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A new relation to find all natural solutions to hyperbola and Diophantine equations using Pell's equation

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Abstract

In this paper a new relation has been found between any natural solution to the hyperbola equation$x^2 - dy^2 = c^2$, where $c \epsilon N$,d a positive square free numberand a form that gives allnatural solutions to Pell'sequation$x^2 - dy^2 = 1$. Which enable us to obtain all natural solutions for the above hyperbola equation thena square natural number$u$ for the Diophantine equation $u = \frac{m^2+n^2}{1+mn}$ where $(m, n) \in N$.

Keywords: Diophantine equation, Pell's equation, continued fraction.

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علاقة جديدةلايجاد كل الحلول الطبيعية لمعادلة القطع الزائد والمعادلة الديفونتية باستخدام معادلة بيل

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الخلاصة

في هذا البحث وجدنا علاقة جديدة بين اي حل طبيعي لمعادلة القطع الزائد $x^2 - dy^2 = c^2$ حيث c عدد طبيعي و d عدد صحيح موجب لا تحتوي مجموعة عواملة عدد مربع،والصيغة التي تعطينا كل الحلول الطبيعية لمعادلة بيل $x^2 - dy^2 = 1$. هذه العلاقة تمكنا من ايجاد كل الحلول الطبيعية لمعادلة القطع الزائد أعلاهثم ايجادكل الازواج الطبيعية (m, n) التي تجعل u عدد طبيعي مربع في المعادلة الديفونتية$u = \frac{m^2+n^2}{1+mn}$.

1. Introduction

For $m, n \in Z$ such that$1 + mn \neq 0$, the Diophantine equation

$$u = \frac{m^2 + n^2}{1 + mn} \quad (1)$$

has infinitely many real solutions[1]. Indeed, when $(m, n) \in Z$ then Eq. (1) has infinitely many rational solutions. The integer numbers (m, n) sometimes give square positive integer number$u$ in Eq. (1).For example,$(m, n) \in \{(-100,0), (7,0), (0, -50), (1,1)\}$.In this paper, Eq. (1) serves as an accurate tool to find all natural pairs(m, n) such that$u$ is a square natural number. This can be proved using theorem (3.11) for some $(m, n) \in N$[1].

For $u \in \{2^2, 3^2, 4^2, 5^2, 6^2, \dots\}$, there are infinitely many natural pairs that can be obtained fromEq. (1)resulting in the following hyperbola equation

$$x^2 - dy^2 = c^2 \quad (2)$$

where$d$ is a positive square free integer and $c \in N$. Note that, there are only one natural pair $(m, n) = (1,1)$ for$u = 1^2$.

We denote by(s_r, t_r), $r = 1,2,3, \dots$to the infinitely many natural solutions of Eq. (2). These solutions have been found through this paper by using the relation in Eq. (13). This relation has been depended on the following equation

$$x^2 - dy^2 = 1 \quad (3)$$

Which is known as Pell's equation and it was named after John Pell. In the seventeenth century Pell [2] searched for integer solution of this type. He was not the first to work on this problem, Fermat [2,3] found the smallest solution for d up to 150,John Wallis[2] solved Eq. (3) for$d$ = 151 or 313.Lagrange[2,3]developedthe general theory of Pell's equation, based on continued fractions and algebraic manipulations with numbers of the form$x + \sqrt{d}y$in (1766–1769).

For Eq.(3), we denote by $(x_r, y_r), r = 1,2,3, \dots$ to all naturalsolutions.The first non-trivial fundamental solution$(x_1, y_1)$for Eq. (3) can be found using the cyclic method [3], or using the slightly less efficient but more regular English method defined in [3,4]. The rest of solutions$(x_r, y_r), r = 2,3,4, \dots$ are easily computed from(x_1, y_1). There are another methods to find this fundamental solution, in this paper we use acontinued fraction method for a real number$\sqrt{d}$ see remark (2.6), (For further details on Pell equation see [3,4,5,7]). In theorem (2.9), (x_1, y_1) has been used to give the form of finding allthe rest natural solutions$(x_r, y_r), r = 2,3,4, \dots$forEq.(3).

We givea new relation in Eq. (13) between one of the natural solutions of Eq. (2) and the form in theorem (2.9) for Eq. (3). By this relationall natural solutions to Eq. (2) and all pairs$(m, n) \in N$ such that u a square number inEq. (1), will becalculated.

