Holomorphically Projective Mappings Equiaffine Spaces Onto Non-Kählerian Spaces

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Abstract

General aspects of the theory of the holomorphically- projective mappings of Kählerian spaces were studied by many authors .

In this paper considered the equations of the mappings of holomorphically-projective of equiaffine spaces onto non- Kählerian spaces in the form of a system of linear Cauchy equations .

1. Introduction

For the beginning we give some basic definitions.

A Riemannian spaces H_n called *almost Hermitian* ([2] , [3])if both metric tensor $g_{ij}(x)$ and almost Hermitian affinor structure $F_i^h(x)$ are determined in it , and satisfies the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h \quad , \quad F_{(i}^\alpha g_{j)\alpha} = 0 \qquad (1)$$

where δ_i^h the Kronecker symbol , (i,j) denoted symmetrization without division .

An almost Hermitian spaces with covariantly constant structure is called *Kählerian spaces*.

A Riemannian spaces K_n is said to be *Kählerian spaces* if there exists , together with the metric tensor $g_{ij}(x)$, a structure $F_i^h(x)$, F is a tensor of type (1 , 1) , with the properties :

$$F_\alpha^h F_i^\alpha = e\,\delta_i^h \;,\quad F_{(ij)} = 0 \quad , F_{i,j}^h = 0 \,, e = \mu 1 \qquad (2)$$

A bijective correspondence between points of Kählerian spaces K_n and $\bar{K}_n$ is called a holomorphically- projective mappings , if any analytically planar curve in K_n is transformed into an analytically planar curve in $\bar{K}_n$. (see [1]-[4])

2. Holomorphically Projective Mappings of Kählerian spaces

A Kählerian space K_n admits a holomorphically projective mapping onto $\overline{K}_n$ if and only if one of the following conditions is satisfied (via diffeomorphism $K_n \to \overline{K}_n$) :

$$\overline{g}_{ij,k} = 2\overline{g}_{ij}\varphi_k + \varphi_{(i}\overline{g}_{j)k} - e\overline{\varphi}_{(i}\overline{F}_{j)k} \ ; \qquad (3)$$

$$a_{ij,k} = \lambda_{(i}g_{j)k} - e\overline{\lambda}_{(i}F_{j)k} \ , \qquad (4)$$

where $\overline{g}_{ij}$ is a metric tensor of $\overline{K}_n$, φ_i , λ_i are gradient vectors , a_{ij} is a tensor , $a_{ij} = a_{ji} = e\,a_{\alpha\beta}F_i^{\alpha}F_j^{\beta}$; $\det\|a_{ij}\| \neq 0$; $\overline{\varphi}_i \equiv \varphi_{\alpha}F_i^{\alpha}$; $\overline{\lambda}_i \equiv \lambda_{\alpha}F_i^{\alpha}$; $\overline{F}_{ij} \equiv \overline{g}_{i\alpha}F_j^{\alpha}$.

The vector λ_i has the following properties:

$$\lambda_{i,j} = \lambda_{j,i} = -e\lambda_{\alpha,\beta}F_i^{\alpha}F_j^{\beta} \ . \qquad (5)$$

From (2) , where $e = \mu 1$, if $e = -1$ then K_n is called an elliptical Kählerian space , and if $e = +1$ then K_n called a hyperbolic Kählerian space.

A curve in K_n defined by the equations $x^h = x^h(t)$ and the conditions

$$d\lambda^h / dt + \Gamma^h_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = \rho_1(t)\lambda^h + \rho_2(t)F^h_{\alpha}\lambda^{\alpha} \ ,$$

where $\lambda^h \equiv dx^h / dt$, ρ_1 , ρ_2 are functions of the argument t .

if $\lambda_i \neq 0$, then the holomorphically projective mapping will called non – trivial .

contracting (4) for $e = -1$, with $g^{ij}(\|g^{ij}\| = \|g_{ij}\|^{-1})$ so the vector $\overline{\lambda}_i$ is a gradient and $\overline{\lambda}_i = 1/4\,\partial_i\left(a_{\alpha\beta}g^{\alpha\beta}\right) \equiv \lambda_{,i}$. If the condition $\overline{\lambda}_{i,j} = \overline{\lambda}_{j,i}$ is hold , it follows that $\overline{\lambda}_{\alpha,\beta}F_i^{\alpha}F_j^{\beta} = 0$.(see[5]-[9])

