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On Nonlinear Integral Operator of Trigonometric Type Using Modified Newton Method

Hameed Husam Hameed

hameedmath71@yahoo.com

Technical Institute of Alsuwiara, The Middle Technical University, Iraq.

Abstarct

The Modified Newton method (MNM) is applied to obtain the approximate solution to the system of nonlinear Volterra type integral equation with the trigonometric kernel function. Modified Newton method (MNM) is used to linearized the system then solved by the Nystrom type Gauss-Legendre quadrature formula (QF). A new majorant function is stated which leads to the increment of convergence interval. The existence and uniqueness of approximate solution are proved. Sufficient condition for the approximate solution is established and their validity is illustrated with example.

Keywords: Modified Newton method, nonlinear operator, system of nonlinear Volterra integral equation, Gauss-Legendre formula.

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الخلاصة

لقد تم تطبيق طريقة نيوتن المطورة لايجاد الحل التقريبي للمعادلات التكاملية غير الخطية مع نواة دالة مثلثية. استخدمت طريقة نيوتن المطورة لامكانية تقريب النظام غير الخطي بنظام خطي وبعدها استخدمت صيغة كاوس- ليجيندر التربيعية لايجاد حل النظام. في هذا البحث طبقنا دالة جديدة لتكون (majorant) حيث ادت الى زيادة فترة تقارب الحل وقد تم اثبات وجود ووحدانية الحل التقريبي مع التاكيد على الشروط الضرورية لتقرب الحل التقربي.

1. Introduction

Consider the nonlinear operator

$$\Upsilon(\chi)=0, \tag{1}$$

where Υ is a differentiable operator from an open set Ω in a Banach space A to a Banach space B. The MNM assures the semi local convergence of the solution of Eq. (1). This method has many theoretical and practical applications and one of these applications is on the system of nonlinear integral equations. For instance, Argyros and Hilout [1] used Lipschitz and centre Lipschitz conditions with recurrent functions to provide a semilocal convergence analysis for Newton's method in order to approximate a local unique solution of an equation in a Banach space. Ezquerro et al. [2] discussed a semilocal convergence of Kantorovich method in Banach space, and solved the two Hammerstein integral equations of the second kind by the MNM. Ezquerro et al. [3] used the majorant principle, which is based on the concept of majorizing sequence given by Kantorovich to find the approximate solution of a particular nonlinear integral equation. Eshkuvatov et al. [4] developed the Newton-Kantorovich method to solve the system of nonlinear integral equations and proved the existence and uniqueness of the solution. Hameed et al. [5] proposed a new majorant function for the Newton- Kantorovich method to solve the system of nonlinear Volterra integral equations with the unknown function in logarithmic form. Saeri-nadjafi and Heidari [8] presented a combination of the MNM and quadrature method to solve the nonlinear integral equation of th Urysohn form in a systematic procedure. Shena and Li [9] established Kantorovich- type convergence criterion for inexact Newton methods which includes the well-known Kantorovich's theorem as a special case.

Consider the system of nonlinear Volterra integral equation of the form

$$\left.\begin{aligned}\xi(t)-\int_{\mu(t)}^{t}\omega_1(t,\tau)\sin\big(\xi(t)\big)=\alpha(t),\\ \xi(t)-\int_{\mu(t)}^{t}\omega_2(t,\tau)\sin\big(\xi(t)\big)=\beta(t),\end{aligned}\right\}$$

(2)

where $t\in[t_0,T]$, $0<t_0\le t\le T$ and $\sin\big(\xi(t)\big)$ is nonlinear continuous differentiable function on $[t_0,T]$ and the known functions $\alpha(t),\beta(t)\in C_{[t_0,T]}$ and $\omega_1(t,\tau),\omega_2(t,\tau)\in C_{[t_0,T]\times[t_0,T]}$. The unknown functions $\xi(t)\in C_{[t_0,T]}$ and $\mu(t)\in C^1_{[t_0,T]}$ are to be determined.

