Hosoya Polynomials of Steiner Distance of Complete m-partite Graphs and Straight Hexagonal chains(*)

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Received on:13/5/2007 ***Accepted on:28/6/2007***

الملخص

تضمن هذا البحث ايجاد متعددات حدود هوسويا لمسافة ستينر $-n$ لكل من بيانات التجزئة $-m$ التام، $K(p_1,p_2,...p_m)$ وبيان سلسلة سداسية مستقيمة $\mathbf{G}_m$. كما اوجدنا القطر $-n$ و دليل وينر $-n$ المسافة ستينر $-n$ لكل من $K(p_1,p_2,...p_m)$ و $\mathbf{G}_m$.

ABSTRACT

The Hosoya polynomials of Steiner distance of complete m-partite graphs $K(p_1,p_2,...p_m)$ and Straight hexagonal chains $\mathbf{G}_m$ are obtained in this paper. The Steiner *n*-diameter and Wiener index of Steiner n-distance of $K(p_1,p_2,...p_m)$ and $\mathbf{G}_m$ are also obtained.

Keywords: Steiner distance, Hosoya polynomial, Steiner *n*-diameter, Wiener index.

1. Introduction

We follow the terminology of [2,3]. For a connected graph $G=(V,E)$ of order *p*, **the *Steiner distance***[8,7] of a non-empty subset $S\subseteq V(G)$ denoted by $d_G(S)$ or simply $d(S)$, is defined to be the size of the smallest connected subgraph *T(S)* of *G* that contains *S*, *T(S)* is called **a *Steiner tree*** of *S*. If |*S*|=*2*, then the definition of the Steiner distance of *S* yields the (ordinary) distance between the two vertices of *S*. For $2\le n\le p$ and |*S*|=*n*, the Steiner distance of *S* is called ***Steiner n-distance*** *of S* in *G*.

The *Steiner n-diameter* of *G* (or the diameter of the Steiner *n*-distance), denoted by $diam_n^* G$ or $\delta_n^*(G)$, is defined to be the maximum Steiner *n*-distance of all *n*-subsets of *V(G)*, that is

$$diam_n^* G = \max\{d_G(S) : S\subseteq V(G), |S|=n\}.$$

Remark 1.1. It is clear that

(1) If $n \geq m$, then $diam_n^* G \geq diam_m^* G$.

(2) If $S' \subseteq S$, then $d_G(S') \leq d_G(S)$.

The ***average Steiner n-distance*** of a graph *G*, denoted by $\mu_n^*(G)$, or average *n*-distance of *G* is the average of the Steiner *n*-distances of all *n*-subsets of *V(G)*, that is

$$\mu_n^*(G) = \binom{p}{n}^{-1} \sum_{\substack{S \subseteq V \\ |S|=n}} d_G(S).$$

If *G* represents a network, then the Steiner *n*-diameter of *G* indicates the number of communication links needed to connect *n* processors, and the average *n*-distance indicates the expected number of communication links needed to connect *n* processors [8].

The *Steiner n-eccentricity* [7] of a vertex $v \in V(G)$, denoted by $e_n^*(v)$, is defined as the maximum of **the Steiner *n*-distances** of all *n*-subsets of *V(G)* containing *v*. **The *Steiner n-radius*** of *G*, denoted by $rad_n^*(G)$, is the minimum of Steiner *n*-eccentricities of all vertices in *G*.

The *Steiner n-distance* of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$ is the sum of the Steiner *n*-distances of all *n*-subsets of *V(G)* containing *v*.

The sum of Steiner *n*-distances of all *n*-subsets of *V(G)* is denoted by $d_n(G)$ or $W_n^*(G)$. Notice that

$$W_n^*(G) = \sum_{\substack{S \subseteq V(G), \\ |S|=n}} d_G(S) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G) = \binom{p}{n} \mu_n^*(G). \quad ....(1.1)$$

The graph invariant $W_n^*(G)$ is called the Wiener index of the Steiner *n*-distance of the graph *G*.

Bounds for the average Steiner *n*-distance of a connected graph *G* of order *p* are given by Danklemann, Oellermann and Swart [4].

