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ORIGINAL STUDY

δ-Small Intersection Graphs of Modules

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Abstract

Let R be a commutative ring with unit and U be a unitary left R-module. The δ-small intersection graph of non-trivial submodules of U, denoted by $\Gamma_\delta(U)$, is an undirected simple graph whose vertices are the non-trivial submodules of U, and two vertices are adjacent if and only if their intersection is a δ-small submodule of U. In this article, we study the interplay between the algebraic properties of U, and the graph properties of $\Gamma_\delta(U)$ such as connectivity, completeness and planarity. Moreover, we determine the exact values of the diameter and girth of $\Gamma_\delta(U)$, as well as give a formula to compute the clique and domination numbers of $\Gamma_\delta(U)$.

Keywords: Module, δ-Small intersection graph, Connectivity, Domination, Planarity

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. Bosak in 1964 [9] introduced the concept of the intersection graph of semigroups. Beck [7] introduced the concept of the zero−divisor graph of rings. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [10]. The intersection graph of ideals of submodules of modules have been investigated in [1]. Numerous other classes of graphs related with algebraic structures have been also actively examined, for instance, see [2−6].

The small intersection graph of a module [13] is another principal graph associated to a ring. The small intersection graph of submodules of a module U, indicated by $\Gamma(U)$ is a graph having the set of all non-trivial submodules of U as its vertex set and two vertices N and L are adjacent if and only if $N \cap L$ is small in U.

Inspired by preceding studies on the intersection graph of algebraic structures, in this paper, we defined $\Gamma_\delta(U)$ the δ-small intersection graph of submodules of a module.

In Section 2, we show that $\Gamma_\delta(U)$ is complete if either U is a module and direct sum of two simple modules or U is δ-hollow module. Also, if U is a δ-supplemented module, then $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$. We proved that if $|\Gamma_\delta(U)| \geq 3$, then $\Gamma_\delta(U)$ is a star graph if and only if $\delta(U)$ is a non-zero simple δ-small submodule of U where every pair of non-trivial submodules of U have non δ-small intersection. We establish that if $|\mathbb{S}_\delta(U)| \in \{1,2\}$ and under some condition, then $\Gamma_\delta(U)$ is a planar graph. Also, $\Gamma_\delta(U)$ is not a planar graph, whenever $|\mathbb{S}_\delta(U)| \geq 3$. In Section 3, we show that if $U = \oplus_{i=1}^{n} U_i$, with U_i are distinct simple left R-module, then $\Gamma_\delta(U)$ is a planar graph if and only if $n \leq 4$.

Throughout this paper R is a commutative ring with identity besides U is a unitary left R-module. We mean a non-trivial submodule of U is a non-zero proper submodule of U. A submodule N (we write $N \leq U$) of U is called small in U (we write $N \ll U$), if for every submodule $L \leq U$, with $N + L = U$ implies that $L = U$. A submodule $L \leq U$ is said to be essential in U, indicated as $L \leq_e U$, if $L \cap N = 0$ for every non-zero submodule $N \leq U$. A module U isnamed singular if $U \cong \frac{K}{L}$ for some module K and an essential submodule $L \leq_e K$. Following Zhou [17], a submodule N of a module U is called a δ-small submodule (we write $N \ll_\delta U$), if, whenever $U = N + X$ with $\frac{U}{X}$ singular, we have $X = U$. It is obvious that every small submodule or projective semisimple submodule of U is δ-small in U. A nonzero R-module U is called hollow [resp., δ-hollow], if every proper submodule of U is small [resp., δ-small] in U [14]. A non-zero module U named local if it is hollow and finitely generated [16]. A submodule P of a module U is

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maximal iff it is not properly contained in any other submodule of U. An R-module U is said to be local if it has a unique maximal submodule. The set is of maximal submodules of U is denoted by $\max(U)$. The Jacobson radical of an R-module U, indicated by $Rad(U)$, is the intersection of all maximal submodules of U. By $\delta(U)$ we will denote the sum of all δ-small submodules of U as in [17, Lemma 1.5 (1)]. Also, $\delta(R) = \delta({}_R R)$. Since $Rad(U)$ is the sum of all small submodules of U, it follows that $Rad(U) \leq \delta(U)$ for a module U. A module U is called δ-local if $\delta(U) \ll_\delta U$ and $\delta(U)$ is maximal [14]. The module U is named simple if it has no proper submodules, and U is said to be semisimple if it is a direct sum of simple submodules. The socle of a module U, denoted by $Soc(U)$, is the sum of all simple submodules of U. The references for module theory are [16,17]; for graph theory is [8].

For a graph Γ, $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and edges, respectively. The set of vertices adjacent to vertex v of the graph Γ is called the neighborhood of v besides indicated by $N(v)$. The order of Γ is the number of vertices of Γ besides we indicated it by $|\Gamma|$. Γ is finite, if $|\Gamma| < \infty$, *else*, Γ is infinite. If u and v are two adjacent vertices of Γ, then we write $u - v$, i.e. $\{u,v\} \in E(\Gamma)$. The degree of a vertex v in a graph Γ, indicated by $\deg(v)$, is the number of edges incident with v. Let u and v be vertices of Γ. An u, $v-$ path is a path (trail) with starting vertex u and ending vertex v. For distinct vertices u and v, $d(u,v)$ is the least length of an u, $v-$ path. If Γ has no such a path, then $d(u,v) = \infty$. The diameter of Γ, indicated by diam (Γ), is the supremum of the set $\{d(x,y)$: u and v are distinct vertices of $\Gamma\}$. A cycle in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. The girth of a graph Γ, indicated by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. A graph is said to be connected (or joined), if there is a path between every pair of vertices of the graph. A joined graph which does not contain a cycle is named a tree. If Γ is a tree consisting of one vertex adjacent to all the others then Γ is named star graph. Γ is complete if it is connected with diam $(\Gamma) \leq 1$. A complete graph with n distinct vertices, indicated by K_n. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph Γ, symbolized by $\omega(\Gamma)$, is called the clique number of Γ.

