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Ahmed H. Alwan
Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

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ORIGINAL STUDY

g-Coatomic Modules

Ahmed H. Alwan

Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

Abstract

Let R be a ring and M be a left R-module. A submodule N of M is said to be g-small in M, if for every submodule $L \leq M$, with $N + L = M$ implies that $L = M$. Then $Rad_g(M) = \sum N \leq M | N$ is a g-small submodule of $M\}$. We call M g-coatomic module whenever $N \leq M$ and $M/N = Rad_g(M/N)$ then $M/N = 0$. Also, R is called right (left) g-coatomic ring if the right (left) R-module R_R ($_R R$) is g-coatomic. In this work, we study g-coatomic modules and ring. We investigate some properties of these modules. We prove $M = \bigoplus_{i=1}^{n} M_i$ is g-coatomic if and only if each M_i $(i = 1, ..., n)$ is g-coatomic. It is proved that if R is a g-semiperfect ring with $Rad_g(R / Rad_g(R)) = 0$, then R is g-coatomic ring.

Keywords: g-small submodule, Coatomic module, g-coatomic module, g-semiperfect module

1. Introduction

Throughout the present paper, all rings are associative rings with identity and all modules are unital right modules.

Let R be a ring and let M be an R-module. We denote a submodule N of M by $N \leq M$. Let M be an R-module and let $N \leq M$. A submodule N of an R-module M is called small in M (we write $N \ll M$), if for every submodule $L \leq M$, with $N + L = M$ implies that $L = M$. A submodule $L \leq M$ is said to be essential in M, denoted as $L \trianglelefteq M$, if $L \cap N = 0$ for every non-zero submodule $N \leq M$. The submodule K is called a generalized small (briefly, g-small) submodule of M if, for every essential submodule T of M such that $M = K + T$ implies that $T = M$, we can write $K \ll_g M$ (in [12], it is called an e-small submodule of M and denoted by $K \ll_e M$). It is clear that every small submodule is a g-small submodule but the converse is not true generally. If T is essential and maximal submodule of M then T is said to be a generalized maximal submodule of M. The intersection of all generalized maximal submodules of M is called the generalized radical of M and denoted by $Rad_g(M)$ that also knows as the sum of all g-small submodules in M [6,12]. For any R-module M, we write $Rad(M)$, $Soc(M)$ and $Z(M)$ for the radical, socle and singular submodule of M, respectively. M is said to be singular(or non-singular) if $M = Z(M)$ (or $Z(M) = 0$). M is called coatomic if every submodule N of M, $Rad(M/N) = M/N$ implies $M/N = 0$, equivalently every proper submodule of M is contained in a maximal submodule of M see ([1], [3,4]). A submodule N of a module M is called δ-small in M, denoted by $N \ll_\delta M$, if $N + K \neq M$ for any proper submodule K of M with M/K singular. Further, for a module M the submodule $\delta(M)$ is generated by all δ-small submodules of M [10]. In [5] M is called δ-coatomic if every submodule N of M, $\delta(M/N) = M/N$ implies $M/N = 0$. The paper deals with g-coatomic modules as a generalization of coatomic modules. We say that a module M is g-coatomic, if every submodule of M is contained in a generalized maximal submodule of M or equivalently, for a submodule $N \leq M$, if $Rad_g(M/N) = M/N$ then $M/N = 0$. In Section 2, some properties of generalized small submodules are given. In Section 3, several basic properties and characterizations of g-coatomic modules and rings are given.

We will refer to [1,2,9] for all undefined notions used in the text, and also for basic facts concerning coatomic and singular modules.

2. g-small submodule and the functor $Rad_g(M)$

In this section, some important properties of generalized small submodules are presented.

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E-mail address: ahmedha_math@utq.edu.iq.

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Definition 2.1. [6,12] Let N be a submodule of a module M. N is said to be g-small, denoted by $N \ll_g M$, in M if, for every essential submodule T of M such that $M = N + T$ implies that $T = M$ (in [12], it is called an e-small submodule of M and denoted by $K \ll_e M$). If N is any small submodule of M, then N is g-small submodule of M. For the reader's convenience, we record here some of the known results which will be used repeatedly in the sequel.

Proposition 2.2. [12, Proposition 2.3] Let N be a submodule of a module M. The following are equivalent.

(1) $N \ll_g M$,
(2) if $M = X + N$, then $M = X \oplus Y$ with M/X a semisimple module and $Y \leq M$.

