Nonstandard Treatment of Two Dimensional Taylor Series with Reminder Formulas

Ibrahim O. Hamad
College of Sciences
University- Salahaddin

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الملخص

إن الهـــدف من هذا البحـــث هو إيجاد صيغ جديدة لمتسلسلة تـــايلور للـــدوال بمتغيرين وذلك باستخدام بعض مفاهيم التحليل غير القياسي الذي أوجده **Robinson** و وضعه **Nilson** بأسلوب منطقي.

ABSTRACT

The aim of this paper is to establish some new two dimensional Taylor series formulas using some concepts of nonstandard analysis given by **Robinson** and axiomatized by **Nelson**

Keyword:, nonstandard analysis, infinitely near, Taylor series.

1- Introduction: -

Let f be a continuous function defined on a domain D and posses its derivatives up to order n in D , then the Taylor development of $f(x)$ about x_o with remainder form is given by:

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(x_o)}{k!}(x-x_o)^k+R_{n-1}(x),$$

where $x_o \in D$ and $R_{n-1}(x)$ is the remainder, which takes one of the following forms:

$$R_{n-1}(x)=\sum_{k=n}^{\infty}\frac{f^{(k)}(x_o)}{k!}(x-x_o)^k$$

$$R_{n-1}(x)=\frac{f^{(n)}(\xi)}{n!}(x-x_o)^n, \qquad \text{for} \qquad \xi\in[x_o,x]$$

$$R_{n-1}(x)=\frac{1}{(n-1)!}\int_{x_o}^{x}(x-t)^{n-1}f^{(n)}(t)\ dt$$

Through this paper we need the following nonstandard concepts:

Every set or element defined in a classical mathematics is called **standard** [1].

Definition 1.1

A real number x is called limited if there exists a positive standard real number r such that $|x| \le r$, otherwise it is called unlimited. The set of all unlimited real numbers is denoted by $\overline{\mathbf{R}}$ [1].

Definition 1.2

A real number x is called **infinitesimal** if $|x| \le r$, for all positive standard real numbers r [1]

Definition 1.3

Two real numbers, x and y are **infinitely close** if $x - y$ is infinitesimal, and is denoted by $x \cong y$ [1].

Definition 1.4

A function f is differentiable at x_o, denoted by $f'(x_o)$, if there exists a standard number λ such that: $f'(x_o) = \lambda \cong \dfrac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$.[3]

2- Higher Order Differentiation

In [2] and [6] a brief introduction of higher order differentiation is given. Suppose that $z = f(x, y)$ is a function of two variables with continuous partial derivatives of first order, then the differentiation of z, denoted by dz, is defined by:

$$dz = df(x, y) = f_x(x, y)dx + f_y(x, y)dy,$$

since dz is also a function of x and y, so if the second order partial derivatives of f exists then differentiation of dz exists, and it is called second order differentiation, which is denoted by d^2z.

It is important to emphasize that the quantities dx and dy are assumed to be constants. Therefore we have:

$$\begin{aligned} d^2z &= d^2f(x, y) = d\left(df(x, y)\right) \\ &= \left(f_{xx}dx + f_{xy}dy\right)dx + \left(f_{yx}dx + f_{yy}dy\right)dy \\ &= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \\ &= \left(D_x dx + D_y dy\right)^2 f(x, y), \text{ where } D_x = \frac{\partial}{\partial x} \end{aligned}$$

that is

$$d^2f(x,y)=\left(D_x dx+D_y dy\right)^2 f(x,y). \qquad \text{…(2.1)}$$

In general

$$d^nf(x,y)=\left(D_x dx+D_y dy\right)^n f(x,y)$$
$$=\sum_{k=0}^{n}\binom{n}{k}D_x^{n-k}D_y^{k}dx^{n-k}dy^{k}f(x,y) \qquad \text{…(2.2)}$$

Consider now $z=f(x,y)$ such that:

$x=u(t)$ and $y=v(t)$ then $df(x,y)$ and $d^2f(x,y),\cdots$ are given as follows:

$$df(x,y)=f_x(x,y)dx+f_y(x,y)dy$$

where dx and dy are differentials of other functions not still constant, therefore

$$d^2f(x,y)=\left(D_x dx+D_y dy\right)^2 f(x,y)+\left(D_x d^2x+D_y d^2y\right)f(x,y)$$

and

$$d^3f(x,y)=\left(D_x dx+D_y dy\right)^3 f(x,y)+f_x d^3x+2f_{xx}d^2x^2+f_y d^3+2f_{yy}d^2y^2+$$
$$3f_{xy}d^2xdy+3f_{xy}d^2y.$$

Therefore

$$d^3f(x,y)=\left(D_x dx+D_y dy\right)^3 f(x,y)+\sum_{i=1}^{2}\binom{2}{i-1}\left(f_{x^i}d^{4-i}x^i+f_{y^i}d^{4-i}y^i\right)+g(D_x,D_y,dx,dy),$$

where $g(D_x,D_y,dx,dy)=3f_{xy}d^2xdy+3f_{xy}dxd^2y$.

The following lemma gives a general form of any compound function $f(x,y)$

Lemma 2.1

Let $f(x,y)$ be a continuous function of two variables x and y such that $x=u(t)$ and $y=v(t)$ where $a\le t\le b$ for $a,b\in\mathbf{R}$, then the n^{th} order differentiation of $f(x,y)$ is given by:

$$d^nf(x,y)=\left(D_x dx+D_y dy\right)^n f(x,y)+\sum_{i=1}^{n-1}\binom{n-1}{i-1}\left(f_{x^i}d^{n+1-i}x^i+f_{y^i}d^{n+1-i}y^i\right)$$
$$+\sum_{k=1}^{n-2}\sum_{j=1}^{n-k-1}\sum_{i=j}^{n-k}\alpha_i\cdot\binom{n}{i}\left(f_{x^jy^k}d^{i-j+1}x^jd^{n-i-k+1}y^k\right)$$

where α_i are real constants

Proof:

Use mathematical induction to get the result.

3- Taylor Expansion of *f*(*x*,*y*)

Let f be a real valued function defined on a domain D , then

$$\Delta f(x_o) = f(x) - f(x_o) = f(x_o + \Delta x) - f(x_o), \qquad \dots(3.1)$$

where $\Delta x = x - x_o$ (later we shall use h $= x - x_o$).

Therefore

$$\Delta f(x_o) = \sum_{k=1}^{\infty} \frac{f^{(k)}(x_o)}{k!} \Delta^k x = \sum_{k=1}^{n-1} \frac{f^{(k)}(x_o)}{k!} \Delta^k x + R_{n-1}(x_o) \qquad \dots(3.2)$$

where $R_{n-1}(x_o) = \frac{f^{(n)}(\xi)}{n!}$ for some $\xi \in [x_o, x]$ [3].

Now by using Definition (1.4) we get that $\Delta y \cong f'(x_o)\Delta x$, and then

$$dy \cong \Delta y \;\Rightarrow\; dy \cong f'(x_o)\Delta x\,, \qquad \dots(3.3)$$

therefore

$$\Delta f(x_o) \cong \sum_{k=1}^{\infty} \frac{d^k f(x_o)}{k!}$$

thus

$$\Delta f(x_o) \cong \sum_{k=1}^{n-1} \frac{d^k f(x_o)}{k!} + R_{n-1}(x_o) \qquad \dots(3.4)$$

where $R_{n-1}(x_o) = \frac{1}{n!} d^n f(\xi)$, for some $\xi \in [x_o, x]$.[4] $\qquad \dots(3.5)$

The formulas (3.4) and (3.5) represent differential formulas of a Taylor series expansion with remainder.

Similarly with a necessary modification we can define a Taylor series expansion of multiple variable functions [2], [5].

Let $z = f(x, y)$ be a function of two variables defined in a rectangular region D such that its n -partial derivatives are defined and continuous in D . By using (3.1) and (3.4) we find that:

$$f(x,y)=\sum_{k=0}^{n-1}\frac{d^k f(x_o,y_o)}{k!}+R_{n-1}(x_o,y_o) \qquad \ldots(3.6)$$

with the assumption that

$f(x_o,y_o)=d^o f(x_o,y_o)$ and $R_{n-1}(x_o,y_o)=\frac{1}{n!}d^n f(\xi,\lambda)$ for some $\xi\in[a,x]$ and $\lambda\in[c,y]$ in $D=\{(x,y): a\le x\le b\ ,\ c\le y\le d\}$.