Lemma 1.1. [17] Let $Z \leq U$. The next are equivalent:

(1) $Z \ll_\delta U$.
(2) If $U = W + Z$, then $U = W \oplus Y$ for a projective semisimple submodule Y with $Y \leq Z$.

Lemma 1.2. [17, Lemma 1.3] Let U be an R-module.

(1) For submodules N, Z, L of U with $Z \leq N$, we have
  i. $N \ll_\delta U$ iff $Z \ll_\delta U$ and $N/Z \ll_\delta U/Z$.
  ii. $N + L \ll_\delta U$ iff $N \ll_\delta U$ and $L \ll_\delta U$.
(2) $Z \ll_\delta U$ and $f : U \to N$ is a homomorphism, then $f(Z) \ll_\delta N$. In particular, if $Z \ll_\delta U \leq N$, then $Z \ll_\delta N$.
(3) Let $Z_1 \leq U_1 \leq U$, $Z_2 \leq U_2 \leq U$ and $U = U_1 \oplus U_2$. Then $Z_1 \oplus Z_2 \ll_\delta U_1 \oplus U_2$ iff $Z_1 \ll_\delta U_1$ and $Z_2 \ll_\delta U_2$.

Lemma 1.3. [17, Lemma 1.5] Let U and N be modules.

(1) $\delta(U) = \sum\{L \leq U | L$ is a δ-small submodule of $U\}$.
(2) If $f : U \to N$ is an R-homomorphism, then $f(\delta(U)) \subseteq \delta(N)$. Also, $\delta\ ({}_R R\ )U \subseteq \delta(U)$.
(3) If $U = \oplus_{i \in I} U_i$, then $\delta(U) = \oplus_{i \in I} \delta(U_i)$.
(4) If every proper submodule of U is contained in a maximal submodule of U, then $\delta(U)$ is the unique largest δ-small submodule of U.

2. Connectedness and completeness

In this Section, we generalizing the definition of [13], we consider a graph $\Gamma_\delta(U)$ as follows:

Definition 2.1. Let U be an R-module. The δ-small intersection graph of U, symbolized by $\Gamma_\delta(U)$, is defined to be a simple graph whose vertices are in one-to-one correspondence with all non-trivial submodules of U and two vertices N and L are adjacent, and we write $N-\ L$, if and only if $N \cap L \ll_\delta U$.

Remark 2.2.

(1) Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$. The nonzero proper submodules of $\mathbb{Z}_6$ are $2\mathbb{Z}_6$ and $3\mathbb{Z}_6$. Obviously, $2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = 0 \ll_\delta \mathbb{Z}_6$ and so $\Gamma_\delta(\mathbb{Z}_6)$ is $2\mathbb{Z}_6 - 3\mathbb{Z}_6$.
(2) It is clear that the graph $\Gamma(U)$ introduced in [13] is a subgraph of $\Gamma_\delta(U)$.
(3) The δ-small submodules of a singular module are small submodules [17]. Clearly when U is a singular module, we get that $\Gamma_\delta(U)$ is the small intersection graph $\Gamma(U)$ of U introduced in [13].

A null graph is a graph whose vertices are not adjacent to each one other (i.e., edgeless graph).

Theorem 2.3. Let U be a not simple module. Then $\Gamma_\delta(U)$ is a null graph if and only if every pair of non-trivial submodules of U, have non δ-small intersection.

Proof. Assume $\Gamma_\delta(U)$ is an edgeless graph. Presume for contrary that there exist $A, B \le U$ such that $A \cap B \ll_\delta U$. At that time $A - B$, hence $\Gamma_\delta(U)$ is not null, which is a contradiction to the hypothesis "$\Gamma_\delta(U)$ is an edgeless graph". The reverse is easy.

Example 2.4. $\Gamma_\delta(\mathbb{Z}_4)$ and $\Gamma_\delta(\mathbb{Z})$ are edgeless graphs.

Proposition 2.5. Let U be an R-module. At that point $\Gamma_\delta(U)$ is complete, if one of the following holds.

(1) If U is δ-hollow.
(2) If $U = U_1 \oplus U_2$ is a module, where U_1 and U_2 are simple R-modules.

Proof. (1) Let U be a δ-hollow module. Presume that A_1, A_2 are two different vertices of the graph $\Gamma_\delta(U)$. From this time A_1 and A_2 are two nonzero δ-small submodules of U. As $A_1 \cap A_2 \le A_i$, for $i = 1, 2$, by Lemma 1.2, $A_1 \cap A_2 \ll_\delta U$, hence $\Gamma_\delta(U)$ is a complete graph.