Theorem 1

A diffeomorphism $f : K_n \to \overline{K}_n$ is holomorphically projective mapping if and only if there exist a solution of the following linear Cauchy equations

$$a_{ij,k} = \overline{\lambda}_{(i}g_{j)k} + \lambda_{(i}F_{j)k} \ , \qquad (6)$$

$$\lambda_{i,j} = \tau F_{ij} + a_{\alpha\beta}M_i^{\alpha\beta} \ , \qquad (7)$$

$$\tau_{,i} = \lambda_{\alpha}M_i^{\alpha} + a_{\alpha\beta}M_i^{\alpha\beta} \ . \qquad (8)$$

On unknown tensor a_{ij} , a vector λ_i , and a function τ , where $a_{ij} = a_{ji}$, $a_{\alpha(i}F_{j)}^{\alpha} = 0$. Here $M_{ij}^{\alpha\beta}$, M_i^{α} , $M_i^{\alpha\beta}$ are tensors determined by formulas below and dependent on g_{ij} , F_i^h of the space K_n.

Proof

Consider the integrability conditions of the equation (4) :

$$a_{\alpha(i}R^{\alpha}_{j)kl} = g_{k(i}\overline{\lambda}_{j),l} - g_{l(i}\overline{\lambda}_{j),k} - F_{k(i}\lambda_{j),l} + F_{l(i}\lambda_{j),k} \ . \qquad (9)$$

Contracting (9) with $F^k_{k'}$ and also $F^l_{l'}$ we obtain two expressions . After removing primes we sum them up . Since in $\overline{K}_n$ it holds

$$R^h_{i\alpha k}F^\alpha_j + R^h_{ij\alpha}F^\alpha_k = 0 \ ,$$

we get

$$F_{k((i\mathrm{T}j)l} - F_{l(i\mathrm{T}j)k} = 0 \quad where \quad \mathrm{T}_{ij} = \bar{\lambda}_{i\,.\,j} + \lambda_{i\,,\alpha}F^\alpha_j \ .$$

It follows that

$$\mathrm{T}_{ij} = \mathrm{T}F_{ij} \qquad \text{, i.e.} \qquad \bar{\lambda}_{i\,,\,j} + \lambda_{i\,,\alpha}F^\alpha_j = \mathrm{T}F_{ij}$$

where T is a function .

Contracting (9) with g^{jk} we then get

$$n\bar{\lambda}_{i\,,\,j} = \mu g_{ij} + \upsilon F_{ij} + a_{\alpha i}R^\alpha_j - a_{\alpha\beta}\,R^{\alpha\,..\,\beta}_{.ij.} \ , \qquad (10)$$

where R^h_{ijk} *and* R_{ij} are Riemannian and Ricci tensors , respectively , the operations of lifting and lowering of indices are induced by the metric tensor , and $\mu\,,\upsilon$ are certain functions .

After symmetrizing (10) we get

$$n\bar{\lambda}_{i\,,\,j} = \mu g_{ij} + \tfrac{1}{2}a_{\alpha(i}R^\alpha_{j)} - a_{\alpha\beta}\,R^{\alpha\,..\,\beta}_{.ij.} \quad , \qquad (11)$$

Substituting (11) into (9) we obtain

$$a_{\alpha\beta}\,M^{\alpha\beta}_{ijkl} = F_{li}\lambda_{j\,,\,k} + F_{lj}\lambda_{i\,,\,k} - F_{ki}\lambda_{j\,,\,l} + F_{kj}\lambda_{i\,,\,l} \qquad (12)$$

Here and in what follows M are tensors determined by $g_{i\,j}$ and F^h_i on K_n. More concretely

$$M^{\alpha\beta}_{ijkl} = \delta^\alpha_{(i}R^\beta_{j)kl} + M^{\alpha\beta}_{1|k(i}g_{j)l} - M^{\alpha\beta}_{1|l(i}g_{j)k} \ ;$$

$$nM^{\alpha\beta}_{1|li} = \tfrac{1}{2}\delta^\alpha_{(i}R^\beta_{j)} - R^{\alpha\,..\,\beta}_{.ij.}$$

where δ^h_i is the Kronecker symbol.