Lemma 2.3. Let M be a module. Then

(1) For submodules N, K, L of M with $K \leq N$, we have
  (a) If $N \ll_g M$, then $K \ll_g M$ and $N/K \ll_g M/K$.
  (b) $N + L \ll_g M$ if and only if $N \ll_g M$ and $L \ll_g M$.
(2) If $K \ll_g M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_g N$. In particular, if $K \ll_g M \leq N$, then $K \ll_g N$.
(3) Let N, K, L, and T be submodules of M. If $K \ll_g L$ and $N \ll_g T$, then $K + N \ll_g L + T$.
(4) Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_g M_1 \oplus M_2$ if and only if $K_1 \ll_g M_1$ and $K_2 \ll_g M_2$.

Proof. See Proposition 2.5 of [12,] or see [6].

Corollary 2.4. [6] Let M be an R-module, $K \ll_g M$ and $L \leq M$. Then $K + L/L \ll_g M/L$.

Definition 2.5. [12] Let M be a module. Define $Rad_g(M) = \cap\{N \trianglelefteq M \mid N \text{ is maximal in } M\}$.
For a module M, the intersection of maximal essential submodules of an R-module M is called a generalized radical of M and denoted by $Rad_g(M)$ (in [12], it is denoted by $Rad_e(M)$). If M have no maximal essential submodules, then we denote $Rad_g(M) = M$. Obviously, $Rad(M) \subseteq \delta(M) \subseteq Rad_g(M)$. For an arbitrary ring R, let $Rad_g(R) = Rad_g(R_R)$.
In the following we use g-small submodules to characterize $Rad_g(M)$.

Theorem 2.6. Let M be an R-modules. Then $Rad_g(M) = \sum_{N \ll_g M} N$.

Proof. [12, Theorem 2.10].

Lemma 2.7. Let M and N be modules. Then

(1) *If* $f : M \to N$ is an R-homomorphism, then $f(Rad_g(M)) \leq Rad_g(N)$.
(2) If every proper essential submodule of M is contained in a maximal submodule of M, then $Rad_g(M)$ is the unique largest g-small submodule of M.

Proof. [12] Corollary 2.11.

Lemma 2.8. If $M = \bigoplus_{i \in I} M_i$ then $Rad_g(M) = \bigoplus_{i \in I} Rad_g(M_i)$.

Proof. See [6, Lemma 4].

Lemma 2.9. Let M be a finitely generated R-module. Then$Rad_g(M) \ll_g M$.

Proof. See [8, Lemma 14].

Remark 2.10. It is clear that, in general, $Rad_g(M)$ need not be g-small in M. But if M is a coatomic module, i.e. every proper submodule of M is contained in a maximal submodule of M, then $Rad_g(M)$ is g-small in M by Lemma 2.7(2).

Remark 2.11. Clearly, for a module M, if $Rad(M)$ is small in M then $M/Rad(M)$ has no nonzero small submodule. Also, in [5, Lemma 1.3(2)] If $\delta(M)$ is δ-small in M, then $\delta(M/\delta(M)) = 0$. However this statement cannot be generalized for $Rad_g(M)$, i.e., if $Rad_g(M) \ll_g M$, maybe $Rad_g(M/Rad_g(M)) \neq 0$. As the following example shows.

Example 2.12. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{24}$. $Rad_g(M) = 2\mathbb{Z}_{24} \ll_g M$. But $\frac{\mathbb{Z}_{24}}{2\mathbb{Z}_{24}} \cong \mathbb{Z}_2$ and $\mathbb{Z}_2 \ll_g \mathbb{Z}_2$.

Lemma 2.13. Let M be a nonsingular module. If $Rad_g(M)$ is g-small in M and $K/Rad_g(M)$ is also g-small in $M/Rad_g(M)$ where $K \leq M$, then K is g-small in M.

Proof. Let $K/Rad_g(M)$ be a g-small submodule of $M/Rad_g(M)$ and $M = K + L$ with $L \trianglelefteq M$. So, $L + Rad_g(M) \trianglelefteq M$. By [2, Proposition 1.21], $M/(L + Rad_g(M))$ is singular, so $M/Rad_g(M)/(L + Rad_g(M))/Rad_g(M)$ is singular. By [2, Proposition 1.21], $(L + Rad_g(M))/Rad_g(M)$ is essential submodule of $M/Rad_g(M)$, and since $M/Rad_g(M) = K/Rad_g(M) + (L + Rad_g(M))/Rad_g(M)$ and $K/Rad_g(M)$ is g-small submodule of $M/Rad_g(M)$, $M = L + Rad_g(M)$. Being $Rad_g(M)$ is g-small in M and $L \trianglelefteq M$, we then have $M = L$ and so K is g-small in M.
Now we give a characterization of $M/Rad_g(M)$.