(2) Assume that $U = U_1 \oplus U_2$ with U_1 besides U_2 are simple R-modules. So, $U_1 + U_2 = U$ and $U_1 \cap U_2 = \{0\}$. Then every non-trivial submodule of U is simple. Let $\mathfrak{A}$, $\mathfrak{B}$ be binary different vertices of $\Gamma_\delta(U)$. At that moment they are the non-trivial submodules of U which are simple besides minimal. Furthermore, $\mathfrak{A} \cap \mathfrak{B} \le \mathfrak{A}, \mathfrak{B}$ and if $\mathfrak{A} \cap \mathfrak{B} \ne (0)$, then minimality of $\mathfrak{A}$ and $\mathfrak{B}$ implies that $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} = \mathfrak{B}$, a contradiction. Thus, $\mathfrak{A} \cap \mathfrak{B} = (0) \ll_\delta U$, henceforth $\Gamma_\delta(U)$ is complete.

By Part 1 of Proposition 2.5, we have the next corollary.

Corollary 2.6. Let R be a ring and U be a module over R. Then the next hold:

(1) If $V(\Gamma(U))$ is a totally ordered set, at that time a graph $\Gamma(U)$ is complete.
(2) If U is a δ-local module, at that point the graph $\Gamma_\delta(U)$ is complete.
(3) Every one nonzero δ-small submodule of U is adjacent to all other vertices of $\Gamma_\delta(U)$ besides the induced subgraphs on the sets of δ-small submodules of U are cliques.

Proof. (1) Suppose $V(\Gamma(U))$ is a totally ordered set. Then all two nontrivial submodules of U are comparable. Evidently, for all $\mathscr{R} \le U$, $\mathscr{R} \ll U$, besides so $\mathscr{R} \ll_\delta U$. Hence, U is a δ-hollow R-module. So, by Proposition 2.5 (1), $\Gamma_\delta(U)$ is complete.

(2) Suppose that U is a δ-local R-module, at that time $\delta(U) \ll_\delta U$ besides $\delta(U)$ is maximal. Now, let $\mathfrak{w}$ be a nonzero submodule of U. To prove that $\mathfrak{w} \le \delta(U)$, by contrary way, assume $\mathfrak{w}$ is not subset of $\delta(U)$, so $\delta(U) + \mathfrak{w} = U$ since $\delta(U)$ is maximal. Hence $\mathfrak{w} = U$ since $\delta(U) \ll_\delta U$, a conflict. Thus, $\mathfrak{w} \le \delta(U)$. So, $\mathfrak{w}$ is δ-small submodule of U. Thus, U is δ-hollow. So, by Proposition 2.5 (1), $\Gamma_\delta(U)$ is complete.

(3) Evident.

Example 2.7. For every $c \in \mathbb{Z}$ with $c \ge 2$ besides for all prime number p, $\mathbb{Z}_{p^c}$ is a local $\mathbb{Z}$-module, then it is hollow and so is δ-hollow. Also, let $R = \mathbb{Z}$, p be a prime and $U = \mathbb{Z}_{p^\infty}$, the Pr ü fer p-group, then every proper submodule of R-module U is δ-small in U. Moreover, $\delta(U) = U$. Hence for every prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is δ-hollow. By Proposition 2.5 (1), $\Gamma_\delta(\mathbb{Z}_{p^c})$ and $\Gamma_\delta(\mathbb{Z}_{p^\infty})$ are complete graphs.

Remark 2.8 [17]**. For a ring** R,

(1) $\delta(R) =$ the intersection of all maximal essential left ideals of R.
(2) $\delta(R) =$ the largest δ-small left ideal of R.
(3) $\delta(R) = R$ if and only if R is a semisimple ring, see [17, Corollary 1.7].

Proposition 2.9. Let R be an integral domain with $\delta(R) \ne 0$ besides let U be a finitely generated torsion-free R-module. Then $\Gamma_\delta(U)$ is connected and $\mathrm{diam}(\Gamma_\delta(U)) \le 2$.

Proof. Since U is finitely generated, then $\delta(U)$ is the largest δ-small submodule of U according to Lemma 1.3(4). Also, the largest δ-small left ideal of R is $\delta(R)$ by Remark 2.8. By Lemma 1.3(2), $\delta(R)U \le \delta(U)$. Thus, $\delta(R)U \ll_\delta U$. Since U is torsion-free and $\delta(R) \ne 0$ then $\delta(R)U \ne 0$. Therefore, $\delta(R)U$ is a vertex in $\Gamma_\delta(U)$. But $X \cap \delta(R)U \ll_\delta U$ for every nonzero submodule X of U by Lemma 1.2(1). So, there exists an edge among vertex $\delta(R)U$ besides X of $\Gamma_\delta(U)$. Also, for all two vertices X, Y in the graph $\Gamma_\delta(U)$, there exists a path $X - \delta(R)U - Y$ of length 2 in $\Gamma_\delta(U)$. This completes the proof.

Theorem 2.10. Let a ring R be a sum $R = \oplus_{i \in I} T_i$ of simple left ideals T_i, $i \in I$. At that point the next statements hold:

(1) $\mathrm{diam}(\Gamma_\delta(R)) = 1$,
(2) The graph $\Gamma_\delta(R)$ is a complete graph.

Proof. (1) Let $R = \oplus_{i \in I} T_i$, where each T_i are simple left ideals, $i \in I$. By Remark 2.8(3), we have $\delta(R) = R$. So, each T_i is δ-small submodule of ${}_R R$. Now, let T_i and T_j are two non-zero ideals of R, then $T_i \cap T_j$ is δ-small in ${}_R R$, and thus, there exists an edge between the vertices T_i and T_j in $\Gamma_\delta(R)$, for all $i, j \in I$. Hence, the graph $\Gamma_\delta(R)$ is connected besides diam $(\Gamma_\delta(R)) = 1$.

(2) It follows from the proof of (1).