2. Preliminaries:

In this section, the basic definitions, theorems and remarks which will be used in this workhave been introduced.

Definition 2.1[3]: The Diophantine equation is a polynomial equation, usually in two or more unknowns, such that only the integersolutions are sought or studied (an integer solution is a solution such that all the unknowns take integer values).

Definition 2.2[1]: The square-free, or quadrate free integer, is an integer which is divisible by no other perfect square than 1. For example, 10 is square-free but 18 is not, as 18 is divisible by 9 $=3^2$.

Definition 2.3[6]: The quadratic Diophantine equation of the form $x^2 - dy^2 = \pm 1$is called a Pell's equation where d is a positive square free integer. In this paperthe Pell equation of the form $x^2 - dy^2 = 1$ was discussed.

Example 2.4:

i. $x^2 - 8y^2 = 1$

ii. $x^2 - 13y^2 = 1$

iii. $x^2 - 13 = -1$

The solutions for equations (i), (ii) are given by$(x, y) = (3,1)$, $(x, y) = (649,180)$ respectively,while equation (iii) does not have any solution; it is not solvable [6].

Definition 2.5[6]: The expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{.+\cfrac{1}{.+\cfrac{1}{.+\cdots}}}}}}$$

wherea_i's are integers, is called the continued fraction expressionto any real number denoted by the notation $[a_0;\ a_1, a_2, a_3, a_4, \ldots, \overline{2a_0, a_1, a_2, a_3, a_4, \ldots, 2a_0, a_1, a_2, a_3, a_4, \ldots}\ ]$.This expression will be used to find the fundamental solution $(x_1, y_1)$for$\sqrt{d}$ in Eq. (3).

The following remark has beenexplained shortly thecontinued fraction method [2] for finding the non-trivial fundamental natural solution(x_1, y_1) toEq. (3).

<u>Remark 2.6:</u> For $\sqrt{d}$inEq. (3) assume that$\alpha_0 = \sqrt{d}$, $a_0 = \lfloor \alpha_0 \rfloor$. In general,

$\alpha_k = a_k + \frac{1}{\alpha_{k+1}}$, $a_k = \lfloor \alpha_k \rfloor$ For $k = 0,1,2,3, \ldots$

We obtain$\sqrt{d} = [a_0;\ a_1, a_2, a_3, a_4, \ldots, \overline{2a_0, a_1, a_2, a_3, a_4, \ldots, 2a_0, a_1, a_2, a_3, a_4, \ldots}]$

For finding(x_1, y_1), only the numbers $[a_0;\ a_1, a_2, a_3, a_4, \dots]$ will be used such that

$$\sqrt{d} = \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \cfrac{1}{\mathrm{a}_3 + \cfrac{1}{\mathrm{a}_4 + \cfrac{1}{.+\cfrac{1}{.+\cfrac{1}{.+1}}}}}}} = \frac{x_1}{y_1}$$

Example 2.7: The continued fraction expression for

$$\sqrt{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{8}{3} = \frac{x_1}{y_1}$$

where, $\sqrt{7} = [a_0;\ a_1, a_2, \dots, \overline{2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \dots, 2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \dots}] = [2; 1,1,1,4,2,1,1,1,4, \dots]$

Remark 2.8:A continued fraction is purely periodic with period m if the initial block of partial quotients $a_0, a_1, \dots, a_{m-1}$ repeats infinitely and no block for length less than m is repeated, and it is periodic with period m if it consists of an initial block of length n followed by a repeating block of length m.Purely periodic continued fraction→ $[\overline{\mathrm{a}_0; \mathrm{a}_1, \dots, \mathrm{a}_{m-1}}]$. Periodic continued fraction → $[\mathrm{a}_0; \mathrm{a}_1, \dots, \mathrm{a}_{t-1}, \overline{\mathrm{a}_t; \mathrm{a}_{t+1}, \dots, \mathrm{a}_{t+m-1}}]$

the length of the period was denoted by t .