Proposition 2.14. Let M be an R-module.

(1) If, for any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_g M_2$, then $M/Rad_g(M)$ is semisimple.

(2) If, for every submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_g M$, then $M/\ Rad_g(M)$ is semisimple.

Proof:

(1) Let $Rad_g(M) \leq N \leq M$. Then $N/\ Rad_g(M) \leq M/\ Rad_g(M)$. By assumption, there exists a submodule A of N such that $M = A \oplus B$ and $N \cap B \ll_g B$ for some submodules B *of* M. So $M/\ Rad_g(M) = N/\ Rad_g(M) \oplus ((B + Rad_g(M)\ /\ Rad_g(M))$.
(2) Let $Rad_g(M) \leq N \leq M$. By hypothesis, there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll_g M$. Then $N \cap K \leq Rad_g(M)$. Hence $M/\ Rad_g(M)$ is semisimple by [7, Proposition 2.1].

3. g-Coatomic modules and rings

In this section, we define g-coatomic modules and g-semiperfect modules. We study properties and characterizations of g-coatomic and g-semiperfect modules. In [5] the authors defined δ-coatomic modules in this vein, we introduce g-coatomic modules.

Definition 3.1. An R-module M is said to be a g-coatomic if every submodule N of M, $Rad_g(M\ /N) = M/N$ implies $M/N\ = 0$. In The ring R is called right (or left) g-coatomic if the right (or left) R-module R_R (or $_R\ R$) is g-coatomic.
We can give another definition of g-coatomic module.

Lemma 3.2. Let M be a module. The following are equivalent.

(1) M is g-coatomic.
(2) Every proper submodule K of M is contained in a generalized maximal submodule.

Proof
$1 \Rightarrow 2$: Let K be any proper submodule of M. By (1), $Rad_g(M\ /K) \neq M/K$. Hence there exists a singular simple module S and homomorphism $f : M/\ K \to S$. Let $Ker(f)\ =\ N/K$. Then N is an essential and maximal submodule in M.
$2 \Rightarrow 1$: Let K be a proper submodule of M. Assume that $Rad_g(M\ /K)\ = M/K$. We prove $M/K\ = 0$. By (2) there exists an essential and maximal submodule N of M such that $K \leq N$. Let p denote the canonical epimorphism from M/K onto M/N. Since $Ker(p)\ = N/K$, $Rad_g(M\ /K) \leq N/K$. By assumption $M/\ K\ = N/K$, and so $M\ = N$. This contradiction completes the proof.

Theorem 3.3. Let M be an R-module with $Rad_g(M) \ll_g M$ and $Rad_g(M\ /\ Rad_g(M))\ = 0$. Then M is g-coatomic if it satisfies one of the following conditions.

(1) $M/Rad_g(M)$ is semisimple.
(2) For every submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_g M$.

Proof

(1) Suppose that $M/\ Rad_g(M)$ is semisimple with $Rad_g(M) \ll_g M$ and $Rad_g(M/\ \ Rad_g(M))\ = 0$. For any submodule N of M, let $Rad_g(M\ /N)\ = M/N$. Since $M/\ Rad_g(M)$ is semisimple, there exists a submodule K of M with $Rad_g(M) \leq K$ and $M/\ Rad_g(M)\ =\ ((N +\ Rad_g(M))\ /\ Rad_g(M)) \oplus K/\ Rad_g(M)$. Then $M = N + K$ and $N \cap K \leq Rad_g(M)$. Hence $M/N\ =\ (N +\ K)/N \cong K/(N \cap K)$. Let p denote the canonical epimorphism $K/(N \cap K) \to K\ /Rad_g(M)$. By Lemma 2.3, $K\ /Rad_g(M)\ = p(K\ /(N \cap K))\ = p(\ Rad_g(K/\ \ (N \cap K))) \leq Rad_g(K/\ Rad_g(M))$, and by assumption, $Rad_g(M\ /\ Rad_g(M))\ =\ 0$, and so $Rad_g(K\ /Rad_g(M))\ =\ 0$. Hence $K/(N \cap K)\ = 0$. Thus $M/N\ = 0$.
(2) Assume that, for every submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_g M$. By Proposition 2.14, $M/Rad_g(M)$ is semisimple. Hence M is g-coatomic by pa rt (1).