Definition 2.11. [12] Let U be a module besides let N and L be submodules of U. L is named a δ-supplement of N in U if $U = N + L$ and $N \cap L \ll_\delta L$ (and so $N \cap L \ll_\delta U$). N is named a δ-supplement submodule if N is a δ-supplement of some submodule of U. U is named a δ-supplemented if every submodule of U has a δ-supplement in U.

Proposition 2.12. Let $\mathscr{h} \leq U$. Then any δ-supplement of $\mathscr{h}$ in U is adjacent to $\mathscr{h}$ in $\Gamma_\delta(U)$.

Proof. Let $\mathscr{h}$ be a submodule of U and let $\mathscr{g}$ δ-supplement of $\mathscr{h}$ in U. Hence $U = \mathscr{h} + \mathscr{g}$ and $\mathscr{h} \cap \mathscr{g} \ll_\delta \mathscr{g}$, and so $\mathscr{h} \cap \mathscr{g} \ll_\delta U$. Thus $\mathscr{g}$ adjacent to $\mathscr{h}$ in $\Gamma_\delta(U)$.

We now state-owned our next result, which gives us certain information on the structure of the δ-small intersection graphs of δ-supplemented modules.

Proposition 2.13. Let U be a δ-supplemented module. Then $\Gamma_\delta(U)$ is connected and $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$.

Proof. Let N, L are submodules of U. Since U is δ-supplemented, then there exists submodule K of U such that $N + K = U$, $N \cap K \ll_\delta K$, and so $N \cap K \ll_\delta U$. One can consider binary likely cases for $N \cap K$.

Case 1: If $N \cap K = (0)$, then $N \oplus K = U$.

Now, if $L \leq N$, then $L \cap K \ll_\delta U$. Thus $L - K - N$ is a path of length 2 in $\Gamma_\delta(U)$. If $L \leq K$, then $L \cap N \ll_\delta U$. Thus N and L are adjacent vertices in the graph $\Gamma_\delta(U)$. Hence, $\Gamma_\delta(U)$ is joined besides $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$.

Case 2: If $N \cap K \neq (0)$. Since $N \cap K$ is a δ-small submodule of U, thus $N - N \cap K - L$ is a path of length 2 in $\Gamma_\delta(U)$. Hence, $\Gamma_\delta(U)$ is joined besides $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$.

The next examples show there are connected graphs $\Gamma_\delta(U)$ with $\mathrm{diam}(\Gamma_\delta(U)) \geq 2$ whenever U is not δ-supplemented.

Example 2.14. (1) The $\mathbb{Z}$-module $U = \oplus_{i=1}^{\infty} U_i$ with each $U_i = \mathbb{Z}_{p^\infty}$ where p is prime number is not δ-supplemented see [12]. It is easy to see that $\Gamma_\delta(U)$ is connected and $\mathrm{diam}(\Gamma_\delta(U)) \geq 2$.

(2) The $\mathbb{Z}$-module $\mathbb{Q}$ is not δ-supplemented see [12]. Now, from [12] that Let $\mathbb{Q}_1 = \{a/b \in \mathbb{Q} \mid 2$ does not divide $b\}$ and $\mathbb{Q}_2 = \{a/b \in \mathbb{Q} \mid 2$ divides $b\}$. Then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$. Since $\mathbb{Q}/\mathbb{Q}_1$ and $\mathbb{Q}/\mathbb{Q}_2$ are singular $\mathbb{Z}$-modules, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are not δ-small submodules in $\mathbb{Q}$. Hence, any proper submodule L of $\mathbb{Q}$ with $\mathbb{Q}_1 \leq L$ we have L is not adjacent to $\mathbb{Q}_1$. So, $\Gamma_\delta(\mathbb{Q}) \geq 2$. But $\Gamma_\delta(\mathbb{Q})$ is connected graph.

Lemma 2.15. Let U be a module.

(1) Let $N \leq U$ be a finitely generated submodule with $N \leq \delta(U)$. Then $N \ll_\delta U$.
(2) Let $N \leq U$ be a semisimple submodule with $N \leq \delta(U)$. Then $N \ll_\delta U$.

Proof. (1) Suppose that $N \leq U$ is finitely generated. Then, $N = \sum_{i=1}^{r} Rn_i$ for some $n_i \in N$, $1 \leq i \leq r$. Since $Rn_i \leq \delta(U)$, $Rn_i \ll_\delta U$. According to Lemma 1.2, $N \ll_\delta U$.

(2) By [15, Lemma 2.2].

Proposition 2.16. For an R-module U with $\Gamma_\delta(U)$ and $\delta(U) \neq (0)$. The following conditions hold:

(1) If N is a direct summand submodule of U with $(0) \neq \delta(N) \ll_\delta U$, then $\Gamma_\delta(U)$ contains at least one cycle of length 3.
(2) If T is a non-trivial semisimple or finitely generated submodule of U contained in $\delta(U)$. At that time $d(T, \delta(U)) = 1$ and $d(T, L) = 1$ for every non-trivial submodule L of U.