The structure of this paper is as follows: In Section 2, we described the MNM. In Section 3, the system of algebraic linear Volterra integral equation using Nystrom Gauss-Legendre QF is described. Section 4 discusses the rate of convergence of the approximate solution of the system (2). An example is provided in Section 5 to show the accuracy and efficiency of the method. Finally, Section 6 concludes the main ideas of the approximate method.

2. Modified Newton method for integral operator

To find the unknown functions $\xi(t)$ and $\mu(t)$ in Eq.(1) we use the notations

$$\left.\begin{aligned}\Upsilon_1(\chi)=\xi(t)-\int_{\mu(t)}^{t}\omega_1(t,\tau)\sin\big(\xi(t)\big)-\alpha(t)=0,\\ \Upsilon_2(\chi)=\xi(t)-\int_{\mu(t)}^{t}\omega_2(t,\tau)\sin\big(\xi(t)\big)-\beta(t)=0,\end{aligned}\right\}$$

(3)

where $\chi=\big(\xi(t),\mu(t)\big)$, $t\in[t_0,T]$ and $0<t_0\le t\le T$, then the system (3) can be written in operator equation

$$\Upsilon(\chi)=\big(\Upsilon_1(\chi),\Upsilon_2(\chi)\big)=0.$$

(4)

Write the initial approximation as

$$\Upsilon'(\chi_0)(\chi-\chi_0)+\Upsilon(\chi_0)=0,\ \chi_0=\big(\xi_0(t),\mu_0(t)\big),$$

(5)

where χ_0 refers to the initial condition and $\xi_0(t), \mu_0(t)$ can be any continuous functions provided that $\mu_0(t) < t$ and $\xi_0(t) \neq 0$ for $t \in [t_0, T]$. The Frechet derivative of $\Upsilon(\chi)$ at the point χ_0 is defined as [7]

$$\Upsilon'(\chi_0)\chi = \begin{pmatrix} \left.\frac{\partial \Upsilon_1}{\partial \xi}\right|_{(\xi_0,\mu_0)} & \left.\frac{\partial \Upsilon_1}{\partial \mu}\right|_{(\xi_0,\mu_0)} \\ \left.\frac{\partial \Upsilon_2}{\partial \xi}\right|_{(\xi_0,\mu_0)} & \left.\frac{\partial \Upsilon_2}{\partial \mu}\right|_{(\xi_0,\mu_0)} \end{pmatrix} \begin{pmatrix} \xi \\ \mu \end{pmatrix} \tag{6}$$

Eqs. (5) and (6) yield

$$\left.\begin{aligned} \left.\frac{\partial \Upsilon_1}{\partial \xi}\right|_{(\xi_0,\mu_0)} (\Delta\xi(t)) + \left.\frac{\partial \Upsilon_1}{\partial \mu}\right|_{(\xi_0,\mu_0)} (\Delta\mu(t)) = -\Upsilon_1(\xi_0(t), \mu_0(t)), \\ \left.\frac{\partial \Upsilon_2}{\partial \xi}\right|_{(\xi_0,\mu_0)} (\Delta\xi(t)) + \left.\frac{\partial \Upsilon_2}{\partial \mu}\right|_{(\xi_0,\mu_0)} (\Delta\mu(t)) = -\Upsilon_2(\xi_0(t), \mu_0(t)), \end{aligned}\right\} \tag{7}$$

where $\Delta\xi(t) = \xi_1(t) - \xi_0(t)$, $\Delta\mu(t) = \mu_1(t) - \mu_0(t)$ and $(\xi_0(t), \mu_0(t))$ is the initial guess. To solve Eq. (7) with respect to $\Delta\xi$ and $\Delta\mu$ we need to compute all the partial derivatives