Theorem 2.9[6]: If $(a, b)$is a solution to $x^2 - dy^2 = 1$ where $a > 1$and $b \geq 1$, then$(x, y)$is also a solution such that

$$x + y\sqrt{d} = \left(a + b\sqrt{d}\right)^j$$

for $j = 1, 2, 3, 4, \dots$,.Similarly, If $(c, k)$is a solution to $x^2 - dy^2 = -1$ where $c > 1$and $k \geq 1$, then $(x, y)$is also a solution such that

$$x + y\sqrt{d} = \left(c + k\sqrt{d}\right)^j$$

for$j = 1, 3, 5, 7, \dots$.

Theorem 2.10[6]: The equation $x^2 - \mathrm{d}y^2 = 1$ is always solvable and the fundamentalsolution is (A_k, B_k) where $k = t$ or $2t$ and $A_k/B_K$is a convergent to$\sqrt{d}$. The equation $x^2 - \mathrm{d}y^2 = 1$ is solvable if and only if the period length of the continued expansion of $\sqrt{d}$ is odd. The fundamental solution is (A_k, B_k) where $k = t$or$t + 1$.

3. The main result

In this section, theorem (3.11) has been proved that u in Eq. (1) is a square natural number for somepairs$(m, n) \in N$, so $u \in \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, \dots\}$. We give a new relation in Eq. (13)to find all natural solutions forEq.(2)then all natural pairs (m, n) in theorem (3.11)toEq.(1).

Theorem 3.11 [1]: A number $u = \frac{m^2+n^2}{1+mn}$ is a square of some natural numbers?

Proof: Let us use indirect way to prove this statement. Supposethat there are some pairs(m, n) of natural numbers such that $\left(\frac{m^2+n^2}{1+mn}\right)$ is a natural numberbut not a square. If $m = n$, then we have

$$\frac{m^2 + n^2}{1 + mn} = \frac{2m^2}{1 + m^2} < 2$$

But the only natural number less than 2 equals 1, and 1 is a square. foreach pair (m, n) such that$m \neq n$ we take $s = \max\{m, n\}$.

Let Xrepresents the set of all numbers $s \in N$ obtained using this way. This set is nonempty because there are some pairs(m, n) as we supposed. By Minimum Principle Theorem, we have $s_0 = \max\{m_0, n_0\}$represents the smallest element belongs to X. Therefore,

$$\frac{{m_0}^2 + {n_0}^2}{1 + m_0 n_0} = u_0 \qquad (4)$$

is a natural number which is not square. It can be proved that $m_0 < n_0$or $m_0 > n_0$, here we take $m_0 < n_0$. The equality in Eq. (4) shows that n_0 is a root of the quadratic equation

$n_0^2 - u_0 m_0 n_0 + m_0^2 - u_0 = 0$, this is the same $x^2 - u_0 m_0 x + m_0^2 - u_0 = 0$.

Let us denote by n_1 to the second root of this equation. Then using Viete's formula, $n_1 + n_0 = u_0 m_0$, from whichn_1 is an integerand $n_0 n_1 = m_0^2 - u_0$.As result, we have

$$n_1 = \frac{m_0^2 - u_0}{n_0} < \frac{n_0^2 - u_0}{n_0} < n_0.$$

It follows that$\max\{m_0, n_1\} < \max\{m_0, n_0\} = s_0$. Moreover,

$$\frac{m_0^2 + n_1^2}{1 + m_0 n_1} = u_0 \qquad (5)$$

and$n_1 \notin Z$ which give $n_1 \in N$and this is contradiction. Because if $n_1 < 0$ then the denominator of the fraction inEq. (5) would be negative and this would give the negativeness ofu_0 and, if $n_1 = 0$ then we would have $0 = n_0 n_1 = m_0^2 - u_0$, and u_0 would be a square.This contradiction shows that

$$u = \frac{m^2 + n^2}{1 + mn} \qquad (6)$$

is a square number. □

<u>Note that</u>Eq. (6) is the same asEq. (1). In this paper the following cases for Eq. (6)have been discussedwhen $u = 1^2, 2^2, 3^2$and similarly, can be done for $u = 4^2, 5^2, 6^2, \ldots$.

<u>Case1</u>: If $u = 1^2$, thenEq. (6)is

$$1^2 = \frac{m^2 + n^2}{1 + mn}$$

$$1 + mn = m^2 + n^2 \Rightarrow \ m^2 - mn + n^2 = 1$$

By multiplying both sides by4, gives$4m^2 - 4mn + 4n^2 = 4$

Add and subtract n^2,yields$4m^2 - 4mn + 4n^2 - n^2 + n^2 = 4$

$$4m^2 - 4mn + n^2 + 3n^2 = 4$$

$$(2m - n)^2 + 3n^2 = 4 \qquad (7)$$

It follows that Eq. (7)is a Diophantine equationand$(m, n) \in N$.

