frac{I}{2D\lambda}y}[d_1\cos\sqrt{\frac{A\lambda^3+H\lambda+J}{D\lambda}-(\frac{I}{2D\lambda})^2}\,y$

$+d_2\sin\sqrt{\frac{A\lambda^3+H\lambda+J}{D\lambda}-(\frac{I}{2D\lambda})^2}\,y]$

; $d_1= A_3\cos f_1$, $d_2=A_3\sin f_1$, where d_1,d_2 and λ are arbitrary constants .

If $\frac{A\lambda^3+H\lambda+J}{D\lambda}\neq(\frac{I}{2D\lambda})^2,\lambda\neq 0$

ii) $Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{I}{2D\lambda}y}$; $A_3=e^g$

If $\frac{A\lambda^3+H\lambda+J}{D\lambda}=(\frac{I}{2D\lambda})^2,\lambda\neq 0$

Where A_3 and λ are arbitrary constants .

Proof : since

$$A(u''+3uu'+u^3)+D(uv'+uv^2)+Hu+Iv+J=0$$

Here we cann´t separate the variables in this equation , so we suppose that $u=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$A\lambda^3+H\lambda+J+Iv+D\lambda v'+D\lambda v^2=0\Rightarrow v'+v^2+\frac{I}{D\lambda}v+\frac{A\lambda^3+H\lambda+J}{D\lambda}=0$$

i) If $\frac{A\lambda^3+H\lambda+J}{D\lambda}\neq(\frac{I}{2D\lambda})^2,\lambda\neq 0$, we get :

$v=r_1\tan(f_1-r_1y)-\frac{I}{2D\lambda}$; $f_1=r_1c_1$, $r_1^2=\frac{A\lambda^3+H\lambda+J}{D\lambda}-(\frac{I}{2D\lambda})^2,\lambda\neq 0$, so ,

$$Z(x,y)=e^{\int\lambda dx+\int(r_1\tan(f_1-r_1y)-\frac{I}{2D\lambda})dy}=e^{\lambda x+\ln|\cos(f_1-r_1y)|-\frac{I}{2D\lambda}y+g}\Rightarrow$$

$$Z(x,y)=A_3\cos(f_1-r_1y)e^{\lambda x-\frac{I}{2D\lambda}y}\quad ;\ A_3=e^g$$

And the complete solution is given by :-

$$Z(x,y)=e^{\lambda x-\frac{I}{2D\lambda}y}[d_1\cos\sqrt{\frac{A\lambda^3+H\lambda+J}{D\lambda}-(\frac{I}{2D\lambda})^2}\,y$$

$$+d_2\sin\sqrt{\frac{A\lambda^3+H\lambda+J}{D\lambda}-(\frac{I}{2D\lambda})^2}\,y]$$

; $d_1= A_3\cos f_1$, $d_2=A_3\sin f_1$

Where d_1,d_2 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

ii) If $\frac{A\lambda^3+H\lambda+J}{D\lambda}=(\frac{I}{2D\lambda})^2,\lambda\neq 0$, we get : $v=\frac{1}{y-c_1}-\frac{I}{2D\lambda}$, therefore

$$Z(x,y)=e^{\int\lambda dx+\int[\frac{1}{y-c_1}-\frac{I}{2D\lambda}]dy}=e^{\lambda x+\ln|y-c_1|-\frac{I}{2D\lambda}y+g}$$

And the complete solution is given by :-

$$Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{I}{2D\lambda}y} \quad ; \quad A_3=e^g$$

Where A_3 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

Example : To solve the P.D.E :

$3Z_{xxx}+\frac{1}{2}Z_{xyy}+Z_x-Z_y-Z=0$, here A=3 , D=$\frac{1}{2}$, H= 1 , I= -1 , J= -1

Since $\frac{A\lambda^3+H\lambda+J}{D\lambda}\neq(\frac{I}{2D\lambda})^2$, then by using the formula (as in case-4- i)

We get the complete solution which is form

$$Z(x,y)=e^{\lambda x+\frac{1}{\lambda}y}[d_1\cos\frac{\sqrt{6\lambda^4+2\lambda^2-2\lambda-1}}{\lambda}y+d_2\sin y\frac{\sqrt{6\lambda^4+2\lambda^2-2\lambda-1}}{\lambda}y]$$

$;\lambda\neq 0$

Where d_1 , d_2 and λ are arbitrary constants

Case (5) : - If A=B=D=0 , so the P.D.E is given by :

$CZ_{xxy}+EZ_{xx}+FZ_{yy}+GZ_{xy}+HZ_x+IZ_y+JZ=0$, then the equation (3.3) becomes

$$C(vu'+vu^2)+E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0$$

And the complete solution is given by :-

i) $$Z(x,y)=e^{\lambda x-\frac{C\lambda^2+G\lambda+I}{2F}y}[d_1\cos\sqrt{\frac{E\lambda^2+H\lambda+J}{F}-(\frac{C\lambda^2+G\lambda+I}{2F})^2}\,y$$
$$+d_2\sin\sqrt{\frac{E\lambda^2+H\lambda+J}{F}-(\frac{C\lambda^2+G\lambda+I}{2F})^2}\,y]$$

; $d_1=A_3\cos f_1$, $d_2=A_3\sin f_1$

If $\frac{E\lambda^2+H\lambda+J}{F}\neq(\frac{C\lambda^2+G\lambda+I}{2F})^2$.