Now putting $h_x=x-x_o$ and $h_y=y-y_o$, and then applying (2.2) and (3.6) we get:

$$f(x,y)=\sum_{s=0}^{n-1}\frac{1}{s!}\sum_{k=0}^{s}\binom{s}{k}D_x^{s-k}D_y^k h_x^{s-k}h_y^k f(x_o,y_o)+R_{n-1}(x_o,y_o) \qquad \ldots(3.7)$$

where $R_{n-1}(x_o,y_o)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}D_x^{n-k}D_y^k h_x^{n-k}h_y^k f(\xi,\lambda)$ for some $\xi\in[a,x]$, $\lambda\in[c,y]$ [6].

Consequentially, with the first formula of (2.2) we can write the exponential Taylor expansion formula of a function of two variables as:

$f(x,y)\cong f(x_o,y_o)+\sum_{s=1}^{n-1}\frac{1}{s!}(D_xh_x+D_yh_y)^s f(x_o,y_o)\cong e^{(D_xh_x+D_yh_y)}f(x_o,y_o)$ for unlimited n .

In the next section we try to deduce new formulas of Taylor series with different forms of remainders.

4- Integral Formula of Taylor Series with Remainders

The integral formula of Taylor series of a function of two variables is based on the line integral on a curve. Let $z=f(x,y)$ be a two variables function whose partial derivatives f_x and f_y are defined and continuous in an open rectangle region D and its differentiation is given by:

$$df(x,y)=f_x dx+f_y dy=Pdx+Qdy\ , \qquad \ldots(4.1)$$

provided that $f(x,y)$ posses its integral line $\int_C df(x,y)$ where *C* is a curve in D . Let $A(x_o,y_o)$ be the initial point of *C* and $B(x,y)$ be the terminal point of *C*, then

$$\int_C df(x,y) = \int_{A(x_o,y_o)}^{B(x,y)} df(x,y), \qquad \ldots(4.2)$$

Therefore $$\int_C (Pdx + Qdy) = f(x,y) - f(x_o,y_o), \qquad \ldots(4.3)$$

provided that the differentiation is not exact whenever we used it , since the line integral of exact differentiation will vanish.

Theorem 4.1

Let $z = f(x,y)$ be a function of two variables whose *n* partial derivatives in *x* and *y* are continuous in an open rectanglular region D such that $f(x,y)$ has a total differential of any order over a sectionally smooth curve *C* contained completely in D with initial point $(x_o,y_o)$and terminal point(x,y). Then the Taylor series of $f(x,y)$ whose integral form of the remainder is given by:

$f(x,y) = f(x_o,y_o) + \sum_{k=1}^{n-1} \frac{1}{2^{k-1}} (\int_C \ )^k d^k f(x_o,y_o) + R_{n-1}(x_o,y_o)$, where

$R_{n-1}(x_o,y_o) = \frac{1}{2^n} \int_C \cdots \int_C d^n f(s,u)$ for some $s \in [a,x]$ and $u \in [c,y]$ in $D = \{(x,y) : a \le x \le b \ , \ c \le y \le d\}$.

Proof:

Since $\int_C df(s,u) = \int_{A(x_o,y_o)}^{B(x,y)} df(s,u) = f(x,y) - f(x_o,y_o)$, then by using (2.1) we get

$$f(x,y) = f(x_o,y_o) + \int_C df(s,u)$$

$$= f(x_o,y_o) + \int_C [df(x_o,y_o) + \tfrac{1}{2} \int_C d^2 f(s,u)]$$

where $$df(x_o.y_o) = df(x,y)\Big|_{\substack{x=x_o \\ y=y_o}} .$$

In general we obtain:

$$f(x,y) = f(x_o,y_o) + \sum_{k=1}^{n-1} \frac{1}{2^{k-1}} (\int_C \ )^k d^k f(x_o,y_o) + R_{n-1}(x_o,y_o),$$

where$R_{n-1}(x_o,y_o) = \frac{1}{2^n} \int_C \cdots \int_C d^n f(s,u)$,for some $s \in [a,x]$ and $u \in [c,y]$.