Let ε^j *and* υ^k be vectors such that $\varepsilon^j\upsilon^k F_{jk} = 1$.

Denote $M_i = \varepsilon^\alpha F_{\alpha i}$.

Contracting (12) with $\varepsilon^i\varepsilon^j\upsilon^k$ we get

$$\varepsilon^\alpha\,\lambda_{\alpha\,,\,i} = \tau M_i + a_{\alpha\beta}\,M^{\alpha\beta}_{2|l} \quad , \qquad (13)$$

where $\tau = \lambda_{\alpha\,,\,\beta}\varepsilon^\alpha\upsilon^\beta$. Contracting (12) with $\varepsilon^j\upsilon^k$ and using (13) we arrive at

$$\lambda_{i\,,\,i} = \tau\,F_{ij} + \mathrm{T}_i M_j + a_{\alpha\beta}\,M^{\alpha\beta}_{3|ij} \quad , \qquad (14)$$

where T_i is a vector . Substituting (14) into (9) we obtain

$$T_i(M_k F_{ij} - M_i F_{kj}) + T_j(M_k F_{li} - M_l F_{ki}) = a_{\alpha\beta} M^{\alpha\beta}_{4|ijkl} \ . \quad (15)$$

Contracting (15) with $\upsilon^i \upsilon^j \upsilon^k \varepsilon^l$ we get $\upsilon^\alpha T_\alpha = a_{\alpha\beta} M^{\alpha\beta}_{5|}$, and contracting (15) with $\upsilon^j \upsilon^k \varepsilon^l$ we get $T_i = a_{\alpha\beta} M^{\alpha\beta}_{6|i}$. Then (14) assumes the form (7) .

Integrability conditions of (7) are

$$F_{ij}\tau_{,j} - F_{ik}\tau_{,j} = a_{\alpha\beta} M^{\alpha\beta}_{7|ijk} + \lambda_\alpha M^{\alpha\beta}_{8|ijk} \ . \quad (16)$$

Contracting (15) with $\varepsilon^i \varepsilon^j \upsilon^k$ we get

$$\tau_{,\alpha}\varepsilon^\alpha = a_{\alpha\beta} M^{\alpha\beta}_{9|} + \lambda_\alpha M^{\alpha\beta}_{10|} \ .$$

Then contracting (15) with $\varepsilon^j \upsilon^i$ we get finally (8) .

The theorem is proved .

The system (6) , (7) and (8) has all most one solution for the initial values in a point x_o: $a_{ij}(x_o)$, $\lambda_i(x_o)$ *and* $\tau(x_o)$. Hence , the general solution of this system on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters .

The integrability conditions of the system (6) , (7) and (8) and their differential prolongation are linear algebraic equations on the components of the unknown tensors a_{ij} , λ_i , *and* τ with coefficients from K_n .

Thus , one can , in principle , solve the problem whether a given m - parabolically – Kählerian space K_n admits or not a holomorphically – projective mapping and its uniqueness .

3. Holomorphically Projective Mappings Equiaffine Spaces Onto Non-Kählerian Spaces

In case

$$F^h_{i,j} \neq 0 \quad (17)$$

and

$$F^h_{i/j} \neq 0 \ , \quad (18)$$

where ‘ , ’ and ‘ / ‘ are covariate derivatives in affine – connection space A_n and a Riemannian space $\overline{H}_n$ respectively .