***Definition* 1.2**[1] Let $C_n^*(G,k)$ be the number of *n*-subsets of distinct vertices of G with Steiner *n*-distance *k*. The graph polynomial defined by

$$H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k) x^k, \quad .... (1.2)$$

where δ_n^* is the Steiner *n*-diameter of *G*; is called the ***Hosoya polynomial of Steiner n-distance of G***.[1].

Then **the *n-Wiener index* of *G***, $W_n^*(G)$ will be

$$W_n^*(G)=\sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k) \qquad \text{……. (1.3)}$$

The following proposition summarizes some properties of $H_n^*(G;x)$.

***Proposition* 1.2.** For $2 \le n \le p(G)$,

(1) $\deg H_n^*(G;x)$ is equal to the Steiner *n*-diameter of *G*.

(2) $H_n^*(G;1)=\sum_{k=n-1}^{\delta_n} C_n^*(G,k)=\binom{p}{n}$, …….(1.4)

(3) $W_n^*(G)=\frac{d}{dx}H_n^*(G;x)\,|_{x=1}$. …….(1.5)

(4) For *n=2*, $H_2^*(G;x)=H(G;x)-p$, …….(1.6)

where *H(G;x)* is the ordinary Hosoya polynomial of *G*.

(5) Each end-vertex of a Steiner tree *T(S)* must be a vertex of *S*.

For $1 \le n \le p$, let $C_n^*(u,G,k)$ be the number of *n*-subsets *S* of distinct vertices of *G* containing *u* with Steiner *n*-distance *k*. It is clear that

$C_1^*(u,G,0)=1$.

Define

$$H_n^*(u,G;x)=\sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)x^k . \qquad \text{……. (1.7)}$$

Obviously, for $2 \le n \le p$

$$H_n^*(G;x)=\frac{1}{n}\sum_{u\in V(G)} H_n^*(u,G;x) . \qquad \text{……. (1.8)}$$

Ali and Saeed [1] were first who studied this distance-based polynomial for Steiner *n*-distances, and established Hosoya polynomials of Steiner n-distance for some special graphs and graphs having some kind of regularity, and for Gutman's compound graphs $G_1 \bullet G_2$ and $G_1 : G_2$ in terms of Hosoya polynomials of G_1 and G_2.

In this paper, we obtain the Hosoya polynomial of Steiner n-distance of a complete m-partite graph $K(p_1, p_2,...p_m)$; and we determine the Hosoya polynomial of Steiner 3-distance of a straight hexagonal chain $\mathbf{G}_m$. Moreover, $diam_n^* K(p_1, p_2,...p_m)$ and $diam_n^* \mathbf{G}_m$ are also determined.

2. Complete m-partite Graphs

A graph *G* is *m-partite graph* [3], $m \ge 1$, if it is possible to partition $V(G)$ into *m* subsets $V_1, V_2, ..., V_m$ (called *partite sets*) such that every edge *e* of *G* joins a vertex of V_i to a vertex of V_j, $i \ne j$. A *Complete m-partite* *graph G* is an *m*-partite graph with partite sets $V_1, V_2, ..., V_m$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = p_i$, then this graph is denoted by $K(p_1, p_2, ... p_m)$.

It is clear that the order, the size and the diameter of $K(p_1, p_2, ..., p_m)$ are $\sum_{i=1}^{m} p_i, \sum_{i \neq j} p_i p_j$, and 2, respectively.

The following proposition determines the diameter of Steiner *n*-distance of $K(p_1, p_2, ..., p_m)$.

***Proposition* 2.1**. For $n \geq 2$, $m \geq 2$, let $p' = \max\{p_1, p_2, ... p_m\}$, then

$$diam_n^* K(p_1, p_2, ... p_m) = \begin{cases} n, & if \ 2 \leq n \leq p', \\ n-1, & if \ p' < n \leq p. \end{cases}$$

Proof. Let *S* be any *n*-subset of the vertices of $K(p_1, p_2, ..., p_m)$. If *S* contains *u*,*v* such that $u \in V_i$ and $v \in V_j$, $i \neq j$, then $\langle S \rangle$ is connected, and so $d(S) = n - 1$.

If $S \subseteq V_i$, for $1 \leq i \leq m$, then $d(S) = n$, namely, the size of $T(S) (\cong K(1, n))$. Therefore, taking $S \subseteq V_{p'}$ and $2 \leq n \leq p'$, we get $diam_n^* K(p_1, p_2, ..., p_m) = n$.