Corollary 4.2

Let $z = f(x, y)$ be a two variables function satisfying the conditions of Theorem 4.1, then:

$$R_{n-1}(x_o, y_o) = \sum_{k=n}^{\infty} \tfrac{1}{2^k} \Big(\int_C\Big)^k d^k f(x_o, y_o)$$

$$= \sum_{k=n}^{\infty} \tfrac{1}{2^k} \Big(\int_C\Big)^k \sum_{i=0}^{k} \Big(\tbinom{k}{i} D_x^{k-i} D_y^{i}\, dx^{k-s} dy^{i} \Big|_{\substack{x = x_o \\ y = y_o}}$$

Proof:

For finding its Taylor expansion, expand f in a Taylor series and use formula (2.2).

Theorem 4.3

Let $f(x, y)$ be a function whose *n* partial derivatives in *x* and *y* are continuous in an open rectanglular region D such that $f(x, y)$ has a total differential of any order over a sectionally smooth curve *C* where *C* is a curve from $A(0, x)$ to $B(0, y)$. Then the Taylor series of $f(x, y)$ with integral form of the remainder is given by:

$$f(x, y) = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \frac{\binom{k}{i}}{2^k (k-i)!\, i!} \left(xD_x\right)^{k-i} \left(yD_y\right)^{i} f(x, y) \Big|_{\substack{x = x_o \\ y = y_o}} + R_{n-1}(x_o, y_o),$$

where

$$R_{n-1} = \frac{1}{2^n (n-1)!} \left[\begin{array}{l} \int_0^x \left(x-u\right)^{n-1} f_{s^n}(u, t)\, du \\ + \int_0^y \left(y-v\right)^{n-1} f_{t^n}(v, s)\, dv \end{array} \right] + \frac{1}{2^n} \sum_{k=0}^{m=n-2} \frac{\binom{m+2}{k+1}}{(m-k)!\, k!} \times \mathbf{I},$$

and $\mathbf{I} = \int_0^y \int_0^x \left(x-u\right)^{m-k} \left(y-v\right)^{k} f_{s^{m+1} t^{k+1}}(u, v)\, du\, dv$

Proof:

Put $f_o = f(X_o) = f(x_o, y_o)$, then using theorem (4.1) we get

$$f = f_o + \sum_{k=1}^{n-1} \tfrac{1}{2^k} \Big(\int_C\Big)^k d^k f_o,$$

$$= f_o + \tfrac{1}{2} \int_{X_o}^{X} f_x(X_o)\, dx + f_y(X_o)\, dy$$

$$+\frac{1}{4}\int_{X_o}^{X}\int_{X_o}^{X}\left\{\begin{array}{l} f_{xx}(X_o)dx^2+2f_{xy}(X_o)dxdy+f_x(X_o)dx \\ +f_{yy}(X_o)dy^2 \end{array}\right\}+\sum_{k=3}^{n-1}\frac{1}{2^k}(\int_C\ )^k d^k f_o$$

$$=f(x_o)+\frac{1}{2}[\int_0^x f_x(x_o)dx+\int_0^y f_y(x_o)dy]$$

$$+\frac{1}{4}[\int_0^x\int_0^x f_{xx}(x_o)dxx+2\int_0^x\int_0^y f_{xy}(x_o)dxdy+\int_0^y\int_0^y f_{yy}(x_o)dyy+\sum_{k=3}^{n-1}\frac{1}{2^k}(\int_C\ )^k d^k f_o$$

$$=f(x_o)+\frac{1}{2}[xf_x(x_o)+yf_y(x_o)]$$

$$+\frac{1}{4}[\frac{x^2}{2}f_{xx}(x_o)+2xyf_{xy}(x_o)+\frac{y^2}{2}f_{yy}(x_o)]+\sum_{k=3}^{n-1}\frac{1}{2^k}(\int_C\ )^k d^k f_o$$