$$\begin{aligned} \left.\frac{\partial \Upsilon_1}{\partial \xi}\right|_{(\xi_0,\mu_0)} &= \lim_{\rho\to 0} \frac{1}{\rho} \left(\Upsilon_1(\xi_0 + \rho\xi, \mu_0) - \Upsilon_1(\xi_0, \mu_0) \right) \\ &= \lim_{\rho\to 0} \frac{1}{\rho} \left[\rho\xi(t) - \int_{\mu_0(t)}^{t} \omega_1(t,\tau) \left(\sin(\xi_0(\tau) + \rho\xi(\tau)) - \sin(\xi_0(\tau)) \right) d\tau \right] \\ &= \xi(t) - \int_{\mu_0(t)}^{t} \omega_1(t,\tau) \cos(\xi_0(\tau)) \xi(\tau)\, d\tau, \end{aligned} \tag{8}$$

$$\begin{aligned} \left.\frac{\partial \Upsilon_1}{\partial \mu}\right|_{(\xi_0,\mu_0)} &= \lim_{\rho\to 0} \frac{1}{\rho} \left(\Upsilon_1(\xi_0, \mu_0 + \rho\mu) - \Upsilon_1(\xi_0, \mu_0) \right) \\ &= \lim_{\rho\to 0} \frac{1}{\rho} \left[\int_{\mu_0(t)}^{\mu_0(t)+\rho\mu(t)} \omega_1(t,\tau) \sin(\xi_0(\tau)) d\tau \right] \\ &= \omega_1(t, \mu_0(t)) \sin(\xi_0(\mu_0(t))) \mu(t), \end{aligned} \tag{9}$$

and with the same procedure we get

$$\left.\frac{\partial \Upsilon_2}{\partial \xi}\right|_{(\xi_0,\mu_0)} = \xi(t) + \int_{\mu_0(t)}^{t} \omega_2(t,\tau)\sin\left(\xi_0(\tau)\right)\xi(\tau)\,d\tau, \tag{10}$$

$$\left.\frac{\partial \Upsilon_2}{\partial \mu}\right|_{(\xi_0,\mu_0)} = -\omega_2(t,\mu_0(t))\sin\left(\xi_0(\mu_0(t))\right)\mu(t). \tag{11}$$

By substituting Eqs (8) –(11) into Eq (7) we arrive at

$$\left.\begin{aligned}
&\Delta\xi(t) - \int_{\mu_0(t)}^{t} \omega_1(t,\tau)\cos(\xi_0(\tau))\Delta\xi(t)\,d\tau + \omega_1(t,\mu_0(t))\sin(\xi_0(\mu_0(t)))\Delta\mu(t) \\
&\quad = \int_{\mu_0(t)}^{t} \omega_1(t,\tau)\sin(\mu_0(\tau))\,d\tau - \mu_0(t) + \alpha(t), \\
&\Delta\xi(t) + \int_{\mu_0(t)}^{t} \omega_2(t,\tau)\cos(\xi_0(\tau))\Delta\xi(t)\,d\tau - \omega_2(t,\mu_0(t))\sin(\xi_0(\mu_0(t)))\Delta\mu(t) \\
&\quad = -\int_{\mu_0(t)}^{t} \omega_2(t,\tau)\sin(\mu_0(\tau))\,d\tau - \mu_0(t) + \beta(t).
\end{aligned}\right\} \tag{12}$$

Since Eq.(12) is a linear Volterra integral equations, it can easily be solved in term of $\Delta\xi(t)$ and $\Delta\mu(t)$ as

$$\left.\begin{aligned}
&\Delta\xi(t) - \int_{\mu_0(t)}^{t} \varpi(t,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi(t)d\tau = \varphi_0(t), \\
&\Delta\mu(t) = \frac{1}{\nu(t)}\left[\Delta\xi(t) + \int_{\mu_0(t)}^{t} \omega_2(t,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi(t)d\tau\right. \\
&\qquad \left. + \int_{\mu_0(t)}^{t} \omega_2(t,\tau)\sin\left(\xi_0(\tau)\right)d\tau + \xi_0(t) - \beta(t)\right],
\end{aligned}\right\} \tag{13}$$