Proof. (1) Since N is a direct summand of U, there is $Z \leq U$ such that $N \oplus Z = U$. Then $\delta(N) \oplus \delta(Z) = \delta(U)$, according to Lemma 1.3. Since $\delta(N) \leq N$ and $N \cap \delta(Z) \leq N \cap Z = (0)$, by the modular law, $\delta(U) \cap N = [\delta(Z) + \delta(N)] \cap N = [\delta(Z) \cap N] + \delta(N) = \delta(N)$. Thus, $\delta(U) \cap N = \delta(N)$. Then $\delta(U) \cap N \ll_\delta U$. Also, $\delta(N) = N \cap \delta(N) \ll_\delta U$ and $\delta(N) = \delta(N) \cap \delta(U) \ll_\delta U$ and we have, $d(N, \delta(U)) = 1$, $d(N, \delta(N)) = 1$ and $d(\delta(N), \delta(U)) = 1$. Hence, $(N, \delta(N), \delta(U))$ is a cycle. Thus, $\Gamma_\delta(U)$ contains at least one cycle of distance 3.

(2) Let $T \leq U$ be a non-trivial semisimple or finitely generated submodule. At that moment by Lemma 2.15, $T \ll_\delta U$. Since $T \leq \delta(U)$, $T = T \cap \delta(U) \ll_\delta U$ and since $T \cap L \leq T$, $T \cap L \ll_\delta U$ for every other non-trivial submodule L of U via Lemma 1.2. Hence $d(\delta(U), T) = 1$ and $d(L, T) = 1$.

Proposition 2.17. Let U be a R-module. If U has at least one non-zero δ-small submodule, at that point $\Gamma_\delta(U)$ is a connected graph besides $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$.

Proof. Let $F \in \Gamma_\delta(U)$ be a non-zero δ-small submodule of U. Let A and B be two non-adjacent vertices of $\Gamma_\delta(U)$. It is clear that $A \cap F \leq F \ll_\delta U$, and $F \cap B \leq F \ll_\delta U$. Thus $A \cap F \ll_\delta U$, and $F \cap B \ll_\delta U$ by Lemma 1.2. So, $A - F - B$ is a trail of length 2. So, $\Gamma_\delta(U)$ is a joined graph besides $\mathrm{diam}(\Gamma_\delta(U)) \leq 2$.

Corollary 2.18. Let $\delta(U) \neq (0)$, if one of the next holds. Then $\Gamma_\delta(U)$ is a joined graph,

(1) There exists a non-trivial submodule of U which is semisimple or finitely generated contained in $\delta(U)$.
(2) U is a finitely generated module.

Proof. (1) It follows from Proposition 2.17 and Lemma 2.15. (2) Clear.

Proposition 2.19. If $\Gamma_\delta(U)$ has no isolated vertex, then $\Gamma_\delta(U)$ is connected and $\mathrm{diam}(\Gamma_\delta(U)) \leq 3$.

Proof. Let A and B be two non-adjacent vertices of $\Gamma_\delta(U)$. Since $\Gamma_\delta(U)$ has no isolated vertex, there exist submodules A_1 and B_1 such that $A \cap A_1 \ll_\delta U$ and

$B \cap B_1 \ll_\delta U$. Now, if $A_1 \cap B_1 \ll_\delta U$, then $A - A_1 - B_1 - B$ is a path of length 3. Otherwise $A - A_1 \cap B_1 - B$ is a trail of size 2. Showed that $\mathrm{diam}(\Gamma_\delta(U)) \le 3$ besides $\Gamma_\delta(U)$ is a joined graph.

Proposition 2.20. Let U be a not simple R-module which is semisimple R-module. At that point the next declarations hold:

(i) $\Gamma_\delta(U)$ has no isolated vertex.
(ii) $\Gamma_\delta(U)$ is joined besides $\mathrm{diam}(\Gamma_\delta(U)) \le 3$.

Proof. (i) Let Z be a vertex of the graph $\Gamma_\delta(U)$. Since U is a semisimple module, then every submodule of U is a direct summand of U by [16, 20.2, p. 166]. Thus there exists a submodule Y of U such that $U = Z \oplus Y$. Hence $Z \cap Y = (0) \ll_\delta U$ besides as a result, there exists an edge among vertex Z of $\Gamma_\delta(U)$ besides another vertex of $\Gamma_\delta(U)$. At that time Z is non-isolated vertex. So, $\Gamma_\delta(U)$ has no isolated vertex.

(ii) By Proposition 2.19 besides Part (i).

Now we use $\mathbb{S}_\delta(U)$ which symbolizes the set of all non-zero δ-small submodules of U.

Proposition 2.21. Let n be a positive integer. In R-module U with $|\mathbb{S}_\delta(U)| = n$ and $|\Gamma_\delta(U)| \ge 2$.

(i) If $N \in \mathbb{S}_\delta(U)$, then deg $(N) \neq 0$.
(ii) $\omega(\Gamma_\delta(U)) \ge n$.
(iii) If $\omega(\Gamma_\delta(U)) < \infty$, then the number of δ-small submodules of U is finite.

Proof. (i) Let $N \in \mathbb{S}_\delta(U)$. Suppose that the order of $\Gamma_\delta(U)$ is $|\Gamma_\delta(U)| = n \ge 2$ where n is integer number. Let K be any non-zero submodule of U. Then $K \cap N \le N \ll_\delta U$. By [17, Lemma 1.3(1)], $K \cap N \ll_\delta U$ and thus an edge exists among vertex N of $\Gamma_\delta(U)$ and another vertex of $\Gamma_\delta(U)$. At that point N is cannot an isolated vertex. Thus, deg $(N) \neq 0$.