The differentiation of (2) covariantly by x^h in A_n with unknown symmetric regular tensor a_{ij} , we obtaining

$$a^{\alpha(i} F^{j)}_{\alpha,k} = 0 \quad (19)$$

and its simplification differentiation

$$\psi^{(i} F^{j)}_\alpha F^\alpha_{\gamma,k} - \psi^\alpha F^{(i}_\alpha F^{j)}_{\gamma,k} + \delta^{(i}_\gamma F^{j)}_\beta F^\beta_{\alpha,k}\psi^\alpha - F^{(i}_\gamma F^{j)}_{\alpha,k}\psi^\alpha = a^{\alpha(i} F^{j)}_{\alpha,k\gamma} \ .(20)$$

First contraction (20) with respect to the indices j and γ, then contracting with F_i^{τ}, replace index τ by i, there

$$F^i_{\alpha,k}\psi^{\alpha} = \frac{-1}{n+2}\left(a^{\alpha\beta}F^{\sigma}_{\alpha,k\beta} + a^{\alpha\sigma}F^{\sigma}_{\alpha,k\sigma}\right)F^i_{\sigma} \qquad (21)$$

Easily , there exist vectors such as ϕ,φ with the following properties :

$$a^i = F^i_{j,k}\phi^k\varphi^j \neq 0 \ , \qquad (22)$$

also $$a^{ij} = a^{ji} \ . \qquad (23)$$

Theorem 2

Affine- connection spaces A_n admits a holomorphically projective mappings onto a non-Kählerian Hermitian spaces $\overline{H}_n$ if and only if the following system of linear differential equations of type Cauchy is solve with respect to unknown function a^{ij} where

$$a^{ij}{}_{,k} = \psi^{\alpha}F^{(i}_{\alpha}\delta^{j)}_k + \psi^{(i}F^{j)}_k \ , \qquad (24)$$

$$\psi^i = a^{\alpha\beta}T^i_{\alpha\beta} \ . \qquad (25)$$

T is a tensor depend on objects in A_n.

The system (24) and (25) dose not have more than one solution for initial conditions

$a^{ij}(x_o) = a_o^{ii}$ under conditions (23) and

$$a^{ij} = a^{\alpha\beta}F^i_{\alpha}F^j_{\beta} \ . \qquad (26)$$

Proof

We shall investigate the differential conditions of (25) by differentiate them covariantly by x^k in A_n and applying (24) simplification them after contracting with respect to the indices j *and* l then contracting with $F_i^{i'}$ with replace the index i' *by* i. We get

$$\psi^{(i}F^{j)}_{\alpha}F^{\alpha}_{l,k} - \psi^{\alpha}F^{(i}_{\alpha}F^{j)}_{l,k} = a^{\alpha\beta}\overset{1}{T}{}^{(ij)}_{\alpha\beta kl} \ , \qquad (27)$$

where

$$\overset{1}{T}{}^{ij}_{\alpha\beta kl} \overset{def}{=} \delta^i_{\alpha}F^j_{\beta,kl} - \frac{1}{n+2}\left(F^i_l\left(F^j_{\delta}F^{\delta}_{\alpha,k\beta} + F^{\gamma}_{\alpha,k\gamma}F^j_{\beta}\right) + \delta^i_l\left(F^j_{\alpha,k\beta} + F^{\gamma}_{\alpha,k\gamma}F^j_{\beta}\right)\right)$$

Under the condition (18) we can obtain the vectors ε *and* η such that $a^i = F^i_{k,l}\varepsilon^l\eta^k \neq 0$. Evidently , the vector a^i is not collinear with the vector $a^{\alpha}F^i_{\alpha}$. Hence , there exist a covector λ_i such that

$\lambda_\alpha a^\alpha = 0 \, , \lambda_\alpha F^\alpha_\beta a^\beta = 1$

Now , contracting (27) first with $\varepsilon^l \eta^k$ then with $\lambda_i \lambda_j$ we find

$\psi^\alpha \lambda_\alpha = a^{\alpha\beta} \overset{1}{T}{}^{ij}_{\alpha\beta kl} \varepsilon^l \eta^k \lambda_i \lambda_j$ and contracting λ_j we get

$$\psi^i = \varepsilon a^i + a^{\alpha\beta} \overset{2}{T}{}^i_{\alpha\beta} \, , \qquad (28)$$

where $\varepsilon \overset{def}{=} \psi^\alpha F^\beta_\alpha \lambda_\beta$ and

$$\overset{2}{T}{}^i_{\alpha\beta} = \left(\overset{1}{T}{}^{(i\gamma)}_{\alpha\beta\, kl} - \overset{1}{T}{}^{(\gamma\delta)}_{\alpha\beta\, kl} \lambda_\delta F^i_\alpha a^\alpha \right) \varepsilon^k \eta^l \lambda_\gamma$$