Since $3n^2 < 4$, then $n = 1$andEq. (7) is

$(2m - n)^2 + 3 = 4 \Rightarrow (2m - 1)^2 = 1 \Rightarrow \quad 2m = \mp 1 + 1,$

hence$m = 1, (m = 0$ neglected as$m \in N$). So the natural solution toEq. (6) when $u = 1^2$ is only the pair $(m, n) = (1,1)$.

<u>Case2</u>: If $u = 2^2$ in Eq. (6), then

$$2^2 = \frac{m^2 + n^2}{1 + mn}$$

$$4 + 4mn = m^2 + n^2 \Rightarrow \ m^2 - 4mn + n^2 = 4$$

Add and subtract $4n^2$,then

$m^2 - 4mn + 4n^2 - 4n^2 + n^2 = 4 \Rightarrow (m - 2n)^2 - 3n^2 = 4$

This equation has infinitely many solutions (s_r, t_r), $r = 1,2,3, \ldots$that is,

for$r = 1,2,3, \ldots$ we have, $m - 2n = s_r$, *and* $n = t_r$ then

$$s_r{}^2 - 3t_r{}^2 = 4 \qquad (8)$$

Equation (8) is ahyperbola equation.

<u>Case3</u>: If $u = 3^2$ in Eq. (6), thus

$$3^2 = \frac{m^2 + n^2}{1 + mn}$$

$$9 + 9mn = m^2 + n^2 \ \Rightarrow m^2 - 9mn + n^2 = 9$$

By multiplying both sides by 4 , then $4m^2 - 36mn + 4n^2 = 36$ Add and subtract $81n^2$, gives$4m^2 - 36mn + 4n^2 + 81n^2 - 81n^2 = 36$

$(2m)^2 - 2(2m)(9n) + 81n^2 - 77n^2$=36

$(2m - 9n)^2 - 77n^2 = 36$

Then for $r = 1,2,3, \ldots$ we obtain,$2m - 9n = s_r$, $n = t_r$ such that

$$s_r{}^2 - 77t_r{}^2 = 36 \qquad (9)$$

Hence Eq. (9) is ahyperbola equation.

In order to find all pairs$(m,n) \in N$ suchthat $u = 2^2, 3^2$in Eq. (6), we have to solve Eq. (8), Eq. (9) in Case2, Case3 respectively which are hyperbolic equations having the same form of Eq. (2).Also Eq. (6) for $u = 4^2, 5^2, 6^2, \ldots$ has the same form of Eq. (2). That is, we need to solveEq. (2).

3.1Finding of the new relation

In the following steps, we give a new relation that will be used to find all natural solutions$(s_r, t_r), r = 1,2,3, \ldots$forEq. (2).

Step1: We are looking for a natural solution to$(x, y) = (s_1, t_1)$toEq. (2).

Solving the equation$x^2 - dy^2 = c^2 \Rightarrow x^2 = c^2 + dy^2 \;\; \Rightarrow \;\; x = \mp\sqrt{c^2 + dy^2}$.

The required natural solution is,$x = \sqrt{c^2 + dy^2}$ (10)

From Eq. (10) the pair$(x, y) = (s_1, t_1)$such that$\gamma = s_1 + \sqrt{d}t_1 \;,\;\; \delta = s_1 - \sqrt{d}t_1$

$$\gamma\delta = (s_1 + \sqrt{d}t_1)(s_1 - \sqrt{d}t_1) = {s_1}^2 - d{t_1}^2 = c^2$$

$$\gamma + \delta = (s_1 + \sqrt{d}t_1) + (s_1 - \sqrt{d}t_1) = 2s_1$$

By this step we have been found one natural solution (s_1, t_1) to Eq. (.2) such that

$$\gamma = s_1 + \sqrt{d}t_1$$

Step2: In general, assume that all natural solutions of Eq. (2) are $(x, y) = (s_r, t_r),\; r = 1,2,3, \ldots$

$$\gamma_r = s_r + \sqrt{d}t_r \,, \delta_r = s_r - \sqrt{d}t_r$$

$$\gamma_r + \delta_r = s_r + \sqrt{d}t_r + s_r - \sqrt{d}t_r = 2s_r$$

$$\gamma_r\delta_r = (s_r + \sqrt{d}t_r)(s_r - \sqrt{d}t_r) = {s_r}^2 - d{t_r}^2 = c^2$$