Lemma 3.4. Let M be a module. Then the following holds.

(1) If $X \leq Rad_g(M)$ and X is g-coatomic, then $X \ll M$.
(2) If M is g-coatomic, then $Rad_g(M) \ll M$. In either case $Rad_g(M) \ll_g M$.

Proof

(1) Suppose that $X \leq Rad_g(M)$ and X is g-coatomic module. Let $M = X + Y$ for some submodule Y of M. We show that $M\ =\ Y$. Suppose that $M \neq Y$. Then $X \neq X \cap Y$. By hypothesis and Lemma 3.2, there exists a maximal submodule X' of X such that $X \cap Y \leq X' \leq X$ and X/X' is singular simple. Hence $M/(X' + Y)$ is singular simple since $X\ /X' \cong (X +\ Y\ )/(X' +\ Y)\ = M/(X' +\ Y)$. It follows that $X' \leq Rad_g(M) \leq X' + Y$ and $X' + Y \leq Rad_g(M) +\ Y \leq X' +\ Y$, and so $M\ = X' +\ Y$. Therefore $X\ = X'$. This contradicts the fact that X' is maximal submodule of X. Thus X is small in M and so g-small in M.
(2) Assume that M is g-coatomic module. Let $M = Rad_g(M) + Y$ for some $Y \leq M$. Assume that $M \neq Y$. By Lemma 3.2, there exists $Y \leq Y' \leq M$ with M/Y'

singular simple. Thus, Y' is a generalized maximal submodule. By Lemma 2.3, $Rad_g(M) \leq Y'$. Hence $M = Y'$. This contradicts the fact that Y' is maximal submodule of M. Hence $Rad_g(M)$ is small in M and so g-small in M.

Theorem 3.5. For an R-module M with $Rad_g(M / Rad_g(M)) = 0$, the following are equivalent.

(1) $M/Rad_g(M)$ is semisimple and every submodule of $Rad_g(M)$ is g-coatomic.
(2) For every submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_g M$, and every submodule of M is g-coatomic.

Proof. Note under the assumptions 1 and 2, $Rad_g(M) \ll_g M$ by Lemma 3.4 and Proposition 2.14.
(1) ⇒ (2) For any submodule A of M, let $M/Rad_g(M) = ((A + Rad_g(M)) / Rad_g(M)) \oplus B/Rad_g(M)$ for some submodule B of M. Then $M = A + B$ and $A \cap B \leq Rad_g(M)$. Since $Rad_g(M) \ll_g M$, by Lemma 2.3, $A \cap B \ll_g M$.
Let X be a submodule of M. We show that X is g-coatomic. Assume that $Rad_g(X/A) = X/A$ for some submodule A of X. Then $M/Rad_g(M) = ((A + Rad_g(M)) / Rad_g(M)) \oplus B/Rad_g(M)$ for some submodule B of M since $M/Rad_g(M)$ is semisimple. Then $M = A + B$ and $A \cap B \leq Rad_g(M)$. It is easy to check that
$(X + Rad_g(M))/(A + Rad_g(M)) = Rad_g((X + Rad_g(M)) / (A + Rad_g(M)))$
$\leq Rad_g(M / (A + Rad_g(M)))$.
$Rad_g(M / (A + Rad_g(M))) \cong Rad_g(B / Rad_g(M)) \leq Rad_g(M / Rad_g(M))$.
By assumption, $Rad_g(M / Rad_g(M)) = 0$. Hence $A + Rad_g(M) = X + Rad_g(M)$, and so $X = A + (X \cap Rad_g(M))$. Then $X/A \cong (X \cap Rad_g(M))/(A \cap Rad_g(M))$. Since every submodule of $Rad_g(M)$ is g-coatomic by hypothesis, $X \cap Rad_g(M)$ is a g-coatomic submodule of $Rad_g(M)$. Since $Rad_g((X \cap Rad_g(M)) / (A \cap Rad_g(M))) = (X \cap Rad_g(M)) / (A \cap Rad_g(M))$, we have that $X \cap Rad_g(M) = A \cap Rad_g(M)$. Hence $A = X$.

(2) ⇒ (1) It is clear by Proposition 2.14.

Proposition 3.6. Let $0 \to K \to M \to N \to 0$ be an exact sequence of modules.

(1) If M is g-coatomic module, then N is g-coatomic.
(2) If K and N are g-coatomic modules, then M is g-coatomic.

In particular, any direct summand of a g-coatomic module is g-coatomic.