Where d_1,d_2 and λ are arbitrary constants

ii) $Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{C\lambda^2+G\lambda+I}{2F}y}$; $A_3=e^g$

If $\frac{E\lambda^2+H\lambda+J}{F}=(\frac{C\lambda^2+G\lambda+I}{2F})^2$.

Where A_3 and λ are arbitrary constants .

Proof:Since

$$C(vu'+vu^2)+E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0$$

Here also we cann´t separate the variables in this equation , so we suppose that $u(x)=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$C\lambda^2 v + E\lambda^2 + Fv' + Fv^2 + G\lambda v + H\lambda + Iv + J = 0$$

$$\Rightarrow v' + v^2 + \frac{C\lambda^2 + G\lambda + I}{F} v + \frac{E\lambda^2 + H\lambda + J}{F} = 0$$

i) If $\frac{E\lambda^2 + H\lambda + J}{F} \neq (\frac{C\lambda^2 + G\lambda + I}{2F})^2$, we get :

$$v = t_1 \tan(f_1 - t_1 y) - \frac{C\lambda^2 + G\lambda + I}{F} \quad ; \quad f_1 = t_1 c_1 \quad ,$$

$$t_1^2 = \frac{E\lambda^2 + H\lambda + J}{F} - (\frac{C\lambda^2 + G\lambda + I}{2F})^2. \quad \text{, so,}$$

$$Z(x,y) = e^{\int \lambda dx + \int (t_1 \tan(f_1 - t_1 y) - \frac{C\lambda^2 + G\lambda + I}{F}) dy} = e^{\lambda x + \ln|\cos(f_1 - t_1 y)| - \frac{C\lambda^2 + G\lambda + I}{F} y + g}$$

$$\Rightarrow Z(x,y) = A_3 \cos(f_1 - t_1 y) e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{F} y} \qquad A_3 = e^g$$

And the complete solution is given by :-

$$Z(x,y) = e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{2F} y} [d_1 \cos \sqrt{\frac{E\lambda^2 + H\lambda + J}{F} - (\frac{C\lambda^2 + G\lambda + I}{2F})^2}\, y$$

$$+ d_2 \sin \sqrt{\frac{E\lambda^2 + H\lambda + J}{F} - (\frac{C\lambda^2 + G\lambda + I}{2F})^2}\, y]$$

; $d_1 = A_3 \cos f_1$, $d_2 = A_3 \sin f_1$

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$

Where d_1,d_2 and λ are arbitrary constants

ii) If $\frac{E\lambda^2 + H\lambda + J}{F} = (\frac{C\lambda^2 + G\lambda + I}{2F})^2$. , we get :

$$v = \frac{1}{y - c_1} - \frac{C\lambda^2 + G\lambda + I}{2F} \quad \text{, therefore}$$

$$Z(x,y) = e^{\int \lambda dx + \int [\frac{1}{y - c_1} - \frac{C\lambda^2 + G\lambda + I}{2F}] dy} = e^{\lambda x + \ln|y - c_1| - \frac{C\lambda^2 + G\lambda + I}{2F} y + g}$$

And the complete solution is given by :-

$$Z(x,y) = A_3 (y - c_1) e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{2F} y} \quad ; \quad A_3 = e^g$$

Where A_3 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$.

Note : If we suppose that $v = \lambda$,we get the complete solution by the same method .

Example : To solve the P.D.E :

$$Z_{xxy} + Z_{xx} - Z_{yy} - Z_x - 2Z_y - 2Z = 0 \text{, here}$$

C=1 , E=1, F= -1 , G= 0 , H= -1 , I= -2 , J= - 2

Since $\frac{E\lambda^2+H\lambda+J}{F} \neq (\frac{C\lambda^2+G\lambda+I}{2F})^2$, then by using the formula (as in case-5-i)

We get the complete solution which is form

$$Z(x,y)=e^{\lambda x+\frac{(\lambda^2-2)}{2}y}[d_1\cos\frac{\sqrt{4+4\lambda-\lambda^4}}{2}y+d_2\sin\frac{\sqrt{4+4\lambda-\lambda^4}}{2}y]$$

Where d_1 , d_2 and λ are arbitrary constants

<u>Case (6) :</u>

If B= 0 , so, the P.D.E is given by : $AZ_{xxx}+CZ_{xxy}+DZ_{xyy}+ EZ_{xx}+FZ_{yy}+GZ_{xy} + HZ_x+IZ_y+JZ =0$,
then the equation (3) becomes

$$A(u''+3uu'+u^3)+C(vu'+vu^2)+D(uv'+uv^2)+$$
$$E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0$$