Step3: Take the following Pell equation:

$$x^2 - dy^2 = 1 \qquad (11)$$

Step4: Assume that (x_1, y_1) is the fundamental natural solution toEq. (11),whichhas been found by continued fraction expression for$\sqrt{d}$ in remark(2.6) such that

$$\alpha = x_1 + \sqrt{d}y_1 \;,\;\; \beta = x_1 - \sqrt{d}y_1$$

$$\alpha\,\beta = (x_1 + \sqrt{d}y_1)(x_1 - \sqrt{d}y_1) = {x_1}^2 - d{y_1}^2 = 1$$

Hereby, we obtain the fundamental natural solution$(x_1, y_1)$for Eq. (11) such that

$$\alpha = x_1 + \sqrt{d}y_1$$

Step5:All natural solutions$(x_r, y_r),\; r = 1,2,3,4, \ldots$for Eq. (11) will be given, as follows: We have in Step4 $\alpha = x_1 + \sqrt{d}y_1$, using theorem (2.9)for $r = 1,2,3, \ldots$ then

$\alpha^r = (x_1 + \sqrt{d}y_1)^r$ (12)is a natural solution forEq. (11) .

Step6: From the use of Step1and Step5, we have $\gamma = s_1 + \sqrt{d}t_1$, and $\alpha^i = (x_1 + \sqrt{d}y_1)^i$respectively.All natural solutions$(s_r\,, t_r)$, $r = 1,2,3,4, \ldots$ to Eq. (2) can be obtained from the following relation

$$s_r + \sqrt{d}t_r = \gamma\alpha^i (13)$$

where$i = 0,1,2,3, \ldots$. Using these solutions to calculated the square number u for $(m, n) \in N$ inEq. (6).

We have been applied the relation in Eq. (13)to the previous cases:

In Case2 we will solveEq. (8) by using Eq. (13)as follows:

By Step1$x^2 - 3y^2 = 4 \Rightarrow x = \sqrt{c^2 + dy^2}$.Here $c^2 = 4$, $d = 3$, therefore, let $y = 2 \Rightarrow x = 4$,resulting$(s_1\,, t_1) = (4,2)$ such that$\gamma = 4 + 2\sqrt{3}$.

By Step3 the Pell equation $x^2 - 3y^2 = 1$ (14)

By Step4 we use the continued fraction expression of $\sqrt{3}$for Eq. (14) to find$(x_1\,, y_1)$:Assume that $\alpha_0 = \sqrt{d} = \sqrt{3} = 1.7$ *and* $a_0 = \lfloor 1.7 \rfloor = 1$

$$\alpha_0 = a_0 + \frac{1}{\alpha_1} \Rightarrow \sqrt{3} = 1 + \frac{1}{\alpha_1} \Rightarrow \alpha_1 = \frac{1}{\sqrt{3} - 1}$$

$$\alpha_1 = \frac{1}{\sqrt{3} - 1} * \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{\sqrt{3} + 1}{2} \Rightarrow a_1 = \lfloor 1.3 \rfloor = 1$$

$$\alpha_1 = a_1 + \frac{1}{\alpha_2} \Rightarrow \frac{\sqrt{3} + 1}{2} = 1 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{2}{\sqrt{3} - 1} * \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{2(\sqrt{3} + 1)}{2} = \sqrt{3} + 1 \Rightarrow a_2 = \lfloor 2.7 \rfloor = 2$$

$$\alpha_2 = a_2 + \frac{1}{\alpha_3} \Rightarrow \sqrt{3} + 1 = 2 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2} = \alpha_1 \Rightarrow a_3 = \lfloor 1.3 \rfloor = 1$$

$$\alpha_3 = a_3 + \frac{1}{\alpha_4} \Rightarrow \frac{\sqrt{3} + 1}{2} = 1 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{2(\sqrt{3} + 1)}{2} = \sqrt{3} + 1 = \alpha_2 \Rightarrow a_4 = \lfloor 2.7 \rfloor = 2$$

And so on then:$1 = a_1 = a_3 = a_5 = a_7 = \cdots$, 2$= a_2 = a_4 = a_6 = \cdots$ .Hence $\sqrt{3} = [1; 1,2,1,2, \ldots]$. The continued fraction expression for

$$\sqrt{3} = 1 + \frac{1}{1} = \frac{2}{1} = \frac{x_1}{y_1}$$

That is, $(x_1, y_1) = (2,1) \;\Rightarrow\; \alpha = 2 + \sqrt{3}$.