(ii) Let $\mathbb{S}_\delta(U) = \{N \mid N \ll_\delta U\}$ and let $|\mathbb{S}_\delta(U)| = n$. Suppose that Z and W are two distinct elements of $\mathbb{S}_\delta(U)$. Then Z and W are non-zero δ-small submodules of U. Thus $Z \cap W \ll_\delta U$ according to [17, Lemma 1.3(1)]. So, Z and W are adjacent vertices. Thus, the induced subgraph on the set $\mathbb{S}_\delta(U)$ is a complete subgraph of $\Gamma_\delta(U)$. From this time, $\omega(\Gamma_\delta(U)) \ge n$.

(iii) It is clear from (ii).

Theorem 2.22. Let $\delta(U)$ be a non-zero simple δ-small submodule of U and let $|\Gamma_\delta(U)| \ge 2$. Then $\Gamma_\delta(U)$ is a star graph whenever $\Gamma_\delta(U)$ is a tree graph.

Proof. Since $\delta(U) \neq 0$, then $\delta(U)$ is a vertex in $\Gamma_\delta(U)$. Now, $\delta(U)$ is simple δ-small, so $\delta(U)$ a unique non-zero δ-small submodule of U. But, $\delta(U) \cap N \ll_\delta U$ for every $\in V(\Gamma_\delta(U))$. Thus then $\Gamma_\delta(U)$ contains a vertex $\delta(U)$ which is adjacent to each other vertex. Now, suppose that $I \neq \delta(U)$ and $J \neq \delta(U)$ are two distinct vertices of $\Gamma_\delta(U)$. Now, if $I \cap J \ll_\delta U$. Then $I - \delta(U) - J$, which is a contradiction since $\Gamma_\delta(U)$ is a tree. Thus, $I \cap J$ is not a δ-small submodule of U. So, I and J are not adjacent. Thus, $\Gamma_\delta(U)$ is star with center $\delta(U)$.

Let Γ be a graph. The chromatic number of Γ is defined to be the smallest number of colors $\chi(\Gamma)$ needed to color the vertices of Γ so that no two adjacent vertices share the same color. One has the next corollary by Theorem 2.22.

Corollary 2.23. Let U be a module with $0 \neq \delta(U) \ll_\delta U$ and $|\Gamma_\delta(U)| \ge 3$. Then the next conditions are equivalent:

(1) $\Gamma_\delta(U)$ is a star graph,
(2) $\Gamma_\delta(U)$ is a tree,
(3) $\chi(\Gamma_\delta(U)) = 2$,
(4) $\delta(U)$ is a simple submodule of U such that every couple of non-trivial submodules of U, have non δ-small intersection.

Proof. (1) → (2) and (2) → (3) The implications are obvious.

(3) → (4) On contrary, suppose $0 \neq K \le \delta(U)$. At that point $K \ll_\delta U$. If $L \in V(\Gamma_\delta(U))$. It is easy to see that $(N, \delta(U), L)$ is a circuit (cycle) of length 3 in $\Gamma_\delta(U)$, which contradicts $\chi(\Gamma_\delta(U)) = 2$. As a result, $\delta(U)$ is simple. Now, take up that $\mathrm{Y}, \varpi \in V(\Gamma_\delta(U))$ such that $\varpi \cap \mathrm{Y} \ll_\delta U$. $(\varpi, \delta(U), \mathrm{Y})$ is a circuit in $\Gamma_\delta(U)$, which contradicts $\chi(\Gamma_\delta(U)) = 2$.

(4) → (1) It is obvious that $\delta(U)$ is adjacent to each other vertex in $\Gamma_\delta(U)$. Now, suppose that $N \neq \delta(U)$ and $L \neq \delta(U)$ are two distinct vertices of $\Gamma_\delta(U)$, such that N and L are adjacent. Thus, $X \cap Y \ll_\delta U$, a contradiction. Hence, $\Gamma_\delta(U)$ is a star graph.

Proposition 2.24. Let U be a module and $|\mathbb{S}_\delta(U)| \ge 1$. If $\Gamma_\delta(U)$ does not contain a cycle, then $\Gamma_\delta(U) = K_1$ or $\Gamma_\delta(U)$ is a star graph.

Proof. Supposing that the graph $\Gamma_\delta(U)$ contains no a cycle. To prove $|\mathbb{S}_\delta(U)| < 2$, by contrary way, let $Z \ll_\delta U$ besides $W \ll_\delta U$. So $\mathrm{Z} + \mathrm{W} \ll_\delta U$ by Lemma 1.2, and hence, $\mathrm{Z} - (\mathrm{Z} + \mathrm{W}) - \mathrm{W}$ is a cycle of length 3, which is a illogicality. Then $|\mathbb{S}_\delta(U)| < 2$. As $|\mathbb{S}_\delta(U)| \ge 1$, then $|\mathbb{S}_\delta(U)| = 1$. Hence, U has a unique non-zero δ-small submodule. Let $N \in \mathbb{S}_\delta(U)$. For every vertex L of $\Gamma_\delta(U)$, if $L = N$, then $\Gamma_\delta(U) \cong K_1$ and if $L \neq N$, as $L \cap N \ll_\delta U$, we deduce $\Gamma_\delta(U) \cong K_2$. Let $\Psi = \{\mathrm{v}_i \mid \mathrm{v}_i \neq N,\ i \in I\}$. At that time every two random distinct vertices v_i and v_j, $i \neq j$, are not adjacent and for $i \neq j$, $\mathrm{v}_i - N - \mathrm{v}_j$ is a path besides hence $\Gamma_\delta(U)$ is a star graph.

Theorem 2.25. Let $\Gamma_\delta(U)$ be a graph of a module U. If $|\mathbb{S}_\delta(U)| \ge 2$, then $\Gamma_\delta(U)$ contains at least one cycle besides $\mathrm{gr}(\Gamma_\delta(U)) = 3$.