Applying (28) to (27) we obtain

$$\psi \left(a^{(i} F^{j)}_\alpha F^\alpha_{l,k} - a^\alpha F^{(i}_\alpha F^{j)}_{l,k} \right) = a^{\alpha\beta} \overset{3}{T}{}^{(ij)}_{\alpha\beta kl} \, , \qquad (29)$$

where

$$\overset{3}{T}{}^{ij}_{\alpha\beta kl} \overset{def}{=} \overset{1}{T}{}^{(ij)}_{\alpha\beta kl} - \overset{2}{T}{}^{(i}_{\alpha\beta} F^{j)}_\gamma F^\gamma_{l,k} + \overset{2}{T}{}^\gamma_{\alpha\beta} F^{(i}_\gamma F^{j)}_{l,k} \, ,$$

The bracket on the left –hand side of (29) must be non-vanishing , otherwise there would be $F^i_{j,k} = 0$, which is in contradiction with (18) . So there exist a tensor field Q^{kl}_{ij} satisfying

$$Q^{kl}_{ij} \left(a^{(i} F^{j)}_\alpha F^\alpha_{l,k} - a^\alpha F^{(i}_\alpha F^{j)}_{l,k} \right) = 1 \, .$$

Hence from (29) it follows that

$$\varepsilon = a^{\alpha\beta} \overset{3}{T}{}^{ij}_{\alpha\beta\, kl} Q^{kl}_{ij}$$

and further

$$\psi^i = a^{\alpha\beta} T^i_{\alpha\beta} \, ,$$

where

$$T^i_{\alpha\beta} \overset{def}{=} \overset{2}{T}{}^i_{\alpha\beta} + a^i \overset{3}{T}{}^{\gamma\delta}_{\alpha\beta\, kl} Q^{kl}_{\gamma\delta} \, . \qquad (30)$$

End of the proof.

The above theorem is a generalization of the results in [2] ,[3] , [4] , [10]

References:

[1] N.S.Sinyukov . Geodesic mappings of Riemannian spaces .Nauka , Moscow, 1979.

[2] K.Yano. Differential geometry on complex and almost complex Spaces. Oxford – London – New York – Paris – Frankfurt ; Pergamon Press. XII , 1965.

[3] J. Mikeš. Geodesic mappings of affine-connected and Riemannian spaces . J. Math. Sci., New York, 1996.

[4] Raad j. k. Al lamy . About 2F- planar mappings of affine – connection spaces . Coll.on Diff. Geom.,Eger (Hungary) 1989.

[5] Raad j. k. Al lamy, I.N. Kurbatova. Invariant geometric objects of 2F – planar mappings of affine connection spaces and Riemannian spaces with affine structure($F^3 = e\delta$) . Dep. f Ukraine,VINIVTI (Kiev) ,1990 , No.1004,UK90.

[6] Raad j. k. Al lamy, Some equations about theory of 2F – Planar mappings Riemannian spaces with ($F^3 = e\delta$); Ukraine , VINIVTI , No 1 1991 .

[7] Raad j. k. Al lamy, Recurrent Riemannian spaces in 2F – Planar mappings Riemannian spaces with ($F^3 = e\delta$)". Ukraine, VINIVTI , No 2 1991 .

[8] Raad j. k. Al lamy, 2F - planar mapping affine connection and Riemannian spaces ; Moscow , MPGU, 1992.

[9] Raad J.K. Al lamy,Josef Mikes,Marie Skodova , On linearly pF – planar mappings , Diff.Geom.and Its Application, Proc.Conf Prague, August 30- sept. 3 , 2004.Charles Univ.Prague (Czech Republic), 2005,347-353

[10] Raad.J.K.AL-lami,MARIE SKODOVA,JOSEF MIKES,ON HOLOMORPHICALLY PROJECTIVE MAPPINGS FROM EQUIAFFINE GENERALLY RECURRENT SPACES ONTO KAHLERIAN SPACES , ARCHIVUM MATHEMATICUM (BRNO) Tomus 42 (2006).