In general applying formula (2.2) to expand each d^n and integrate the result term by term we obtain:

$$f=\sum_{k=0}^{n-1}\sum_{i=0}^{k}\frac{\binom{k}{i}}{2^k(k-i)!i!}\left(xD_x\right)^{k-i}\left(yD_y\right)^i f(x,y)\Big|_{\substack{x=x_o\\ y=y_o}},$$

for determination of R_{n-1},follows from Theorem 4.1, thus

$$R_{n-1}=\frac{1}{2^n}\int_C\cdots\int_C d^n f(s,t)\quad \text{for some } s\in[0,x] \text{ and } t\in[0,y].$$

Therefore

$$R_{n-1}=\frac{1}{2^n}\int_C\cdots\int_C\sum_{i=0}^{n}\binom{n}{i}D_s^{n-i}D_t^i f(s,t)ds^{n-s}dt^i$$

$$=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\int_C\cdots\int_C D_s^{n-i}D_t^i f(s,t)ds^{n-s}dt^i, \qquad \ldots(4.4)$$

for the last formula(4.4) we use integration by part to get the first and final terms of R_{n-1} then using the result obtained by calculating the values of between in terms of R_{n-1} to get the final result of R_{n-1} as follows:

$$R_{n-1} = \frac{1}{2^n (n-1)!}\left[\begin{array}{l}\int_0^x (x-u)^{n-1} f_{s^n}(u,t)du \\ +\int_0^y (y-v)^{n-1} f_{t^n}(v,s)dv\end{array}\right] + \frac{1}{2^n}\sum_{k=0}^{m=n-2} \frac{\binom{m+2}{k+1}}{(m-k)!k!}\times \mathbf{I},$$

where

$$\mathbf{I} = \int_0^y \int_0^x (x-u)^{m-k} (y-v)^k f_{s^{m+1}t^{k+1}}(u,v)dudv$$

Theorem 4.4

Let $z = f(x,y)$ be a function whose *n* partial derivatives in *x* and *y* are continuous in an open rectangular region D such that $f(x,y)$ has a total differential of any order over a sectionally smooth curve *C* where *C* is a curve whose parametric equations are given by $x = h(t), \quad y = g(t)$ $\alpha \le t \le \beta \quad \alpha,\beta \in \mathbf{R}$, where the initial point is $A(x_o,y_o) = (h(\alpha),g(\alpha))$ and the terminal point is $B(x,y) = (h(t),g(t))$ for some $t \in [\alpha,\beta]$. Therefore the Taylor series of f whose remainder is given by:

$$f(x(t),y(t)) = f(x(\alpha),y(\alpha)) + s'(\alpha)(t-\alpha) + \frac{s''(\alpha)}{2}(t-\alpha)^2 + \cdots + \frac{s^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$$
$$+ R_{n-1}(x_o,y_o),$$

where

$$R_{n-1}(x_o,y_o) = \frac{1}{(n-1)!}\int_\alpha^t (t-u)s^{(n)}(u)du$$

And $s(u)$ is the integral of the quantity $P(x(u),y(u))x'(u) + Q(x(u),y(u))y'(u)$

Proof:

We have $\int_C Pdx + Qdy = \int_{(x_o,y_o)}^{(x,y)} Pdx + Qdy$

$$= \int_\alpha^t [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]dt,$$

where $\alpha < t \le \beta$.

Now using equation (4.3) to get

$$f\left(x(t),y(t)\right)=f\left(x(\alpha),y(\alpha)\right)+\int_{\alpha}^{t}s'(u)du$$

Then applying integration by part on the last equation *n*-times we get:

$$f\left(x(t),y(t)\right)=f\left(x(\alpha),y(\alpha)\right)+\sum_{k=1}^{n-1}\frac{s^{(k)}(\alpha)}{k!}+R_{n-1},$$

where

$$R_{n-1}(x_o,y_o)=\frac{1}{(n-1)!}\int_{\alpha}^{t}(t-u)s^{(n)}(u)du$$

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