Maximal Generalization of Pure Ideals

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الملخص

الغرض من هذا البحث هو دراسة الحلقات التي تكون فيها كل المثاليات اليمنى العظمى معممة يسرى، كما تم تعريف هذه الحلقات على أنها من النمط MGP وتم عرض بعض خواصها الأساسية وعلاقتها مع الحلقة المنتظمة بقوة ، حلقة منتظمة بضعف وحلقة كاش .

ABSTRACT

The purpose of this paper is to study the class of the rings for which every maximal right ideal is left GP-ideal. Such rings are called MGP-rings and give some of their basic properties as well as the relation between MGP-rings, strongly regular ring, weakly regular ring and kasch ring.

1- Introduction :

Throughout this paper, R denotes as associative ring with identity. An ideal I of a ring R is said to be right(left) pure if for every $a\in I$, there exists $b\in I$ such that a=ab (a=ba). This concept was introduced by Fieldhouse [6], [7], Al-Ezeh [2],[3] and Mahmood [9].

Recall that:-

1- A ring R is regular if for every $a\in R$ there exists $b\in R$ such that a=aba, if $a=a^2 b$, R is called strongly regular.

2-A ring without non-zero nilpotent elements is called reduced.

3-For any element $a\in R$, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively.

4-A ring R is said to be a left(right) uniform ring if and only if every non-zero left(right) ideals is essential .

5-Following [10], a ring R is said to be semi commutative if xy=0 implies that xRy=0, $x,y\in R$. Clearly every reduced ring is semi commutative. It is easy to see that R is semi commutative if and only if every left(right) annihilator in R is a two-sided ideal.

6-Y(R), J(R) are respectively the right singular ideal and the Jacobson radical of R.

2- MGP-rings

In this section, the concept of maximal GP-ideals is introduced and we use it to define MGP-rings .We study such rings and give some of their basic properties.
Following [8], an ideal I of a ring R is said to be right (left) GP-ideal (generalized pure ideal), if for every a in I, there exists b in I and a positive integer n such that $a^n = a^n b$ ($a^n = b a^n$).

Definition 2.1 :

A ring R is called a right (left) MGP-ring if and only if every maximal right (left) ideal is left (right) GP-ideal.

Example:

Let Z_{12} be the ring of the integers module 12.
Then the maximal ideals , I={0,3,6,9} , J={0,2,4,6,8,10} are GP-ideals.

The following theorem gives some interesting characteristic properties of right MGP-rings. Before that we need the next lemma in our proof.

Lemma 2.2:

Let a be a non zero element of a ring R and let $l(a) = 0$. Then for every positive integer n, $l(a^n) = 0$.
Proof: obvious #

Theorem 2.3 :

If R is a right MGP-ring and every ideal is principal, then any left regular element is right invertible .
Proof :

Let $0 \neq c \in R$, such that $l(c) = 0$. If $cR \neq R$, then there exists a maximal right ideal M containing cR . Since R is right MGP-ring, then M is a left GP-ideal, there exists $d \in M$ and a positive integer n, such that $c^n = dc^n$ and $d = cx$, for some $x \in R$.

So $(1-cx) \in l(c^n)$,Since $l(c) = 0$, then by Lemma (2.2) we have $l(c^n) = 0$, thus $cx = 1 \in M$, this contradicts $cR \neq R$. Therefore $cR = R$, and hence c is a right invertible. #

Lemma 2.4 :

Let R be a reduced ring. Then for every $a \in R$, and every positive integer n , $a^n R \cap r(a^n) = 0$.
Proof: See [8]

Proposition 2.5 :

Let R be a reduced, MGP-ring. Then for every a in R and a positive integer n, r (a^n) is a direct summand of R .

Proof :

To prove r (a^n) is a direct summand, we claim that a^n R + r (a^n) = R. If this is not true, let M be a maximal right ideal containing a^n R + r (a^n). Since R is MGP-ring , so $(a^n)^m$ = b $(a^n)^m$ for some b $\in$ M and a positive integer m, this implies (1-b) $\in$ l(a^{nm}) = r (a^n) $\subseteq$ M (R is reduced) , and so 1$\in$ M,a contradiction. Hence a^nR+r(a^n) =R .