If $n > p'$, then *S* must contain vertices from at least two different partite sets. This completes the proof. $T(S) (\cong K(1, n))$ ■

***Theorem* 2.2**. For $n, m \geq 2$,

$$H_n^*(K(p_1, p_2, ..., p_m); x) = C_1 x^{n-1} + C_2 x^n,$$

in which

$$C_1 = \binom{p}{n} - \sum_{i=1}^{m} \binom{p_i}{n}, \ C_2 = \sum_{i=1}^{m} \binom{p_i}{n}.$$

Proof. From Proposition 2.1, for each *n*-subset *S*,

$$n - 1 \leq d(S) \leq n.$$

For each *n*-subset $S \subseteq V_i$, $1 \leq i \leq m$, $d(S) = n$, thus the numbers of such *n*-subset is C_2. Since, the number of all *n*-subsets is $\binom{p}{n}$, then C_1 is as given in the statement of this theorem. ■

The next corollary follows directly from Theorem 2.2.

***Corollary* 2.3.** For $n, m \geq 2$,

$$W_n^*(K(p_1, p_2, ..., p_m)) = (n-1)\binom{p}{n} + \sum_{i=1}^{m} \binom{p_i}{n},$$

$$\mu_n^*(K(p_1,p_2,\ldots,p_m)) = n-1+\frac{\sum_{i=1}^{m}\binom{p_i}{n}}{\binom{p}{n}}.$$

■

Remark. By combinatorial argument one can easily show that

$$\sum_{i=1}^{m}\binom{p_i}{n} < \binom{p}{n},\ m \ge 2$$

Thus for $m \ge 2$,

$$\mu_n^*(K(p_1,p_2,\ldots,p_m) < n .$$

A complete m-partite graph is called a *regular compete m-partite graph*[3], if $p_i = t$ for all i, and it will be denoted by $K_{m(t)}$. The Hosoya polynomial and the Wiener index of Steiner n-distance of $K_{m(t)}$ are given in the following corollary. Its proof follows easily from Theorem 2.2.

***Corollary* 2.4**. For $2 \le n \le p = mt$

(1) $H_n^*(K_{m(t)};x) = m\binom{t}{n}x^n + \left[\binom{mt}{n} - m\binom{t}{n}\right]x^{n-1}$.

(2) $W_n^*(K_{m(t)}) = (n-1)\binom{mt}{n} + m\binom{t}{n}$.

■

3. Straight Hexagonal Chains

A cycle of length 6 can be drawn as a regular hexagon. **A *Straight Hexagonal Chains*** $\mathbf{G}_m$, $m \ge 2$, is a graph consisting of a chain of m hexagons such that every two successive hexagons have exactly one edge in common in the form shown in Fig. 3.1.

It is clear that

$$p(\mathbf{G}_m) = 4m+2,\ q(\mathbf{G}_m) = 5m+1$$

One can easily show that

$$diam\,\mathbf{G}_m = 2m+1. \qquad \ldots\ldots (3.1)$$

The graph $\mathbf{G}_m$ is known to Chemists [5,6] as benzenoid chain of m hexagonal rings.

We shall find a formula for the diameter of the Steiner n-distance of the graph $\mathbf{G}_m$ for some values of n. The vertices of $\mathbf{G}_m$ are labeled as shown in Fig. 3.1.