where

$$\varphi_0(t)=\int_{\mu_0(t)}^{t}\varpi(t,\tau)\sin\left(\xi_0(t)\right)d\tau-\xi_0(t)+\frac{\phi(t)\beta(t)}{1+\phi(t)}+\frac{\alpha(t)}{t+\phi(t)}, \tag{14}$$

$$\varpi(t)=\frac{\omega_1(t)-\phi(t)\omega_2(t)}{1+\phi(t)}, \tag{15}$$

$$\phi(t)=\frac{\omega_1(t,\mu_0(t))}{\omega_2(t,\mu_0(t))}, \tag{16}$$

$$\nu(t)=\omega_2(t,\mu_0(t))\sin(\xi_0(t)). \tag{17}$$

By continuing this process, a sequence of approximate solution $\left(\xi_m(t),\mu_m(t)\right)$ can be evaluated from the equation

$$\Upsilon'\left(\chi_0\right)\Delta\chi_m+\Upsilon\left(\chi_{m-1}\right)=0,\quad m=1,2,\ldots$$

which is equivalent to the system

$$\left.\begin{aligned}&\Delta\xi_m(t)-\int_{\mu_0(t)}^{t}\varpi(t,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi_m(t)d\tau=\varphi_{m-1}(t),\\&\Delta\mu_m(t)=\frac{1}{\nu(t)}\left[\Delta\xi_m(t)+\int_{\mu_0(t)}^{t}\omega_2(t,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi_m(t)d\tau\right.\\&\qquad\left.+\int_{\mu_{m-1}(t)}^{t}\omega_2(t,\tau)\sin\left(\xi_{m-1}(\tau)\right)d\tau+\xi_{m-1}(t)-\beta(t)\right],\end{aligned}\right\} \tag{18}$$

where

$$\Delta\xi_m(t)=\xi_m(t)-\xi_{m-1}(t),\quad \Delta\mu_m(t)=\mu_m(t)-\mu_{m-1}(t),\ m=2,3,\ldots \tag{19}$$

and

$$\varphi_{m-1}(t)=\int_{\mu_{m-1}(t)}^{t}\varpi(t,\tau)\sin\left(\xi_{m-1}(t)\right)d\tau-\xi_{m-1}(t)+\frac{\phi(t)\beta(t)}{1+\phi(t)}+\frac{\alpha(t)}{t+\phi(t)}.$$

Solving Eq.(18) for $\Delta\xi_m(t)$ and $\Delta\mu_m(t)$ we obtain a sequence of approximate solution $\left(\xi_m(t),\mu_m(t)\right)$

3. Gauss-Legendre Quadrature Method

For the approximate solution of the linear system (18) we introduce a grid points $\Omega_1 = \left\{ t_i : t_i = t_0 + h\dfrac{T-t_0}{n},\ i = 2,3,\ldots,n \right\}$ where n refer to the number of partitions in $[t_0, T]$. Then from the system (18) we obtain

$$\left.\begin{aligned} &\Delta\xi_m(t_i) - \int_{\mu_0(t_i)}^{t_i} \varpi(t_i,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi_m(t_i)d\tau = \varphi_{m-1}(t_i), \\ &\Delta\mu_m(t_i) = \frac{1}{\nu(t_i)}\left[\Delta\xi_m(t_i) + \int_{\mu_0(t_i)}^{t_i} \omega_2(t_i,\tau)\cos\left(\xi_0(\tau)\right)\Delta\xi_m(\tau)d\tau \right. \\ &\left. + \int_{\mu_{m-1}(t_i)}^{t_i} \omega_2(t_i,\tau)\sin\left(\xi_{m-1}(\tau)\right)d\tau + \xi_{m-1}(t_i) - \beta(t_i) \right], \end{aligned}\right\} \quad (20)$$

and

$$\varphi_{m-1}(t_i) = \int_{\mu_{m-1}(t_i)}^{t_i} \varpi(t_i,\tau)\sin\left(\xi_{m-1}(t_i)\right)d\tau - \xi_{m-1}(t_i) + \frac{\phi(t_i)\beta(t_i)}{1+\phi(t_i)} + \frac{\alpha(t_i)}{t+\phi(t_i)}.$$