Proof

(1) We may suppose that $K \leq M$ and $N = M/K$. Let U be a submodule of N. Suppose that $Rad_g(N / U) = N/U$. Then we find submodule L of M with $L/K = U$. Then $Rad_g(M / L) = M/L$. Since M is a g-coatomic module, $M/L = 0$. This implies that $N/U = 0$. It follows that N is g-coatomic.
(2) Assume that K and N are g-coatomic modules. Let L be any proper essential submodule of M.

Case I. $M/K = (L + K)/K$. Then $M = L + K$. Since K is g-coatomic, there exists a generalized maximal submodule K' of K such that $K \cap L \leq K' \leq K$ and K/K' singular simple. Since $K/K' \cong (K + L)/(K' + L) = M/(K' + L)$, $M/(K' + L)$ is singular simple. Thus, $K' + L$ is generalized maximal submodule of M with $L \leq K' + L$. Hence M is g-coatomic by Lemma 3.2.

Case II. $M/K \neq (L + K)/K$. Then $M \neq L + K$. Since N is g-coatomic and $N \cong M/K$, there exists a submodule K'/K of M/K such that $(M/K)/(K'/K) \cong M/K'$ is singular simple and $(L + K)/K \leq K'/K$. Thus, K' is generalized maximal submodule of M with $L \leq K'$. Then M is g-coatomic by Lemma 3.2.

Proposition 3.7. Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of modules M_i ($i = 1, \ldots, n$). Then M is g-coatomic if and only if each M_i ($i = 1, \ldots, n$) is g-coatomic.

Proof. It is sufficient by induction on n to prove this is the case when $n = 2$. Let M_1 and M_2 be g-coatomic modules and $M = M_1 \bigoplus M_2$. We consider the following exact sequence;

$$0 \to M_1 \to M = M_1 \bigoplus M_2 \to M_2 \to 0$$

Hence, $M = M_1 \bigoplus M_2$ is g-coatomic module if and only if M_1 and M_2 are g-coatomic modules by Proposition 3.6.

Definition 3.8. A pair (P, f) is called a projective g-cover of the module M if P is projective right R-module and f is an epimorphism of P onto M with $Ker(f) \ll_g P$.

Lemma 3.9. Let $M = A + B$. If M/A has a projective g-cover, then B contains a submodule A' of A such that $M = A + A'$ and $A \cap A' \ll_g A'$.

Proof. Let $\pi : B \to M/A$ the natural homomorphism and $f : P \to M/A$ be a projective g-cover. Since P is projective, there exists $g : P \to B$ such that $\pi \circ g = f$ and $Ker(f)$ is g-small in P. Then $(\pi \circ g)(P) = f(P)$ and $A \cap g(P) = g(Ker(f))$. Hence $M = A + g(P)$ and $A \cap g(P) = g(Ker(f))$. Since $Ker(f) \ll_g P$, so $g(Ker(f)) \ll_g g(P)$ and thus $A \cap g(P) \ll_g g(P)$.

Lemma 3.10. Let A be any submodule of M. Assume that M/A has a projective g-cover. Then there exists a submodule A' such that $M = A + A'$ and $A \cap A' \ll_g A'$.

Proof. Let $B = M$ in Lemma 3.9.

Definition 3.11. A projective module M is called g-semiperfect if every homomorphic image of M has a projective g-cover.

Lemma 3.12. For any projective R-module M, the following are equivalent:

(1) M is g-semiperfect.
(2) For any $N \leq M$, M has a decomposition $M = M_1 \bigoplus M_2$ for some submodules M_1, M_2 with $M_1 \leq N$ and $M_2 \cap N \ll_g M_2$.

proof. The proof is similar to that of Lemma 2.4 in [10] for δ-semiperfect modules.

Theorem 3.13. Let M be a g-semiperfect module such that $Rad_g(M) \ll_g M$ and $Rad_g(M/Rad_g(M)) = 0$. Then M is g-coatomic.

Proof. Let M be a g-semiperfect module. Let $A \leq M$. By Lemma 3.10, there exists a submodule A' such that $M = A + A'$ such that $A \cap A' \ll_g A'$. So by Theorem 3.3, M is g-coatomic.

Proposition 3.14. For any ring R, $Rad_g(R)$ is g-small in R.

Proof. Let I be an essential right ideal in R ($I \trianglelefteq R$). Assume that $R = Rad_g(R) + I$. Suppose that I is proper and let K be a maximal right ideal containing I. Then K generalized maximal right ideal of R. Hence $Rad_g(R) \leq K$, this is a contradiction. Thus for any $I \trianglelefteq R$ such that $R = Rad_g(R) + I$ we have $R = I$. By definition $Rad_g(R) \ll_g R$.