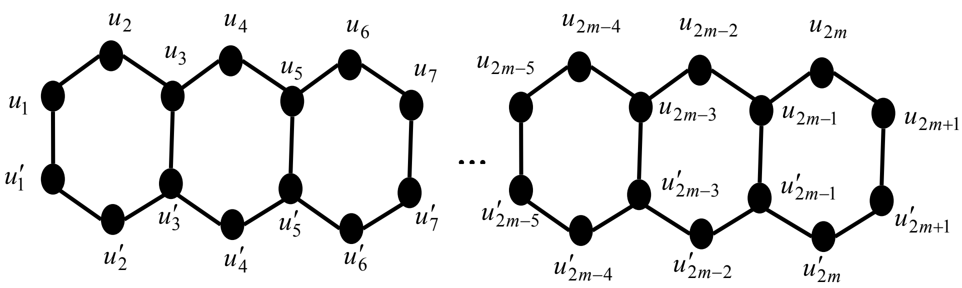

Fig. 3.1 $\mathbf{G}_m$

Proposition **3.1**. For $m \geq 1$, $2 \leq n \leq m+2$,

$$diam_n^* \mathbf{G}_m = 2m + n - 1 .$$

Proof. It is clear that for n=2,

$$diam \mathbf{G}_m = d(u_1, u_{2m+1}') = 2m + 1 .$$

If n= 3, we find that a 3-subset S' of maximum Steiner distance is

$$S' = \{u_1, u_{2m+1}, u_{2m}' \},$$

and so,

$$diam_3^* \mathbf{G}_m = d_3(S') = 2m + 2 .$$

For n=4, we notice that a 4-subset S'' of maximum Steiner distance is

$$S'' = \{u_1, u_{2m+1}', u_{2m}, v\},$$

in which

$$v \in \{u_2', u_4', ..., u_{2m-2}'\} .$$

Thus

$$diam_4^* \mathbf{G}_m = d_4(S'') = 2m + 3$$

Hence, in general for an n-subset S, $2 \leq n \leq m+2$, of maximum Steiner n-distance, we have the following cases:

(1) If n is even, then S consists of the first n vertices from the sequence:

$$u_1, u_{2m+1}', u_{2m}, u_{2m-2}', u_{2m-4}, u_{2m-6}', ..., \begin{cases} u_2', & \text{if } m \text{ is even}, \\ u_4', & \text{if } m \text{ is odd}. \end{cases}$$

When m is even, a Steiner tree, $T(S)$ of such S consists of a ($2m$+1)-path, say, $u_1, u_2, u_3, ..., u_{2m+1}, u_{2m+1}'$ together with $\frac{n-2}{2}$ paths each of length 2, namely $(u_{2m-1}, u_{2m-1}', u_{2m-2}'), (u_{2m-5}, u_{2m-5}', u_{2m-6}'), ...$. Therefore, the size of $T(S)$ is

$$(2m+1) + 2\left(\frac{n-2}{2}\right) = 2m + n - 1 .$$

When m is odd $T(S)$ has the same structure as for the case of even m, and so have size 2m+n-1.

(2) If n is odd, then S consists of the first n vertices from sequence:

$$u_1, u_{2m+1}, u'_{2m}, u_{2m-2}, u'_{2m-4}, u_{2m-6}, u'_{2m-8}, \ldots, \begin{cases} u'_2, & \text{if } m \text{ is odd}, \\ u'_4, & \text{if } m \text{ is even.} \end{cases}$$

When m is odd, a Steiner tree $T(S)$ of such S consists of a $2m$-path, say, $(u_1, u_2, \ldots, u_{2m}, u_{2m+1})$ together with $\frac{n-1}{2}$ paths each of length 2, namely $(u_{2m+1}, u'_{2m+1}, u'_{2m})$, $(u_{2m-3}, u'_{2m-3}, u'_{2m-4})$, ... Therefore, the size of $T(S)$ is

$$2m + 2\left(\frac{n-1}{2}\right) = 2m + n - 1.$$

When m is even, $T(S)$ has the same structure as for odd case of m, and so has size $2m+n-1$.

■

***Proposition* 3.2.** For $m \geq 3$, $m+3 \leq n \leq 2m$,

$$diam_n^* \mathbf{G}_m = 3m + \left\lfloor \frac{n-m}{2} \right\rfloor.$$

Proof. An n-subset S of vertices, $m+3 \leq n \leq 2m$ which has maximum Steiner n-distance consists of $m+2$ vertices described in the proof of Proposition 3.1 together with other n-m-2 vertices chosen in pairs, each pair consists of 2 vertices, belonging to a hexagon, one of degree 2 and the other of degree 3. For instance, when n and m are even, the added *(n-m-2)* vertices are $u'_{2m}, u_{2m-1}; u_{2m-2}, u'_{2m-3}; \ldots$. Each such pair of vertices gives one edge added to the size of $T(S')$, $|S'| = m+2$. Therefore the Steiner n-distance of S is

$$2m + (m+2-1) + \left\lfloor \frac{n-m-2}{2} \right\rfloor.$$

■

Remark. For $m \geq 2$, $n=p$-2,

$$diam_n^* \mathbf{G}_m = n = 4m.$$

Thus, for $2m+1 \leq n \leq 4m$,

$$3m + \left\lfloor \frac{n-m}{2} \right\rfloor \leq diam_n^* \mathbf{G}_m \leq p-2,$$

and

$$diam_n^* \mathbf{G}_m = p-1, \text{ for } n=p\text{-1 or } p.$$

We now find the Hosoya Polynomial of the Steiner 3-distance of $\mathbf{G}_m$.

