Adjoint representations for SU(2), su(2) and sl(2)

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Abstract

This work, presents four kinds of adjoint representations Ad_1, Ad_2, ad_1 and ad_2 for the special unitary matrix Lie group SU(2) and the special unitary, special linear matrix Lie algebras *su(2)* and *sl(2)*. In the first two we assume the vector spaces as the matrix Lie algebras *su(2)* and *sl(2),* later cases obtained by exploiting the action of *su(2)* and *sl(2)* on themselves. Also, we compute their direct sums $\mathrm{Ad}_1 \oplus \mathrm{Ad}_2$ and $\mathrm{ad}_1 \oplus \mathrm{ad}_2$.The results have been displayed as Tables in a nice form.

الخلاصـة

في عملنا هذا، قدمنا اربعة انواع من التمثيلات الملحقة، ad_1 Ad_1 , Ad_2 , وad_2 لزمرة لي المصفوفية الواحدية الخاصة وجبور لي المصفوفية الخطية الخاصة *su(2)* و *sl(2)*، في الاولين افترضنا فضاءات المتجهات هي جبور لي المصفوفية *su(2)* و*sl(2)* . الحالتين الآخرتين وجدت من خلال فعل الجبريين *su(2)* و *sl(2)* على نفسهما. كذلك قمنا بحساب الجمعين المباشرين $\mathrm{Ad}_1 \oplus \mathrm{Ad}_2$ و النتائج عرضت كجداول منسقة.

INTRODUCTION

In 1896 Frobenius created the general theory of representations. Representation theory of Lie groups can be described as the seek for all possible behaviors of a given group when acting on a vector space[4]. Mahmoud A. A. Sbaih and his colleagues [7] gave a new representation for the Lie unimodular group SU(4). Adjoint representation plays a fundamental rule in the Lie algebra theory because it enables us to transform its problems into a problem in linear algebra representations. Adjoint representations Ad_1 and Ad_2 associated to the conjugation actions of the special unitary matrix Lie group SU(2) on the matrix Lie algebras *su(2)* and *sl(2),* the adjoint representations of the matrix Lie algebras ad_1 and ad_2 associated to their actions on themselves are obtained. Moreover, their direct sums $\mathrm{Ad}_1 \oplus \mathrm{Ad}_2$ and $\mathrm{ad}_1 \oplus \mathrm{ad}_2$ are computed in details.

Preliminaries

Definition 1.1 [2]: A matrix Lie group G is a closed subgroup of the general linear group $GL(n, \mathbb{C})$, that is every sequence $\{A_m\}_{m=1}^{\infty}$ of matrices in G with $A_m \to A \in M_n(\mathbb{C})$ satisfy either $A \in G$ or A is not invertible.

The general linear group $GL(n, \mathbb{C})$ itself and most of its subgroups are matrix Lie groups. In particular those which we are considered in our present work, namely; $SL(n, \mathbb{C})$, and $SU(n, \mathbb{C})$.See [1-5].

Definition 1.2 [8]: A finite dimensional real (complex) representation of matrix Lie group G is a Lie group homomorphism $\Pi: G \to GL(V)$, where V is a finite dimensional real (complex) vector space with dim $V \geq 1$.

The adjoint map of matrix Lie group G into the general linear group acting on the space $\mathfrak{g}$ form a representation called the adjoint representation of G usually denoted by Ad, where $\mathrm{Ad}: G \to GL(\mathfrak{g})$, defined by the formula $\mathrm{Ad}_A(X) = AXA^{-1}$, for $A \in G, X \in \mathfrak{g}$.