Now, since a^n R $\cap$ r (a^n) =0 ,Lemma(2.4) ,then r (a^n) is a direct summand. #

Recall that, a ring R is called a right (left) MP-ring if every maximal right (left) ideal is a left (right) pure.

We consider the condition (*): R satisfies l(b^n) $\subseteq$r(b) for any b $\in$ R and a positive integer n .

Theorem 2.6 :

Let R be a ring satisfying (*). Then R is a right MGP-ring if and only if R is strongly regular .

Proof:

If this is not true let R be a right MGP-ring and let b be any element in R .We shall prove that bR + r (b) =R .

If this is not true let M be a maximal right ideal containing bR+r(b) . Since R is an MGP-ring , then there exists a $\in$ M and a positive integer n such that b^n = ab^n which implies that (1-a) $\in$ l(b^n) $\subseteq$r(b) $\subset$ M , thus

1 $\in$ M ,a contradiction . Therefore bR+ r(b) = R .

In particular, b u + v = 1 , for some u $\in$ R , v$\in$r(b) .

So b = b^2 u , therefore R is strongly regular .

Conversely; assume that R is strongly regular, then by [1], R is regular and reduced. Also by [9] , R is an MP-ring and semi commutative, then R is an MGP –ring . #

Proposition 2.7 :

Let R be a right MGP-ring satisfying (*). Then Y(R) = 0.

Proof:

If Y(R) $\neq$ 0 ,then by a Lemma (7) of [10]; there exists 0 $\neq$ a $\in$Y (R) with a^2 = 0 .From Theorem (2.6) R is strongly regular , that is a = a^2 b , for some b $\in$ R . Hence a = 0, contradiction. Therefore Y(R) = 0. #

Proposition 2.8 :

If R is a right MGP – ring, then any reduced principal right ideal of R is a direct summand .

Proof: Let I = aR be a reduced principal right ideal of R .If aR+r(a) $\neq$ R، then there exists a maximal right ideal M of R containing aR+r(a).

Now , since R is a right MGP-ring and a $\in$ M , then there exists b $\in$ M and a positive integer n such that $a^n = b\, a^n$,and hence $(1-b)a^n= 0$. Since I is reduced then we have (1-b) $\in$ $l(a^n) = r(a^n) = r(a) \subset M$, this implies that 1 $\in$ M , which contradicts M $\neq$ R. Therefore, aR+r(a) = R , thus $a = a^2 c$ for some $c \in R$. If we set $d=a^2 \in I$, then $a=a^2 d$. implies that a = ada and hence aR = e R , where e = ad is an idempotent element . Then by [6], aR is a direct summand. #

Proposition 2.9 :

Let R be a right MGP-ring satisfying (*). If $a^n b = 0$, for any a,b $\in$ R and a positive integer n, then $r(a^n) + r(b) = R$.

Proof: Assume that $r(a^n) + r(b) \neq R$. Let M be a maximal right ideal containing $r(a^n) + r(b)$. Since R is a right MGP-ring and $a^n b = 0$ implies that $b \in r(a^n) \subseteq M$, there exists c $\in$ M and a positive integer m such that $b^m = cb^m$, so(1- c) $\in$ $l(b^m) \subseteq r(b) \subset M$,which implies that $1 \in M$,which is a contradiction . Therefore $r(a^n) + r(b) = R$.

Theorem 2.10 :

Let R be a uniform semi commutative, MGP-ring and every ideal is principal. Then R is a division ring .

Proof : Let 0 $\neq$ a $\in$ R and aR $\neq$ R ,and let M be a maximal right ideal containing aR .Since R is an MGP-ring, then there exists b $\in$ aR $\subseteq$ M ، and a positive integer n such that $a^n= ba^n$.

This implies that $a^n = aca^n$, for some c $\in$ R. Since R is uniform so every ideal is an essential ideal.