By Step6 using Eq. (13) gives $\gamma\alpha^i = s_r + \sqrt{3}t_r$ where $i = 0,1,2,3, \dots$, and $r = 1,2,3, \dots$

$$\gamma\alpha^i = \left(4 + 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)^i = s_r + \sqrt{3}t_r$$

For $i = 0,\; r = 1 \Rightarrow \gamma\alpha^0 = \left(4 + 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)^0 = s_1 + \sqrt{3}t_1$

$$4 + 2\sqrt{3} = s_1 + t_1\sqrt{3} \Rightarrow \;(s_1, t_1) = (4,2)$$

For $i = 1,\; r = 2 \Rightarrow \gamma\alpha^1 = \left(4 + 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)^1 = s_2 + \sqrt{3}t_2$

$$= 14 + 8\sqrt{3} = s_2 + t_2\sqrt{3} \Rightarrow \;(s_2, t_2) = (14,8)$$

For $i = 2,\; r = 3 \;\Rightarrow \gamma\alpha^2 = \left(4 + 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)^2 = s_2 + \sqrt{3}t_2$

$$\gamma\alpha^2 = \left(4 + 2\sqrt{3}\right)\left(7 + 4\sqrt{3}\right) = s_3 + \sqrt{3}t_3$$

$$= 52 + 30\sqrt{3} = s_3 + t_3\sqrt{3} \Rightarrow \;(s_3, t_3) = (52,30).$$

And so on for $i = 3,4,5,6, \dots$ and $r = 4,5,6, \dots$ we will get the rest of all natural solutions (s_r, t_r), for Eq. (8). By using these solutions, all natural pairs (m, n) to Eq. (6) such that $u = 2^2$ will be obtained as follows:

From previous supposed we have, $m - 2n = s_r,\; n = t_r,$ where $r = 1,2,3, \dots$, that is,

For $r = 1, (s_1, t_1) = (4,2) \Rightarrow \;(m, n) = (8, 2) \;\Rightarrow\; u = \frac{m^2+n^2}{1+mn} = \frac{68}{17} = 2^2$

For $r = 2, (s_2, t_2) = (14,8) \;\Rightarrow (m, n) = (30, 8) \;\Rightarrow\; u = \frac{m^2+n^2}{1+mn} = \frac{964}{241} = 2^2$

For $r = 3, (s_3, t_3) = (52,30) \Rightarrow (m, n) = (112,30) \Rightarrow\; u = \frac{m^2+n^2}{1+mn} = \frac{13444}{3361} = 2^2$

And so on for $i = 3,4,5, \dots, r = 4,5,6, \dots$ giving all the other natural pairs (m, n) for Eq. (6) such that $u = 2^2$.

In Case3 we solve Eq. (9) by using Eq. (13) then:

By Step1 $x^2 - 77y^2 = 36 \Rightarrow x = \sqrt{c^2 + dy^2}$. Since $c^2 = 36, d = 77.$ Let $y = 3$ $then$ $x = 27$, we get $(s_2\,, t_2) = (27,3)$ such that $\gamma = 27 + 3\sqrt{77}$.