Definition 3.15. A ring R is named g-semiperfect if every finitely generated right R-module has a projective g-cover. The ring R is g-semiperfect if and only if the regular module R_R is g-semiperfect.
R is g-semiperfect if $R/Rad_g(R)$ is semisimple and idempotents in $R/Rad_g(R)$ can be lifted modulo $Rad_g(R)$.

Proposition 3.16. Let R be a g-semiperfect ring with $Rad_g(R/Rad_g(R)) = 0$. Then R is left and right g-coatomic ring.

Proof. R is right g-coatomic ring from Theorem 3.13 and Proposition 3.14. By symmetry, R is also left g-coatomic ring.

Theorem 3.17: let R be a ring. Then each right ideal I of R with $Rad_g(R/I) = R/I$ is direct summand.

Proof: Let I be a right ideal of R. Assume that $Rad_g(R/I) = R/I$. Then all maps from R/I to singular simple right R-modules is zero. Assume that I is an essential right ideal. Let K be a maximal right ideal containing I. Then R/K is singular simple right R-module. Since R/K is an image of R/I and $Rad_g(R/I) = R/I$, $R = K$. This is a contradiction. Hence I is not essential. Let L be a maximal right ideal with respect to the property $I \cap L = 0$. Then $I \oplus L$ is essential in R. Assume that $I \oplus L$ is proper. Let T be a maximal right ideal containing $I \oplus L$. Then R/T is singular simple image of R/I. This is a contradiction again. Thus $R = I \oplus L$.
The following result is well known and also easy to prove.

Theorem 3.18: The following are equivalent for a ring R.

(1) R is semisimple artinian.
(2) Every maximal right ideal of R is a direct summand of R_R.

Proof: It follows from [11, Lemma 2.1].

Remark 3.19: If I is an essential right ideal in the ring R, then R/I is singular right R-module. The converse is also true. In module case it takes the form: for a nonsingular module M and $N \leq M$, M/N is singular if and only if N is essential in M [2, Proposition 1.21]. Any maximal right ideal in a ring is essential right ideal or direct summand. For g-coatomic rings, this is not the case in general for maximal right ideals.

Theorem 3.20: Let R be a right g-coatomic ring. Then

(1) Every simple right R-module is singular.
(2) Every maximal right ideal in R is essential right ideal.

Proof:

(1) Let I be a maximal right ideal in R. If $Rad_g(R/I) = R/I$, by hypothesis $R = I$. It is not possible. So $Rad_g(R/I) = 0$. Then there exists a nonzero homomorphism $f : R/I \to S$ where S is a singular simple right R-module. Hence f is an isomorphism and so R/I is singular right R-module.
(2) Let I be a maximal right ideal in R. We claim that I is an essential right ideal. Assume that I is not essential right ideal and let $R = I \oplus K$ for some right ideal K. If $Rad_g(R/I) = R/I$, by hypothesis $R = I$. It is not possible. Hence $Rad_g(R/I) \neq R/I$. By (1), R/I is nonzero singular simple right R-module. By Remark 3.19, I is an essential right ideal of R. This contradicts the assumption. Therefore I is direct summand.

Examples 3.21:

(1) Consider the integers $\mathbb{Z}$ as $\mathbb{Z}$-module. Then $Rad_g(\mathbb{Z}) = 0$ and for any prime integer p,

$Rad_g(\mathbb{Z}/p\mathbb{Z}) = 0$ since $\mathbb{Z}/p\mathbb{Z}$ is singular simple $\mathbb{Z}$-module. Hence $\mathbb{Z}$ is g-coatomic $\mathbb{Z}$-module. But the rational numbers $\mathbb{Q}$ as $\mathbb{Z}$-module is not g-coatomic since every cyclic submodule of $\mathbb{Q}$ is small and so $Rad_g(\mathbb{Q}) = \mathbb{Q}$.

(2) Let M be a local module with unique maximal submodule $Rad(M) = Rad_g(M)$. Then M is g-coatomic.

(3) Let M denote the $\mathbb{Z}$-module $\mathbb{Z}$. By Lemma 3.12, M is not g-semiperfect module. Since every proper submodule is contained in an essential maximal submodule, by Lemma 3.2, M is g-coatomic.

Conflicts of interest

There is no conflict of interest.

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