Definition 1.3 [5]

A representation of the Lie algebra $\mathfrak{g}$ is a (finite-dimensional) real or complex vector space V together with a homomorphism of Lie algebra i.e. π: $\mathfrak{g} \longrightarrow gl(V)$ is a representation of Lie algebra $\mathfrak{g}$. If π is a linear map satisfying the following:
$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$; for all x, y $\in \mathfrak{g}$.
The adjoint map of matrix Lie algebra $\mathfrak{g}$ into general linear algebra acting on the space $\mathfrak{g}$ is a representation of $\mathfrak{g}$, called the adjoint representation of $\mathfrak{g}$, denoted by $\mathrm{ad}: \mathfrak{g} \to gl(\mathfrak{g})$ and defined as: $\mathrm{ad}_X(Y) = [X, Y]$, for all $X, Y \in \mathfrak{g}$.

Main Results

Theorem 2.1: Let G be a matrix Lie group, V a vector space over a field F such that Π: G $\to$ GL(V) is a representation of G over V then Π can be completely determined by generators of G and basis of V.

Proof: Let $\{S_1, \ldots, S_n\}$ be a generators of G, $\{v_1, \ldots, v_r\}$ be a basis of V then $\prod_{S_i} \in GL(V)\ \forall i = 1 \ldots n$. Suppose $A \in G$ then $A = S_1^{n_1} * \ldots * S_j^{n_j}$ for some $j \in \{1, \ldots, n\}$ and $n_k \in \{1, \ldots, j\}, k \in Z$. For each $X \in V$, $X = \sum_{i=1}^{r} c_i v_i$ for some $c_i \in F$ we have:

$$\prod_A(X) = \prod_{S_1^{n_1} * \ldots * S_j^{n_j}}(X)$$

$$= \prod_{S_1^{n_1} * \ldots * S_j^{n_j}}\left(\sum_{i=1}^{r} c_i v_i\right)$$

$$= c_1\left[\prod_{S_1^{n_1} * \ldots * S_j^{n_j}}(v_1)\right] + c_2\left[\prod_{S_1^{n_1} * \ldots * S_j^{n_j}}(v_2)\right] + \cdots + c_r\left[\prod_{S_1^{n_1} * \ldots * S_j^{n_j}}(v_r)\right]$$

$$= c_1\left[\prod_{S_1^{n_1}}(v_1).\prod_{S_2^{n_2}}(v_1) \ldots \prod_{S_j^{n_j}}(v_1)\right] + c_2\left[\prod_{S_1^{n_1}}(v_2).\prod_{S_2^{n_2}}(v_2) \ldots \prod_{S_j^{n_j}}(v_2)\right] + \cdots + c_r\left[\prod_{S_1^{n_1}}(v_r).\prod_{S_2^{n_2}}(v_r) \ldots \prod_{S_j^{n_j}}(v_r)\right]$$

With the action of G on V we are done. □

Corollary 2.2: For any matrix Lie group G the adjoint representation
$\mathrm{Ad}: G \to GL(\mathfrak{g})$ is completely determined by generators and basis of G and $\mathfrak{g}$ respectively. For a given matrix Lie group G we can associate matrix Lie algebra as follows:

Definition 2.3 [3]: Matrix Lie algebra $\mathfrak{g}$ of matrix Lie group G is the set of all matrices A such that e^{At} is in G for all real numbers t. that is:

$$\mathfrak{g} = \{\mathbf{A} \in \mathbf{M_{n\times n}} \big| \mathbf{e^{At}} \in \mathbf{G}, \mathbf{t} \in \mathbb{R}\}.$$

Lemma 2.4:

Let V be a vector space over a field F, then $\sum_{i=1}^{m} x_i \sum_{j=1}^{n} y_j = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j$ for $x_i, y_j \in V$.

Proof:
$\sum_{i=1}^{m} x_i \sum_{j=1}^{n} y_j = [\sum_{i=1}^{m} x_i](y_1 + y_2 + \ldots + y_n)$
$=(x_1+x_2+ \ldots +x_m)y_1+(x_1+x_2+ \ldots +x_m)y_2+ \ldots +(x_1+x_2+ \ldots +x_m)y_n$
$= x_1y_1 + x_1y_2 + \ldots + x_1y_n + x_2y_1 + x_2y_2 + \ldots + x_2y_n + \ldots + x_my_1 + x_my_2 + \ldots + x_my_n = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j$.