By Step٣ the Pell equation $x^2 - 77y^2 = 1$ (15)

By Step٤ we use the continued fraction expression of $\sqrt{3}$ for Eq. (15) to find $(x_1\,, y_1)$: Assume that $\alpha_0 = \sqrt{d} = \sqrt{77} = 8.77 \;\Rightarrow a_0 = \lfloor 8.77 \rfloor = 8$

$$\alpha_0 = a_0 + \frac{1}{\alpha_1} \;\Rightarrow\; \sqrt{77} = 1 + \frac{1}{\alpha_1} \;\Rightarrow\; \alpha_1 = \frac{1}{\sqrt{77} - 8}$$

$$\alpha_1 = \frac{1}{\sqrt{77} - 8} * \frac{\sqrt{77} + 8}{\sqrt{77} + 8} = \frac{\sqrt{77} + 8}{13} \;\Rightarrow\; a_1 = \lfloor 1.29 \rfloor = 1$$

$$\alpha_1 = a_1 + \frac{1}{\alpha_2} \Rightarrow \frac{\sqrt{77}+8}{13} = 1 + \frac{1}{\alpha_2} \Rightarrow \frac{1}{\alpha_2} = \frac{\sqrt{77}+8}{13} - 1 = \frac{\sqrt{77}-5}{13}$$

$$\alpha_2 = \frac{13}{\sqrt{77}-5} * \frac{\sqrt{77}+5}{\sqrt{77}+5} = \frac{\sqrt{77}+5}{4} = 3.44 \Rightarrow a_2 = \lfloor 3.44 \rfloor = 3$$

$$\alpha_2 = a_2 + \frac{1}{\alpha_3} \Rightarrow \frac{\sqrt{77}+5}{4} = 3 + \frac{1}{\alpha_3} \Rightarrow \frac{1}{\alpha_3} = \frac{\sqrt{77}+5}{4} - 3 = \frac{\sqrt{77}-7}{4}$$

$$\alpha_3 = \frac{4}{\sqrt{77}-7} * \frac{\sqrt{77}+7}{\sqrt{77}+7} = \frac{\sqrt{77}+7}{7} = 2.25 \Rightarrow a_3 = \lfloor 2.25 \rfloor = 2$$

$$\alpha_3 = a_3 + \frac{1}{\alpha_4} \Rightarrow \frac{\sqrt{77}+7}{7} = 2 + \frac{1}{\alpha_4} \Rightarrow \frac{1}{\alpha_4} = \frac{\sqrt{77}+7}{7} - 2 = \frac{\sqrt{77}-7}{7}$$

$$\alpha_4 = \frac{7}{\sqrt{77}-7} * \frac{\sqrt{77}+7}{\sqrt{77}+7} = \frac{\sqrt{77}+7}{4} = 3.9 \Rightarrow a_4 = \lfloor 3.9 \rfloor = 3$$

$$\alpha_4 = a_4 + \frac{1}{\alpha_5} \Rightarrow \frac{\sqrt{77}+7}{4} = 3 + \frac{1}{\alpha_5} \Rightarrow \frac{1}{\alpha_5} = \frac{\sqrt{77}+7}{4} - 3 = \frac{\sqrt{77}-5}{4}$$

$$\alpha_5 = \frac{4}{\sqrt{77}-5} * \frac{\sqrt{77}+5}{\sqrt{77}+5} = \frac{\sqrt{77}+5}{13} = 1.05 \Rightarrow a_5 = \lfloor 1.05 \rfloor = 1$$

$$\alpha_5 = a_5 + \frac{1}{\alpha_6} \Rightarrow \frac{\sqrt{77}+5}{13} = 1 + \frac{1}{\alpha_6} \Rightarrow \frac{1}{\alpha_6} = \frac{\sqrt{77}+7}{13} - 1 = \frac{\sqrt{77}-7}{13}$$

$$\alpha_6 = \frac{13}{\sqrt{77}-7} * \frac{\sqrt{77}+7}{\sqrt{77}+7} = \sqrt{77}+8 = 16.77 \Rightarrow a_6 = \lfloor 16.77 \rfloor = 16$$

$$\alpha_6 = a_6 + \frac{1}{\alpha_7} \Rightarrow \sqrt{77}+8 = 16 + \frac{1}{\alpha_7} \Rightarrow \frac{1}{\alpha_7} = \sqrt{77}-8$$

$$\alpha_7 = \frac{1}{\sqrt{77}-8} * \frac{\sqrt{77}+8}{\sqrt{77}+8} = \frac{\sqrt{77}+8}{13} \Rightarrow a_7 = \lfloor 1.29 \rfloor = 1 = a_1$$

$$\alpha_7 = a_7 + \frac{1}{\alpha_8} \Rightarrow \frac{\sqrt{77}+8}{13} = 1 + \frac{1}{\alpha_8} \Rightarrow \frac{1}{\alpha_8} = \frac{\sqrt{77}+8}{13} - 1 = \frac{\sqrt{77}-5}{13}$$

$$\alpha_8 = \frac{13}{\sqrt{77}-5} * \frac{\sqrt{77}+5}{\sqrt{77}+5} = \frac{\sqrt{77}+5}{4} \Rightarrow a_8 = \lfloor 3.44 \rfloor = 3 = a_2$$