And the complete solution is given by :-

i)

$$Z(x,y)=e^{\lambda x-\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)}y}[d_1\cos\sqrt{\frac{A\lambda^3+E\lambda^2+H\lambda+J}{D\lambda+F}-(\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)})^2}\,y+$$
$$d_2\sin\sqrt{\frac{A\lambda^3+E\lambda^2+H\lambda+J}{D\lambda+F}-(\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)})^2}\,y]$$

$d_1= A_3 \cos f_1$, $d_2=A_3 \sin f_1$

If $\frac{A\lambda^3+E\lambda^2+H\lambda+J}{D\lambda+F} \neq (\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)})^2, \lambda \neq \frac{-F}{D}$

Where d_1,d_2 and λ are arbitrary constants .

ii) $Z(x,y)=A_3(y-c_1)e^{\lambda x-\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)}y}$; $A_3=e^g$

If $\frac{A\lambda^3+E\lambda^2+H\lambda+J}{D\lambda+F} = (\frac{C\lambda^2+G\lambda+I}{2(D\lambda+F)})^2, \lambda \neq \frac{-F}{D}$

Where A_3 and λ are arbitrary constants .

<u>Proof :</u> since

$$A(u''+3uu'+u^3)+C(vu'+vu^2)+D(uv'+uv^2)+$$
$$E(u'+u^2)+F(v'+v^2)+Guv+Hu+Iv+J=0$$

Here we cann´t separate the variables in this equation , so we suppose that u(x)= λ where λ is an arbitrary constant, then the last equation becomes

$$A\lambda^3 + C\lambda^2 v + D\lambda v' + D\lambda v^2 + E\lambda^2 + Fv' + Fv^2 + G\lambda v + H\lambda + Iv + J = 0$$

$$\Rightarrow v' + v^2 + \frac{C\lambda^2 + G\lambda + I}{D\lambda + F} v + \frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} = 0$$

i) If $\frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} \neq (\frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2$, we get :

$$v = w_1 \tan(f_1 - w_1 y) - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} \qquad ; \qquad f_1 = w_1 c_1$$

$$w_1^2 = \frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} - (\frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2 \text{ ,so,}$$

$$Z(x,y) = e^{\int \lambda dx + \int (w_1 \tan(f_1 - w_1 y) - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)}) dy} = e^{\lambda x + \ln|\cos(f_1 - w_1 y)| - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} y + g}$$

$$\Rightarrow Z(x,y) = A_3 \cos(f_1 - w_1 y) e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} y} \qquad ; A_3 = e^g$$

And the complete solution is given by :-

$$Z(x,y) = e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} y} [d_1 \cos \sqrt{\frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} - (\frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2}\, y$$
$$+ d_2 \sin \sqrt{\frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} - (\frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2}\, y]$$

; $d_1 = A_3 \cos f_1$, $d_2 = A_3 \sin f_1$

Where d_1, d_2 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$.

ii) If $\frac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} = (\frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2$, we get : $v = \frac{1}{y - c_1} - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)}$

, therefore

$$Z(x,y) = e^{\int \lambda dx + \int [\frac{1}{y - c_1} - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)}] dy} = e^{\lambda x + \ln|y - c_1| - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} y + g}$$

And the complete solution is given by :-

$$Z(x,y) = A_3 (y - c_1) e^{\lambda x - \frac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)} y} \quad ; \quad A_3 = e^g$$

Where A_3 and λ are arbitrary constants

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$.

<u>Example</u> : To solve the P.D.E :

$Z_{xxx} - 2Z_{xxy} + Z_{xyy} + 2Z_{xx} + Z_{yy} - Z_{xy} + 2Z_x - Z_y - 4Z = 0$, here

$A = 1, C = -2, D = 1, E = 2, F = 1, G = -1, H = 2, I = -1, J = -4$

Since $\dfrac{A\lambda^3 + E\lambda^2 + H\lambda + J}{D\lambda + F} \neq (\dfrac{C\lambda^2 + G\lambda + I}{2(D\lambda + F)})^2$, then by using the formula (as in case-4- i)

We get the complete solution which is form

$$Z(x,y) = e^{\lambda x + \frac{2\lambda^2 + \lambda + 1}{2\lambda + 2} y}[d_1 \cos \sqrt{\frac{16\lambda^3 + 11\lambda^2 - 10\lambda - 17}{4\lambda^2 + 8\lambda + 4}} y +$$

$$d_2 \sin y \sqrt{\frac{16\lambda^3 + 11\lambda^2 - 10\lambda - 17}{4\lambda^2 + 8\lambda + 4}} y]; \lambda \neq -1$$

Where d_1 , d_2 and λ are arbitrary constants

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