Theorem 2.5:

The adjoint representation of matrix Lie algebra $\mathfrak{g}$ given in definition (1.3) is completely determined by elements of its basis.

Proof: Let $B=\{x_1, x_2, \ldots, x_n\}$ be a basis for a given Lie algebra. Take $X \in \mathfrak{g}$, then $X = \sum_{i=1}^{n} c_i x_i$ for some $c_i \in F$.
Now, $\forall y \in \mathfrak{g}, y = \sum_{j=1}^{n} k_j x_j$ for some $k_j \in F$.

$$\mathrm{ad}_x(y) = [x, y] = xy - yx = \sum_{i=1}^{n} c_i x_i \sum_{j=1}^{n} k_j x_j - \sum_{j=1}^{n} k_j x_j \sum_{i=1}^{n} c_i x_i$$

by lemma (2.4) $= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i k_j x_i x_j - \sum_{j=1}^{n}\sum_{i=1}^{n} k_j c_i x_j x_i$

last expression depends only on the action of $\mathfrak{g}$ on itself by the elements B. □

We define $\mathrm{ad}_1: su(2) \longrightarrow \mathfrak{g}l(su(2))$ and $\mathrm{ad}_2: sl(2) \longrightarrow \mathfrak{g}l(sl(2))$. Then theorem (2.5) ,shows that those representations can be completely determined using basis and generators of *su(2)* and $sl(2)$.

Definition 2.6 [5]: Let G be a matrix Lie group and Let $\Pi_1, \Pi_2, \ldots, \Pi_m$ be a representations of Lie group G acting on a vector spaces $V_1, V_2, \ldots, V_m$. Then, the direct sum of $\Pi_1, \Pi_2, \ldots, \Pi_m$ is a representation $\Pi_1 \oplus \ldots \oplus \Pi_m$ of G acting on the space $V_1 \oplus \ldots \oplus V_m$, defined by:

$[\Pi_1 \oplus \ldots \oplus \Pi_m(A)](v_1, \ldots, v_m) = (\Pi_1(A)v_1, \ldots, \Pi_m(A)v_m)$ for all $A \in G$.

Definition 2.7 [5]: If $\mathfrak{g}$ is a Lie algebra and $\pi_1, \pi_2, \ldots, \pi_n$ are representations of $\mathfrak{g}$ acting on V1, V2,. . ., Vn then we define the direct sum representation of $\pi_1, \pi_2, \ldots, \pi_n$ acting on $V1 \oplus V2 \oplus \ldots \oplus Vn$ by:

$(\pi_1 \oplus \pi_2 \ldots \oplus \pi_n)(V1, V2, \ldots, Vn) = (\pi_1(X)V1, \pi_2(X)V2, \ldots, \pi_n(X)Vn)$, for all $X \in \mathfrak{g}$ where $(V1, V2, \ldots, Vn) \in V1, V2, \ldots, Vn$.

Theorem 2.8: Let $\{\pi_i\}_{i=1}^{n}$ be a representations of a Lie algebra $\mathfrak{g}$ on the vector space $\{V_i\}_{i=1}^{n}$ over a field F. The direct sum $\oplus_{i=1}^{n}\pi_i$ in definition (2.7) below is completely determined by the elements of the basis of $\oplus_{i=1}^{n} V_i$.