And so on then $a_9 = a_3 = 2$, $a_{10} = a_4 = 3$, $a_{11} = a_5 = 1$, …, that is,

$\sqrt{77} = [8; 1,3,2,3,1,16,1,3,2,3,1,16, ...]$. The continued fraction expression for

$$\sqrt{77} = 8 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{1}}}}} = 8 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{4}}}} = 8 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{\frac{9}{4}}}} = 8 + \cfrac{1}{1 + \frac{1}{\frac{31}{9}}} = 8 + \frac{31}{40}$$

$$= \frac{351}{40} = \frac{x_1}{y_1}$$

Thus$(x_1, y_1) = (351{,}40) \Rightarrow \alpha = 351 + 40\sqrt{77}$

ByStep6using Eq. (13) gives$s_r + \sqrt{77}t_r = \gamma\alpha^i$where $i = 0,1,2,3, \dots$,and $r = 1,2,3, \dots$

$$\gamma\alpha^i = (27 + 3\sqrt{77})(351 + 40\sqrt{77})^i = s_r + \sqrt{77}t_r$$

If $i = 0,\ r = 1 \Rightarrow \gamma\alpha^0 = (27 + 3\sqrt{77})(351 + 40\sqrt{77})^0 = s_r + \sqrt{77}t_r$

$$27 + 3\sqrt{77} = s_1 + t_1\sqrt{77} \Rightarrow (s_1, t_1) = (27{,}3)$$

If $i = 1,\ r = 1 \Rightarrow \gamma\alpha^1 = (27 + 3\sqrt{77})(351 + 40\sqrt{77})^1 = s_r + \sqrt{77}t_r$

$$= 18717 + 2133\sqrt{77} = s_2 + t_2\sqrt{77} \Rightarrow (s_2, t_2) = (18717{,}2133)$$

If $i = 2,\ r = 3 \Rightarrow \gamma\alpha^2 = (27 + 3\sqrt{77})(351 + 40\sqrt{77})^2 = s_r + \sqrt{77}t_r$

$= 887202 + 101106\sqrt{77} = s_3 + t_3\sqrt{3} \Rightarrow (s_3, t_3) = (887202{,}101106)$.

And so on for $i = 3,4,5,6, \dots$and $r = 4,5,6, \dots$we will get the rest of all natural solutions(s_r, t_r), for Eq. (9).By using these solutions, all natural pairs (m, n) to Eq. (6) such that $u = 3^2$will be obtained as follows:

As we supposed that, $2m - 9n = s_r,\ n = t_r$, where $r = 1,2,3, \dots$, gives,

For$r = 1, (s_1, t_1) = (27{,}3) \Rightarrow (m, n) = (27{,}3\ ) \Rightarrow u = \frac{m^2+n^2}{1+mn} = \frac{738}{82} = 9$

For$r = 2, (s_2, t_2) = (14{,}8) \Rightarrow (m, n) = (18975{,}2133\ ) \Rightarrow u = \frac{m^2+n^2}{1+mn} = \frac{363917538}{40435288} = 9$

For $r = 3, (s_3, t_3) = (52{,}30) \Rightarrow (m, n) = (13307787{,}1497363) \Rightarrow u = \frac{m^2+n^2}{1+mn} = \frac{179339290791138}{19926587865682} = 9$.

And so on for $i = 3,4,5, \dots, r = 4,5,6, \dots$ giving all the other natural pairs$(m, n)$for Eq. (6) such that$u = 3^2$.

<u>Conclusion:</u> We give a new relation (13) between any natural solutions to the hyperopia Eq. (2) and the Pell equationEq. (3). By using this relation we have been found all natural solutions to Eq. (2) then these solutions have been used to find all natural pairs (m, n)such that$u$ is <u>a natural square number</u> in Eq. (1).The Diophantine Eq. (1) has been discussedwhen$u = 1^2, 2^2, 3^2$ similarly for$u = 4^2, 5^2, 6^2, \dots$

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