***Theorem* 3.3.** For $m \geq 3$, we have the following reduction formula for $H_3^*(\mathbf{G}_m; x)$,

$$H_3^*(\mathbf{G}_m; x) = 2H_3^*(\mathbf{G}_{m-1}; x) - H_3^*(\mathbf{G}_{m-2}; x) + F_m(x),$$

where $F_m(x) = 2x^{2(m-1)}[(2m-3) + (9m-11)x + (13m-9)x^2 + (7m-1)x^3 + mx^4]$

Proof. Let S be any 3-subset of $V(\mathbf{G}_m)$. We refer to Fig 3.1, and denote

$A=\{u_1,u_2,u_1',u_2'\}$, $A'=\{u_{2m},u_{2m+1},u_{2m}',u_{2m+1}'\}$,
$B=\{u_3,u_5,...,u_{2m-1}\}$, $B'=\{u_3',u_5',...,u_{2m-1}'\}$,
$C=\{u_4,u_6,...,u_{2m-2}\}$ and $C'=\{u_4',u_6',...,u_{2m-2}'\}$.

For all possibilities of $S\subseteq V(\mathbf{G}_m)-A$ (or $S\subseteq V(\mathbf{G}_m)-A'$), we have the corresponding polynomial $H_3^*(\mathbf{G}_{m-1};x)$. And for all possibilities of $S\subseteq V(\mathbf{G}_m)-\{A\cup A'\}$, the corresponding polynomial is $H_3^*(\mathbf{G}_{m-2};x)$.
Thus

$$H_3^*(\mathbf{G}_m;x)=2H_3^*(\mathbf{G}_{m-1};x)-H_3^*(\mathbf{G}_{m-2};x)+F_m(x),$$

in which $F_m(x)$ is the Hosoya polynomial corresponding to all 3-subsets of vertices that each contains at least one vertex from A and at least one vertex from A'. Therefore $F_m(x)$ can be spilt into two polynomials $F_1(x)$ and $F_2(x)$, where $F_1(x)$ is the Hosoya Polynomial of all 3-subsets S that each contains one vertex from A, one vertex from A' and one vertex from $W=B\cup B'\cup C\cup C'$, and $F_2(x)$ is the Hosoya polynomial corresponding to all 3-subsets S such that $S\subseteq A\cup A'$, $S\cap A\neq\varphi$ and $S\cap A'\neq\varphi$.

(I) Now, to find $F_1(x)$, we consider the following subcases:

(a) If $S=\{u_1,u_{2m},y\}$ or $\{u_1',u_{2m}',y\}$, then

(1) When $y\in B\cup C$, there are *(2m-3)* such subsets S each of 3-distance *(2m-1)*.
(2) When $y\in B'$, there are *(m-1)* such subsets S, each of 3-distance $2m$.
(3) When $y\in C'$, there are *(m-2)* such subsets S, each of 3-distance $2m+1$.

Therefore, for all such possibilities of S, $S=\{u_1,u_{2m},y\}$ or $\{u_1',u_{2m}',y\}$, $y\in W$, the corresponding polynomial is

$$P_1(x)=2x^{m-1}[(2m-3)+(m-1)x+(m-2)x^2]$$

(b) If $S=\{u_1,u_{2m+1},y\}$ or $\{u_1',u_{2m+1}',y\}$, for all $y\in W$, then the corresponding polynomial can be obtained by a similar way of *(a)* as given below

$$P_2(x)=2x^m[(2m-3)+(m-1)x+(m-2)x^2]$$

(c) If $S=\{u_1,u_{2m}',y\}$ or $\{u_1',u_{2m},y\}$, $y\in W$, then the corresponding polynomial is