Proof: Let $B_i = \{b_{ij}\}_{j=1}^{f_i}$ be a basis of V_i, $i \in [1, \ldots, n]$ where $f_i = \mathrm{Dim}(V_i), \forall i$ fix $X \in \mathfrak{g}, \forall\, Y \in \oplus_{i=1}^{n} V_i$, $Y = (y_1, y_2, \ldots, y_n)$ with $y_i \in V_i$.

we have : $y_i = \sum_{j=1}^{f_i} c_{ij} b_{ij}$ and

$[\oplus_{i=1}^{n} \pi_i(x)]Y = [\oplus_{i=1}^{n} \pi_i(x)]\left(\sum_{j=1}^{f_1} c_{1j} b_{1j}, \ldots, \sum_{j=1}^{f_n} c_{nj} b_{nj}\right)$, by definition (2.6)

$$= \left(\pi_1(x)\left(\sum_{j=1}^{f_1} c_{1j} b_{1j}\right), \ldots, \pi_n(x)\sum_{j=1}^{f_n} c_{nj} b_{nj}\right)$$

$$= \left(\sum_{j=1}^{f_1} c_{1j}\pi_1(x)(b_{1j}), \ldots, \sum_{j=1}^{f_n} c_{nj}\,\pi_n(x)(b_{nj})\right).$$

By the action of $\mathfrak{g}$ on $V_i\, \forall i$ last expression belongs to $\oplus_{i=1}^{n} V_i$. Matrix Lie algebras are vector spaces, we consider the associated matrix Lie algebras *sl(2)* and *su(2)* of the matrix Lie groups SL(2), SU(2) respectively. In the rest of this section, we compute adjoint representations of SU(2) acting on *sl(2)* and *su(2)* and the adjoint representations of *su(2)* and *sl(2),* then find their direct sum, we have:

Case (I): $\mathbf{Ad_1}: \mathbf{SU(2)} \to \mathbf{GL(\boldsymbol{su}(2))}$

First recall that a square matrix A is called Hermitian if $A = A^*$, where($A^* = \overline{A^{tr}}$ is the adjoint matrix of A).

The unitary group U(n) is a subgroup of GL(n,ℂ) satisfy:

$$\mathbf{U(n)} = \{\mathbf{A_{n\times n}} \in \mathbf{GL(n, ℂ)} \,|\, \mathbf{A.A^* = I_n, i.e, A^* = A^{-1}}\}$$

The special unitary group **SU(n)** is a set of all **n×n** unitary matrices with determinant one ,this is a subgroup of **U(n)** ,and hence **GL(n,ℂ)**.

$\mathbf{SU(n)} = \{\mathbf{A} \in \mathbf{U(n)} \,|\, |\mathbf{A}| = \mathbf{1}\}$, see [6].

SU(2) is the set of all two dimensional, complex unitary matrices with generators is the set of three linearly independent, traceless $\mathbf{2 \times 2}$ Hermitian matrices; $F_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$,

$F_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$.

the Lie algebra of SU(n) is the space of all $n \times n$ complex matrices A such that $A^* = -A$ and $\mathrm{trace}(A) = 0$, denoted $su(n)$.

$$\boldsymbol{su(n)} = \{\mathbf{A_{n\times n}} \in \mathbf{GL(n, ℂ)} \,|\, \mathbf{A^* = -A, trace(A) = 0}\}.$$

The basis for *su(2)* is:

$H_1 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $H_3 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}$. Therefore using corollary (2.2) we can compute Ad_1 as follows:

$Ad_{1A}(U) = AUA^{-1} = AUA^*$ for all $A \in \mathbf{SU(2)}$ and $U \in \boldsymbol{su}(\mathbf{2})$. Our computations illustrated in Table (1) below.

Basis of *su(2)* / **Generators of SU(2)**	$\mathbf{H_1}$	$\mathbf{H_2}$	$\mathbf{H_3}$

$\mathbf{F_1}$	$\frac{-1}{4}H_1$	$\frac{-1}{4}H_2$	$\frac{1}{4}H_3$
$\mathbf{F_2}$	$\frac{-1}{4}H_1$	$\frac{1}{4}H_2$	$\frac{-1}{4}H_3$
$\mathbf{F_3}$	$\frac{1}{4}H_1$	$\frac{-1}{4}H_2$	$\frac{-1}{4}H_3$