It is well known that one of the powerful technique to approximate the integrals in the system (20) is Gauss-Legendre QF. We know that the Legendre polynomials $P_n(t)$ are orthogonal on $[-1,1]$ with weight $\omega = 1$. Therefore Gauss-Legendre QF [6, pp. 318]

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} \theta_i f(s_i) + R_i(f) \quad (21)$$

where

$$\theta_i = \frac{2}{\left(1-s_i^2\right)\left[P_n'(s_i)\right]^2},\ \sum_{i=1}^{n}\theta_i = 2,\ P_n(s_i) \equiv 0, i = 1,2,\ldots,n,$$

s_i is a root of Legendre polynomial $P_n(t)$ with the error term

$$R_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{2n}(\zeta),\ -1 < \zeta < 1.$$

The Gauss- Legendre QF formula for arbitrary interval $[a,b]$ has the form

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\sum_{i=1}^{n}\theta_i f(t_i) + R_i(f), \tag{22}$$

where the nods $t_i = \left(\frac{b-a}{2}\right)s_i + \left(\frac{b+a}{2}\right)$. Now, let us introduce a subgrid $\{\Omega_2\}$ at each subinterval $[y_0(t_i), t_i]$ and $[y_{m-1}(t_i), t_i]$ of the interval $[t_0, T]$, such that

1. for the interval $[y_0(t_i), t_i]$ we chose the grid points as

$$\tau_{i(0)}^{j} = \frac{t_i - y_0(t_i)}{2}s_j + \frac{t_i + y_0(t_i)}{2}, \; j = 1,2,\ldots,l; \; i = 1,2,\ldots,n. \tag{23}$$

2. for the interval $[y_{m-1}(t_i), t_i]$ grid points are chosen as

$$\tau_{i(m-1)}^{j} = \frac{t_i - y_{m-1}(t_i)}{2}s_j + \frac{t_i + y_{m-1}(t_i)}{2}, \; j = 1,2,\ldots,l; \; i = 1,2,\ldots,n; m = 1,2,\ldots \tag{24}$$

where l is the number of points in each subinterval $[y_0(t_i), t_i]$ and $[y_{m-1}(t_i), t_i]$ and s_j are the zeros of Legendre polynomial $P_n(x)$ over the interval $[-1,1]$ and $\tau_{i(0)}^{j} \neq t_i$, $\tau_{i(m-1)}^{j} \neq t_i$. Extending Gauss-Legendre QF in Eq. (22) to the integral on each interval $[y_0(t_i), t_i]$ and $[y_{m-1}(t_i), t_i]$ in the system (20), we get

$$\left.\begin{aligned}
&\Delta\xi_m(\tau_i^{\kappa}) - \frac{t_i - y_0(t_i)}{2}\sum_{j=1}^{l}\varpi(\tau_i^{\kappa}, \tau_{i(0)}^{j})\cos\left(\xi_0(\tau_{i(0)}^{j})\right)\Delta\xi_m(\tau_{i(0)}^{j})\theta_j = \varphi_{m-1}(\tau_i^{\kappa}) \\
&\Delta\mu_m(\tau_i^{\kappa}) = \frac{1}{\nu(\tau_i^{\kappa})}\Bigg[\Delta\xi_m(\tau_i^{\kappa}) + \frac{t_i - y_{m-1}(t_i)}{2}\sum_{j=1}^{l}\omega_2(\tau_i^{\kappa}, \tau_{i(m-1)}^{j})\cos\left(\xi_0(\tau_{i(0)}^{j})\right)\Delta\xi_m(\tau_{i(0)}^{j})\theta_j \\
&\quad + \frac{t_i - y_{m-1}(t_i)}{2}\sum_{j=1}^{l}\omega_2(\tau_i^{\kappa}, \tau_{i(m-1)}^{j})\sin\left(\xi_{m-1}(\tau_{i(m-1)}^{j})\right)\theta_j + \xi_{m-1}(\tau_i^{\kappa}) - \beta(\tau_i^{\kappa})\Bigg], \\
&i = 1,2,\ldots,n; \; \kappa = 1,2,\ldots,l; \; m = 1,2,\ldots
\end{aligned}\right\} \tag{25}$$