Let $x \in r(ar) \cap a^n R$ .Then acx =0 and $x= a^n z$ for some $z \in R$, so $aca^n z= 0$, yields $a^n z=0=x$ .Therefore, $r(ac) \cap a^n R = 0$,since R is a uniform ring and $a^n R \neq 0$,then r(ac) = 0 . Since R is semi commutative , l(ac) = 0 , then by Theorem (2.3) ac is a right invertible element ,so there exists $v \in R$ such that acv = 1 . Hence a (cv) = 1 $\in$ M ,which is a contradiction. Therefore aR = R .

Now, since ar=1 (aR=R),we have ara=a which implies that (1-ra) $\in$ $r(a)=l(a) \subseteq l(ar)=r(ar)=0$.Therefore, (1-ra)=0 ,whence ra=1,so a is a left invertible .Thus R is a division ring. #

3-The relation between MGP-rings and other rings

In this section we give further properties of the MGP-rings and link between MGP- rings and other rings .

We shall begin this section with the following result, which gives the connection between MGP-rings and weakly regular rings.

Following [11], a ring R is a right (left) weakly regular if I^2 =I for each right (left) ideal I of R . Equivalently, if $a \in aRaR$ ($a \in RaRa$) for every a in R . Then R is called weakly regular.

Theorem 3.1 :

Let R be a right MGP-ring and satisfying (*). Then R is a reduced weakly regular ring .

Proof : Let a be a non zero element in R with $a^2 = 0$. Let M be a maximal right ideal containing r (a). Since $a \in r(a) \subseteq M$ and R is an MGP-ring , then there exists $b \in M$ and a positive integer n such that $a^n = ba^n$, which implies that $(1- b) \in l(an) \subseteq r(a) \subset M$, yielding $1 \in M$,which is a contradiction .

Therefore, a = 0, and hence R is a reduced ring .We show that $RxR + r(x) = R$, for any $x \in R$.

Suppose that there exists $y \in R$ such that $RyR+ r(y) \neq R$.
Then there exists a maximal right ideal M of R containing RyR+r(y). Since R is a right MGP-ring , there exists a in M and a positive integer n such that $y^n = a y^n$ implying that $(1-a) \in l(y^n) \subseteq r(y) \subset M$, whence $(1-a) \in M$ and so $1 \in M$ implies that M=R , which is a contradiction.
Therefore, RxR +r(x) =R, for any $x \in R$.
Hence R is a right weakly regular ring. Since R is reduced, it also can be easily verified that R is a weakly regular ring. #

Definition 3.2: [9]

A ring R is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator .

Theorem 3.3 :

Every semi commutative right MGP-ring is a right Kasch ring .

Proof: Let M be any maximal right ideal of R and let Y(R) be the right singular ideal of R .

If $M \cap Y(R) = 0$, then for any $y \in Y(R)$, $y \notin M$, this implies that r(y) is an essential right ideal of R .

Let $x \in r(y) \cap r(1-y)$, then yx= 0 and (1-y) x = 0 yields x=yx=0.

Therefore $r(y) \cap r(I\text{-}y) = 0$, whence $r(1\text{-}y) = 0$. Since R is semi commutative ring, then we have $l(1\text{-}y) = 0$.

By Theorem (2.3),(1-y) is an invertible element of R. Hence $y \in J \subset M$,a contradiction.

Thus $M \cap Y(R) \neq \phi$. Let $0 \neq a \in M \cap Y(R)$.

Since R is an MGP-ring, then there exists $b \in M$ and a positive integer n such that $a^n = ba^n = ara^n$ .We claim that $r(ar) \cap a^n R = 0$.
If not, let $d \in r(ar) \cap a^n R$ .Then $ar\ d = 0$ and $d = a^n x$ for some $x \in R$, so $ara^n x = 0$ implies that $a^n x = 0 = d$. Therefore, $r(ar) \cap a^n R = 0$.But r(ar) is essential ,then $a^n R = 0$ and hence $a^n x = 0$, for all $x \in R$ implies that $a^n \in l(x) = r(x)$. Therefore, $M = r(x)$.Thus R is a right Kasch ring. #

Corollary 3.4:

Let R be a reduced MGP-ring. Then R is a Kasch ring .

Proof: Since R is a reduced right MGP-ring .Then by Theorem (3.3) R is a Kasch ring .#

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