Table (1) Adjoint representation Ad_1 of SU(2) acting on space *su(2)*

Case (II): $\mathbf{Ad_2}: \mathbf{SU(2)} \to \mathbf{GL}(\boldsymbol{sl}(\mathbf{2}))$

The associated Lie algebra of the matrix Lie group $\mathbf{SL(n,\mathbb{C})}$ is the space of all $\mathbf{n \times n}$ complex matrices with trace zero, denoted by $\boldsymbol{sl}(\mathbf{n},\mathbb{C})$.

$$\boldsymbol{sl}(\mathbf{n},\mathbb{C}) = \{\mathbf{A} \in \mathbf{M_{n\times n}}(\mathbb{C}) | \mathbf{trace(A) = 0}\}.$$

The following matrices form a basis for $\boldsymbol{sl}(\mathbf{2},\mathbb{C})$: $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Using the formula:

$Ad_{2A}(V) = AVA^{-1} = AVA^{*}$ for all $A \in \mathbf{SU(2)}$ and $V \in \boldsymbol{sl}(\mathbf{2})$, we get Table (2) below.

Basis of *sl(2)* / Generators of SU(2)	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{F_1}$	$\frac{-1}{4}X_1$	$\frac{-1}{4}X_3$	$\frac{1}{4}X_2$
$\mathbf{F_2}$	$\frac{-1}{4}X_1$	$\frac{-1}{4}X_3$	$\frac{1}{4}X_3$
$\mathbf{F_3}$	$\frac{1}{4}X_1$	$\frac{1}{4}X_2$	$\frac{1}{4}X_3$

Table (2) Adjoint representation of SU(2) acting on *sl(2)*.

Case III: $\mathbf{ad_1}: \boldsymbol{su}(\mathbf{2}) \longrightarrow \mathbf{gl}(\boldsymbol{su}(\mathbf{2}))$.

Basis of *su(2)* / Generators of *su(2)*	$\mathbf{H_1}$	$\mathbf{H_2}$	$\mathbf{H_3}$
$\mathbf{H_1}$	0	H_3	$-H_2$
$\mathbf{H_2}$	$-H_3$	0	H_1
$\mathbf{H_3}$	H_2	$-H_1$	0

Table (3): Adjoint representation ad_1 of *su(2)* acting on itself.

Case IV: $\mathbf{ad_2}: \boldsymbol{sl}(\mathbf{2}) \longrightarrow \mathbf{gl}(\boldsymbol{sl}(\mathbf{2}))$.

Basis of *sl(2)* / Generators of *sl(2)*	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$
$\mathbf{x_1}$	0	$2x_2$	$-2x_3$
$\mathbf{x_2}$	$-2x_2$	0	X_1
$\mathbf{x_3}$	$2x_3$	$-x_1$	0

Table (4): Adjoint representation ad_2 of *sl(2)* acting on itself.

Case V: $\mathbf{Ad_1 \oplus Ad_2}: \mathbf{SU(2)} \longrightarrow \mathbf{GL}(\boldsymbol{su}(\mathbf{2}) \oplus \boldsymbol{sl}(\mathbf{2}))$

Let SU(2) be the special unitary matrix Lie group and Let Ad_1, Ad_2 be an adjoint representations of SU(2) acting on vector spaces *su(2)*,*sl(2)* respectively. Then according to definition (2.6) the direct sum of Ad_1,Ad_2 is a representation $Ad_1 \oplus Ad_2$ of SU(2) acting on the space $su(2) \oplus sl(2)$ is defined by:

$$\mathbf{Ad_1 \oplus Ad_2}: \mathbf{SU(2)} \longrightarrow \mathbf{GL}(\boldsymbol{su}(\mathbf{2}) \oplus \boldsymbol{sl}(\mathbf{2}))$$

$[Ad_1 \oplus Ad_2(F)](H, X) = (Ad_1(F)H, Ad_2(F)X)$ for all $F \in G, H \in su(2), X \in sl(2)$.