where

$$\varphi_{m-1}(\tau_i^{\kappa}) = \frac{t_i - y_{m-1}(t_i)}{2}\left[\sum_{j=1}^{l}\varpi(\tau_i^{\kappa}, \tau_{i(m-1)}^{j})\sin\left(\xi_{m-1}(\tau_{i(m-1)}^{j})\right)\theta_j - \xi_{m-1}(\tau_i^{\kappa}) + \frac{\phi(\tau_i^{\kappa})\beta(\tau_i^{\kappa})}{1+\phi(\tau_i^{\kappa})} + \frac{\alpha(\tau_i^{\kappa})}{t+\phi(\tau_i^{\kappa})}\right]$$

.

The first equation of the system (25) is a linear algebraic system of $n\times l$ equations and $n\times l$ unknowns. If the matrix of this system is non singular then it has a unique solution in terms of $\Delta x_m(\tau_i^\kappa)$, $i=1,2,\dots,n$, $\kappa=1,2,\dots,l$. The values of $\Delta y_m(\tau_i^\kappa)$ can be easily determined by computing the second equation of (25). From Eq. (19) it follows that

$$\Delta\xi_m(\tau_i^\kappa)=\xi_m(\tau_i^\kappa)-\xi_{m-1}(\tau_i^\kappa),\ \ \Delta\mu_m(\tau_i^\kappa)=\mu_m(\tau_i^\kappa)-\mu_{m-1}(\tau_i^\kappa),\ \ m=2,3,\dots \qquad (26)$$

Since the values of the functions $\xi_m(\tau_i^\kappa)$ and $\mu_m(\tau_i^\kappa)$ are known at l Legendre grid points in each subintervals $\left(y_0(t_i),t_i\right)$ and $\left(y_{m-1}(t_i),t_i\right)$ for each m, the values of unknown functions $\xi(t_i)$ and $\mu(t_i)$ can be found by using Newton forward interpolation formula [2, pp. 110], i.e

$$\begin{aligned}\xi_m(t)\equiv P_l(t)&=\xi_m(\tau_i^l)+\xi_m(\tau_i^l,\tau_i^{l-1})(t-\tau_i^l)+\xi_m(\tau_i^l,\tau_i^{l-1},\tau_i^{l-2})(t-\tau_i^l)(t-\tau_i^{l-1})\\&+\cdots+\xi_m(\tau_i^l,\tau_i^{l-1},\tau_i^{l-2},\dots,\tau_i^1)(t-\tau_i^l)(t-\tau_i^{l-1})\cdots(t-\tau_i^1),\end{aligned} \qquad (27)$$

$$\begin{aligned}\mu_m(t)\equiv P_l(t)&=\mu_m(\tau_i^l)+\mu_m(\tau_i^l,\tau_i^{l-1})(t-\tau_i^l)+\mu_m(\tau_i^l,\tau_i^{l-1},\tau_i^{l-2})(t-\tau_i^l)(t-\tau_i^{l-1})\\&+\cdots+\mu_m(\tau_i^l,\tau_i^{l-1},\tau_i^{l-2},\dots,\tau_i^1)(t-\tau_i^l)(t-\tau_i^{l-1})\cdots(t-\tau_i^1),\end{aligned} \qquad (28)$$

with the error

$$\left\|\xi_m(t)-P_l(t)\right\|\le\frac{\mathrm{N}_1}{(l+1)!},$$

$$\left\|\mu_m(t)-P_l(t)\right\|\le\frac{\mathrm{N}_2}{(l+1)!},$$

where

$$\left.\begin{aligned}N_1&=\max\left\{\left|\xi_m^{l+1}(\zeta)\right|\left|(t-\tau_i^l),\dots,(t-\tau_i^1)\right|\right\}\\N_2&=\max\left\{\left|\mu_m^{l+1}(\zeta)\right|\left|(t-\tau_i^l),\dots,(t-\tau_i^1)\right|\right\}\end{aligned}\right\},\ \zeta\in(t_0,T).$$

4. Convergence analysis

Depending on the general theorems of modified Newton method [7, pp. 532] for the convergence, we establish the following theorems with regard to the successive approximations which are characterized by system (18).