Proof. Presume that $|\mathbb{S}_\delta(U)| \geq 2$. At that time U has at least two nonzero δ-small submodules, at a guess T_1 and T_2. Since $T_1 \cap T_2 \leq T_i$, for $i = 1, 2$, by Lemma 1.2, $T_1 \cap T_2 \ll_\delta U$. Also, $T_1 \cap (T_1 \cap T_2) \ll_\delta U$ and $T_2 \cap (T_1 \cap T_2) \ll_\delta U$. We consider two probable cases for $T_1 \cap T_2$.

Case 1: If $T_1 \cap T_2 \neq (0)$, then $d(T_1, T_2) = 1$, $d(T_1, T_1 \cap T_2) = 1$ and $d(T_2, T_1 \cap T_2) = 1$. Thus $(T_1, T_1 \cap T_2, T_2)$ is a cycle of size 3. Also by Lemma 1.2, $T_1 + T_2 \ll_\delta U$ and since $T_1 \cap (T_1 + T_2) \ll_\delta U$ and $T_2 \cap (T_1 + T_2) \ll_\delta U$, $(T_1, T_1 + T_2, T_2)$ is a cycle of length 3. Similarly, $(T_1 \cap T_2, T_1, T_1 + T_2)$ and $(T_1 \cap T_2, T_2, T_1 + T_2)$ are cycles of length 3 and $(T_1, T_1 + T_2, T_2, T_1 \cap T_2, T_1)$ is a cycle of length 4.

Case 2: If $T_1 \cap T_2 = (0)$, then $(T_1, T_1 + T_2, T_2)$ is a cycle of size 3 in the graph $\Gamma_\delta(U)$. As a result, $\Gamma_\delta(U)$ contains at least one cycle and so $\mathrm{gr}(\Gamma_\delta(U)) = 3$.

Example 2.26. Let $U = Z \oplus F \oplus K$ be a semisimple module. Then, the subgraph $Z - F - K - Z$ is a clique. Also, $\mathrm{gr}(\Gamma_\delta(U)) = 3$.

Let Γ is a joined graph and let X is a vertex of Γ, X is named a cut vertex of Γ if there are vertices Z besides W of Γ such that X is in every one Z, $W-$path. Equally, X is a cut vertex of Γ if $\Gamma - \{X\}$ is not joined for a joined graph Γ.

Proposition 2.27. $\Gamma_\delta(U)$ has no cut vertex whenever $|\mathbb{S}_\delta(U)| \geq 2$.

Proof. Take up T a cut vertex of $\Gamma_\delta(U)$, as a result $\Gamma_\delta(U) \setminus \{T\}$ is not joined. As a result there exist vertices F, K with T lies on every single trail from F to K. Since $|\mathbb{S}_\delta(U)| \geq 2$, then U has at least two nonzero δ-small submodules, assume $(0) \neq N_1 \ll_\delta U$, $(0) \neq N_2 \ll_\delta U$. Thus $F \cap N_1 \ll_\delta U$, $N_1 \cap N_2 \ll_\delta U$ and $N_2 \cap K \ll_\delta U$. $F - N_1 - N_2 - K$ is a trail in $\Gamma_\delta(U) \setminus \{T\}$, a illogicality. As a result $\Gamma_\delta(U)$ has no cut vertex.

3. Domination and planarity of $\Gamma_\delta(U)$

In this Section, we study domination number and the planarity of $\Gamma_\delta(U)$. We recall that for a graph Γ, a subset D of the vertex-set of Γ is called a dominating set (or DS) if every vertex not in D is adjacent to a vertex in D. The domination number, $\gamma(\Gamma)$, of Γ is the minimum cardinality of a dominating set of Γ, [11]. Here, a subset D of the vertex set $V(\Gamma_\delta(U))$ is a DS iff for any nontrivial submodule N of U there is a L in D such that $N \cap L \ll_\delta U$.

Lemma 3.1. The next hold for an R-module U with $|\Gamma_\delta(U)| \geq 2$:

(1) If $D \subseteq V(\Gamma_\delta(U))$ with either there exists a vertex $X \in D$ which $X \cap Y = (0)$, for every one vertex $Y \in V(\Gamma_\delta(U)) \setminus D$ or D contains at least one δ-small submodule of U. Then D is a DS in $\Gamma_\delta(U)$.

(2) If $|\mathbb{S}_\delta(U)| \geq 1$, then for each $Z \neq 0$ with $Z \ll_\delta U$, $\{Z\}$ is a DS besides $\gamma(\Gamma_\delta(U)) = 1$.

Proposition 3.2. Let $U = N \oplus L$ be an R-module, where N and L are simple R-modules. Then $\gamma(\Gamma_\delta(U)) = 1$.

Proof. Assume $U = N \oplus L$, with N and L are simple R-modules. By Proposition 2.3 (1), is a complete graph $\Gamma_\delta(U)$. Let α be a random vertex of $\Gamma_\delta(U)$. At that time for every different vertex Y of $\Gamma_\delta(U)$, $\alpha \cap Y \ll_\delta U$, so $\{\alpha\}$ is a DS besides $\gamma(\Gamma_\delta(U)) = 1$.

Proposition 3.3. Let $\delta(U) \neq 0$ of a finitely generated R-module U. Then $\{\delta(U)\}$ is a dominating set of $\Gamma_\delta(U)$ and so the graph $\Gamma_\delta(U)$ is joined (=connected).