Together with the results obtained in case (I) and Case (II) we have:

$$[Ad_1 \oplus dA_2(F_i)](H_j, X_k) = (Ad_1(F_i)H_j, dA_2(F_i)X_k), \quad 1 \le i, j, k \le 3.$$

Generators Basis of SU(2) / Basis of $su(2) \oplus sl(2)$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$
$(\mathbf{H_1}, \mathbf{X_1})$	$\left(\frac{-1}{4}H_1, \frac{-1}{4}X_1\right)$	$\left(\frac{-1}{4}H_1, \frac{-1}{4}X_1\right)$	$\left(\frac{1}{4}H_1, \frac{1}{4}X_1\right)$

$(\mathbf{H_1},\mathbf{X_2})$	$\left(\frac{-1}{4}H_1,\frac{1}{4}X_3\right)$	$\left(\frac{-1}{4}H_1,\frac{-1}{4}X_3\right)$	$\left(\frac{1}{4}H_1,\frac{1}{4}X_2\right)$
$(\mathbf{H_1},\mathbf{X_3})$	$\left(\frac{-1}{4}H_1,\frac{1}{4}X_2\right)$	$\left(\frac{-1}{4}H_1,\frac{1}{4}X_3\right)$	$\left(\frac{1}{4}H_1,\frac{1}{4}X_3\right)$
$(\mathbf{H_2},\mathbf{X_1})$	$\left(\frac{-1}{4}H_2,\frac{-1}{4}X_1\right)$	$\left(\frac{1}{4}H_2,\frac{-1}{4}X_1\right)$	$\left(\frac{-1}{4}H_2,\frac{1}{4}X_1\right)$
$(\mathbf{H_2},\mathbf{X_2})$	$\left(\frac{-1}{4}H_2,\frac{1}{4}X_3\right)$	$\left(\frac{1}{4}H_2,\frac{-1}{4}X_3\right)$	$\left(\frac{-1}{4}H_2,\frac{1}{4}X_2\right)$
$(\mathbf{H_2},\mathbf{X_3})$	$\left(\frac{-1}{4}H_2,\frac{1}{4}X_2\right)$	$\left(\frac{1}{4}H_2,\frac{1}{4}X_3\right)$	$\left(\frac{-1}{4}H_2,\frac{1}{4}X_3\right)$
$(\mathbf{H_3},\mathbf{X_1})$	$\left(\frac{1}{4}H_3,\frac{-1}{4}X_1\right)$	$\left(\frac{-1}{4}H_3,\frac{-1}{4}X_1\right)$	$\left(\frac{-1}{4}H_3,\frac{1}{4}X_1\right)$
$(\mathbf{H_3},\mathbf{X_2})$	$\left(\frac{1}{4}H_3,\frac{1}{4}X_3\right)$	$\left(\frac{-1}{4}H_3,\frac{-1}{4}X_3\right)$	$\left(\frac{-1}{4}H_3,\frac{1}{4}X_2\right)$
$(\mathbf{H_3},\mathbf{X_3})$	$\left(\frac{1}{4}H_3,\frac{1}{4}X_2\right)$	$\left(\frac{-1}{4}H_3,\frac{1}{4}X_3\right)$	$\left(\frac{-1}{4}H_3,\frac{1}{4}X_3\right)$

Table (5) Direct sum of adjoint representations$(Ad_1 \oplus Ad_2)$.