$$P_3(x)=4(2m-3)x^{2m}.$$

(d) If $S=\{u_1,u_{2m+1}',y\}$ or $\{u_1',u_{2m+1},y\}$, $y\in W$, then the corresponding polynomial is

$$P_4(x)=4(2m-3)x^{2m+1}.$$

(e) If $S=\{u_2,u_{2m},y\}$ or $\{u_2',u_{2m}',y\}$, for all $y\in W$, then the corresponding polynomial is

$$P_5(x)=2x^{2m-2}[(2m-3)+(m-1)x+(m-2)x^2].$$

(f) If $S=\{u_2,u_{2m+1},y\}$ or $\{u'_2,u'_{2m+1},y\}$, for all $y\in W$, then the corresponding polynomial is

$$P_6(x)=2x^{2m-1}[(2m-3)+(m-1)x+(m-2)x^2].$$

(g) If $S=\{u_2,u'_{2m},y\}$ or $\{u'_2,u_{2m},y\}$, for all $y\in W$, then the corresponding polynomial is

$$P_7(x)=4(2m-3)x^{2m-1}.$$

(h) If $S=\{u_2,u'_{2m+1},y\}$ or $\{u'_2,u_{2m+1},y\}$, for all $y\in W$, then the corresponding polynomial is

$$P_8(x)=4(2m-3)x^{2m}.$$ Therefore

$$F_1(x)=\sum_{i=1}^{8}P_i(x)$$
$$=2x^{2m-2}[(2m-3)+(2m-13)x+(13m-19)x^2+(7m-11)x^3$$
$$+(m-2)x^4].$$

(II) To find $F_2(x)$, let S consists of two vertices from A and one vertex from A', or one vertex from A and two vertices from A'. Thus we have $2\binom{4}{2}\binom{4}{1}=2(24)$ possibilities for the 3-subsets *S*, 24 of them give the same Hosoya polynomials for the other 24 cases. These 24 cases are listed in the following table with their Steiner 3-distances:

Table 3.1

no.	3-*subsets S*	*Steiner distances*	*no.*	3-*subsets S*	*Steiner distances*
1.	$\{u_1,u_2,u_{2m}\}$	*2m-1*	13.	$\{u'_1,u'_2,u'_{2m}\}$	*2m-1*
2.	$\{u_1,u_2,u_{2m+1}\}$	*2m*	14.	$\{u'_1,u'_2,u_{2m+1}\}$	*2m*
3.	$\{u_1,u_2,u'_{2m}\}$	*2m*	15.	$\{u'_1,u'_2,u'_{2m}\}$	*2m*
4.	$\{u_1,u_2,u'_{2m+1}\}$	*2m+1*	16.	$\{u'_1,u'_2,u'_{2m+1}\}$	*2m+1*
5.	$\{u_1,u'_1,u_{2m}\}$	*2m*	17.	$\{u_2,u'_2,u_{2m}\}$	*2m*
6.	$\{u_1,u'_1,u_{2m+1}\}$	*2m+1*	18.	$\{u_2,u'_2,u_{2m+1}\}$	*2m+1*
7.	$\{u_1,u'_1,u'_{2m}\}$	*2m*	19.	$\{u_2,u'_2,u'_{2m}\}$	*2m*
8.	$\{u_1,u'_1,u'_{2m+1}\}$	*2m+1*	20.	$\{u_2,u'_2,u'_{2m+1}\}$	*2m+1*
9.	$\{u_1,u'_2,u_{2m}\}$	*2m+1*	21.	$\{u'_1,u_2,u_{2m}\}$	*2m+1*
10.	$\{u_1,u'_2,u_{2m+1}\}$	*2m+2*	22.	$\{u'_1,u_2,u_{2m+1}\}$	*2m+2*
11.	$\{u_1,u'_2,u'_{2m}\}$	*2m*	23.	$\{u'_1,u_2,u'_{2m}\}$	*2m*
12.	$\{u_1,u'_2,u'_{2m+1}\}$	*2m+1*	24.	$\{u'_1,u_2,u'_{2m+1}\}$	*2m+1*