Proof. Assume $\Re \in \Gamma_\delta(U)$. If $\Re$ is δ-small then $\delta(U)$ is adjacent to $\Re$. Now, if $\Re$ is not δ-small. Since $\delta(U) \neq 0$ in finitely generated module, at that point $\delta(U) \ll_\delta U$. So, $\Re \cap \delta(U) \ll_\delta U$. So, $\Re$ is adjacent to $\delta(U)$. This implies that $\{\delta(U)\}$ is a dominating set of $\Gamma_\delta(U)$, so $\Gamma_\delta(U)$ is connected as obligatory.

Theorem 3.4. Let $|\mathbb{S}_\delta(U)| \geq 2$ besides $|\Gamma_\delta(U)| \geq 3$ of a module U. We have:

(1) If μ and λ are two δ-small submodules of U then there exists $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$.
(2) The graph $\Gamma_\delta(U)$ has at least one triangle.

Proof. It is clear.

Proposition 3.5. The next statements are equivalent for an R-module U:

(1) If $\{\mu, \lambda\} \in E(\Gamma_\delta(U))$, then there is no $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$.
(2) U has at most one nonzero δ-small submodule such that $\hbar \cap h$ is not a δ-small for every couple of non-δ-small nontrivial submodules $\hbar, h$ of U.
(3) The graph $\Gamma_\delta(U)$ has no triangle.

Proof. (1) $\Rightarrow$ (2) Take up that for all two adjacent vertices of $\Gamma_\delta(U)$, there is no $\psi \in V(\Gamma_\delta(U))$ with $\psi \in N(\mu) \cap N(\lambda)$. Assume there exist nonzero submodules $N_1 \ll_\delta U$ and $N_2 \ll_\delta U$. Since $N_1 \cap N_2 \ll_\delta U$, they are adjacent vertices of the graph $\Gamma_\delta(U)$ besides too, there is no $\psi \in V(\Gamma_\delta(U))$ such that $\psi \in N(\mu) \cap N(\lambda)$, which is a illogicality by Theorem 3.4(1).

(2) $\Rightarrow$ (3) Presume there is no nonzero δ-small submodules in U. As $\hbar \cap h$ is not δ-small for every couple of non-δ-small nontrivial submodules $\hbar, h$ of U, $\Gamma_\delta(U)$ has no triangle. Besides, Let S be the unique nonzero δ-small submodule of U. At that point for every three random vertices N_1, N_2, and N_3 of the graph $\Gamma_\delta(U)$, at least two of them are not δ-small. Let $S = N_1$. As $N_2 \cap N_3$ is not a δ-small

submodule of U, then $N_2 - S - N_3$ is a path. Also if $S \neq N_i$, for $i = 1,2,3$. Since $N_i \cap N_j$ is not a δ-small submodule of U, for $i,j = 1,2,3$ and $i \neq j$, then N_1, N_2, and N_3 are not adjacent vertices in the graph $\Gamma_\delta(U)$. Hence, the graph $\Gamma_\delta(U)$ has no any triangle.

(3) $\Rightarrow$ (1) It is clear.

Proposition 3.6. Let $\delta(U) \neq 0$ of a finitely generated R-module U, then the graph $\Gamma_\delta(U)$ has a triangle.

Proof. Since U is finitely generated, from this time $(0) \neq \delta(U) \ll_\delta U$ according to Lemma 1.3(4). Now consider two possible cases for $\delta(U)$.

Case I: If $\delta(U)$ is a simple submodule of U, because $\delta(U) = \sum_{i \in \Lambda} U_i$, where $U_i \ll_\delta U$, $\forall i \in \Lambda$, we choose $\Gamma = \sum_{i \in \Lambda - \{1\}} U_i$. Then $\{U_1, \delta(U), \Gamma\}$ is a triangle in $\Gamma_\delta(U)$.

Case II: If $\delta(U)$ is a non-simple submodule of U, at that point there exists a non-trivial submodule $Z \leq U$ which $Z \subset \delta(U)$. Since $\delta(U) \ll_\delta U$, then, $Z \ll_\delta U$. Thus for each vertex H of $\Gamma_\delta(U)$, $\{Z, \delta(U), H\}$ is a triangle in $\Gamma_\delta(U)$.

Definition 3.7. [8] If a graph Γ has a drawing in a plane without crossings, then Γ is said to be planar.

Theorem 3.8. [8, Th. 10.30] A graph is planar if it contains no subdivision of either K_5 or $K_{3,3}$.

Proposition 3.9. If $|\mathbb{S}_\delta(U)| = 1$ or $|\mathbb{S}_\delta(U)| = 2$, and the intersection of every pair of non-small submodules of U is a non-small submodule, then $\Gamma_\delta(U)$ is a planar graph.

Proof. Similar to that in [13, Theorem 2.15].

Proposition 3.10. For any module U, if $|\mathbb{S}_\delta(U)| \geq 3$, then $\Gamma_\delta(U)$ is not a planar graph.

Proof. Suppose $|\mathbb{S}_\delta(U)| \geq 3$. Then U has at least three nonzero δ-small submodules, at a guess M, N and P. Any one of the vertices $M+N$, $N+P$ and $M+P$ are non-zero submodules and adjacent to all of submodules M, N and P in $\Gamma_\delta(U)$. $\Gamma_\delta(U)$ contains a complete graph K_5 for example the subgraph induced on the set $\{M, N, P, M+N, N+P\}$. By Th. 3.8, $\Gamma_\delta(U)$ is not planar.

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