Case VI: $\mathbf{ad_1 \oplus ad_2}: \boldsymbol{su(2) \oplus sl(2)} \longrightarrow \mathbf{gL}(\boldsymbol{su(2) \oplus sl(2)})$

Let $g = su(2) \oplus sl(2)$ is a Lie algebra and ad_1 is an adjoint representation of *su(2)* acting on vector space *su(2)* and ad_2 is adjoint representation of *sl(2)* acting on vector space *sl(2)*, then we define the direct sum $ad_1 \oplus ad_2$ acting on $su(2) \oplus sl(2)$ by :

$$[ad_1 \oplus ad_2(H_i,X_i)](H_i,X_i) = [ad_{1H_i} \oplus ad_{2X_i}](H_i,X_i) = \left(ad_{1H_i}(H_i), ad_{2X_i}(X_i)\right), \text{ where } 1 \le i \le 3.$$

Together with the results obtained in case (III) and Case (IV) we have

The sum $ad_1 \oplus ad_2$ illustrated by the following:

1- $[ad_1 \oplus ad_2(H_1,X_1)](H_1,X_1) = [ad_{1H_1} \oplus ad_{2X_1}](H_1,X_1) = \left(ad_{1H_1}(H_1), ad_{2X_1}(X_1)\right) = (0,0).$

2- $[ad_1 \oplus ad_2(H_1,X_1)](H_1,X_2) = [ad_{1H_1} \oplus ad_{2X_1}](H_1,X_2) = \left(ad_{1H_1}(H_1), ad_{2X_1}(X_2)\right) = (0,2X_2).$

3- $[ad_1 \oplus ad_2(H_1,X_1)](H_1,X_3) = [ad_{1H_1} \oplus ad_{2X_1}](H_1,X_3) = \left(ad_{1H_1}(H_1), ad_{2X_1}(X_3)\right) = (0,-2X_3).$

4- $[ad_1 \oplus ad_2(H_1,X_1)](H_2,X_1) = [ad_{1H_1} \oplus ad_{2X_1}](H_2,X_1) = \left(ad_{1H_1}(H_2), ad_{2X_1}(X_1)\right) = (H_3,0).$

5- $[ad_1 \oplus ad_2(H_1,X_1)](H_2,X_2) = [ad_{1H_1} \oplus ad_{2X_1}](H_2,X_2) = \left(ad_{1H_1}(H_2), ad_{2X_1}(X_2)\right) = (H_3,2X_2).$

6- $[ad_1 \oplus ad_2(H_1,X_1)](H_2,X_3) = [ad_{1H_1} \oplus ad_{2X_1}](H_2,X_3) = \left(ad_{1H_1}(H_2), ad_{2X_1}(X_3)\right) = (H_3,-2X_3).$

7- $[ad_1 \oplus ad_2(H_1,X_1)](H_3,X_1) = [ad_{1H_1} \oplus ad_{2X_1}](H_3,X_1) = \left(ad_{1H_1}(H_3), ad_{2X_1}(X_1)\right) = (-H_2,0).$

8- $[ad_1 \oplus ad_2(H_1,X_1)](H_3,X_2) = [ad_{1H_1} \oplus ad_{2X_1}](H_3,X_2) = \left(ad_{1H_1}(H_3), ad_{2X_1}(X_2)\right) = (-H_2,2X_2).$

9- $[ad_1 \oplus ad_2(H_1,X_1)](H_3,X_3) = [ad_{1H_1} \oplus ad_{2X_1}](H_3,X_3) = \left(ad_{1H_1}(H_3), ad_{2X_1}(X_3)\right) = (-H_2,-2X_3).$

10- $[ad_1 \oplus ad_2(H_1,X_2)](H_1,X_1) = [ad_{1H_1} \oplus ad_{2X_2}](H_1,X_1) = \left(ad_{1H_1}(H_1), ad_{2X_2}(X_1)\right) = (0,-2X_2).$

11- $[ad_1 \oplus ad_2(H_1,X_2)](H_1,X_2) = [ad_{1H_1} \oplus ad_{2X_2}](H_1,X_2) = \left(ad_{1H_1}(H_1), ad_{2X_2}(X_2)\right) = (0,0).$

12- $[ad_1 \oplus ad_2(H_1,X_2)](H_1,X_3) = [ad_{1H_1} \oplus ad_{2X_2}](H_1,X_3) = \left(ad_{1H_1}(H_1), ad_{2X_2}(X_3)\right) = (0,X_1).$

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