Therefore, there are 4 subsets S of 3-distance ($2m$-1), 20 of 3-distance $2m$, 20 subsets of 3-distance ($2m+1$) and 4 subsets of 3-distance ($2m+2$). Thus,

$$F_2(x) = 4x^{2m-1}(1+5x+5x^2+x^3)\,.$$

Adding $F_1(x)$ to $F_2(x)$ we get $F_m(x)$ as given in the statement of the theorem.■

Remark. Hosoya Polynomials of Steiner 3-distance of $\mathbf{G}_1$ and $\mathbf{G}_2$ are obtained by direct calculation as shown below:

$$H_3^*(\mathbf{G}_1;x) = 6x^2+12x^3+2x^4\,,$$

and

$$H_3^*(\mathbf{G}_2;x) = 15x^2+36x^3+38x^4+27x^5+4x^6\,.$$

The reduction formula given in Theorem 3.3 can be solved to obtain the following useful formula.

***Corollary* 3.4.** For $m \ge 3$

$$H_3^*(\mathbf{G}_m;x) = 3(3m-1)x^2+12(2m-1)x^3+2(18m-17)x^4$$
$$+27(m-1)x^5+4(m-1)x^6+\sum_{k=0}^{m-3}(k+1)F_{m-k}(x)\,,$$

where

$$F_{m-k}(x) = 2x^{2(m-k-1)}[(2m-2k-3)+(9m-9k-11)x+(13m-13k-9)x^2$$
$$+(7m-7k-1)x^3+(m-k)x^4]\,.$$

Proof. From Theorem 3.3,

$$\begin{aligned}
H_3^*(\mathbf{G}_m;x) &= 2H_3^*(\mathbf{G}_{m-1};x)-H_3^*(\mathbf{G}_{m-2};x)+F_m(x)\\
&= 2[2H_3^*(\mathbf{G}_{m-2};x)-H_3^*(\mathbf{G}_{m-3};x)+F_{m-1}(x)]-H_3^*(\mathbf{G}_{m-2};x)+F_m(x)\\
&= 3H_3^*(\mathbf{G}_{m-2};x)-2H_3^*(\mathbf{G}_{m-3};x)+F_m(x)+2F_{m-1}(x)\\
&= 3[\,2H_3^*(\mathbf{G}_{m-3};x)-H_3^*(\mathbf{G}_{m-4};x)+F_{m-2}(x)]\\
&\quad -2H_3^*(\mathbf{G}_{m-3};x)+F_m(x)+2F_{m-1}(x)\\
&= 4H_3^*(\mathbf{G}_{m-3};x)-3H_3^*(\mathbf{G}_{m-4};x)+\sum_{k=0}^{2}(k+1)F_{m-k}(x)\\
&= (m-1)H_3^*(\mathbf{G}_2;x)-(m-2)H_3^*(\mathbf{G}_1;x)+\sum_{k=0}^{m-3}(k+1)F_{m-k}(x) \qquad \ldots(3.1)
\end{aligned}$$

From the remark above, we have

$$H_3^*(\mathbf{G}_2;x) = 15x^2+36x^3+38x^4+27x^5+4x^6\,,$$

and

$$H_3^*(\mathbf{G}_1;x) = 6x^2+12x^3+2x^4$$

Substituting in (3.1) and simplifying, we get the required result. ■

The 3-Wiener index of $\mathbf{G}_m$ is given in the following corollary.

***Corollary* 3.5.** For $m \geq 3$,

$$W_3^*(\mathbf{G}_m) = \frac{4}{3}m(m-2)(8m^2 + 35m + 83) + 225m - 1$$

***Proof*.** It is known that

$$W_3^*(\mathbf{G}_m) = \frac{d}{dx} H_3^*(\mathbf{G}_m; x)\Big|_{x=1}$$

Hence $W_3^*(\mathbf{G}_m) = 393m - 337 + 2\sum_{k=0}^{m-3}[64k^3 + (116 - 128m)k^2 + (64m^2 - 180m + 68)k$

$+ 8(16m^2 - 13m + 4)]$

Now, using the fact that

$$\sum_{k=0}^{m-3} k = \frac{1}{2}(m-3)(m-2), \sum_{k=0}^{m-3} k^2 = \frac{1}{6}(m-3)(m-2)(2m-5) \quad \sum_{k=0}^{m-3} k^3 = \left\{\frac{1}{2}(m-3)(m-2)\right\}^2,$$

and simplifying we get the required result. ■

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