Consider the following classed of functions:

- $C_{[t_0,T]}$ the set of all continuous functions $R(t)$ defined on the interval $[t_0,T]$,
- $C_{[t_0,T]\times[t_0,T]}$ the set of all continuous functions $S(t,\tau)$ defined on the region $[t_0,T]\times[t_0,T]$,
- $\bar{C}=\{\chi:\chi=(\xi(t),\mu(t)):\xi(t),\mu(t)\in C_{[t_0,T]}\}$
- $\widehat{C}_{[t_0,T]}=\{\mu(t)\in C^1_{[t_0,T]}:\mu(t)<t\}$.

In addition, define the following norms

$\|\xi\|=\max_{t\in[t_0,T]}|\xi(t)|$, $\|\Delta\chi\|_{\bar{C}}=\max\left\{\|\Delta\xi\|_{C_{[t_0,T]}},\|\Delta\mu\|_{C_{[t_0,T]}}\right\}$, $\|\chi\|_{\bar{C}}=\max\left\{\|\xi\|_{C_{[t_0,T]}},\|\xi'\|_{C_{[t_0,T]}}\right\}$,

$\|\bar{\chi}\|_{\bar{C}}=\max\left\{\|\bar{\xi}\|_{C_{[t_0,T]}},\|\bar{\mu}\|_{C_{[t_0,T]}}\right\}$, $\|\omega_1(t,\tau)\|=H_1$, $\|\omega'_{1\tau}(t,\tau)\|=H'_1$, $\|\omega_2(t,\tau)\|=H_2$,

$\|\omega'_{2\tau}(t,\tau)\|=H'_2$, $\|\phi(t)\|=\max_{t\in[t_0,T]}|G(t)|=c_1$, $\|\phi(t)\|=\max_{t\in[t_0,T]}|\phi(t)|=c_1$,

$\left\|\frac{1}{1+\phi(t)}\right\|=\max_{t\in[t_0,T]}\left|\frac{1}{1+\phi(t)}\right|=c_2$ $\min_{t\in[t_0,T]}|\mu_0(t)|=H_3$, $\|\xi'_0\|=\max_{t\in[t_0,T]}|\xi'_0(t)|=H'_3$,

$\|\alpha\|=\max_{t\in[t_0,T]}|\alpha(t)|=H_4$, $\|\beta\|=\max_{t\in[t_0,T]}|\beta(t)|=H_5$.

Let

$\eta_1=\max\{H_1(T-H_3),H_1,H'_1+H_1H'_3,H_2(T-H_3),H_2,H'_2+H_2H'_3\}$.

(29)

Let us introduce the real valued function

$$\Psi(t)=(t-t_0)^2+(\varepsilon+\sigma)(t-t_0)+\varepsilon\sigma,$$

(30)

where ε and σ are real coefficients.

Theorem 1 : Let the nonlinear operator $P(\chi)=0$ in Eq. (4) is defined in open set $\Theta=\{\chi\in C([t_0,T]):\|\chi-\chi_0\|<R\}$ and has continuous second derivative in closed set $\Theta_0=\{\chi\in C([t_0,T]):\|\chi-\chi_0\|\le r\}$ such that $T=t_0+r<t_0+R$. Assume the following conditions are satisfied

1- $\|\Gamma_0\Upsilon(\chi_0)\|\le\frac{\varepsilon\sigma}{\varepsilon+\sigma}$, $\Gamma_0=[\Upsilon'(\chi_0)]^{-1}$, 2- $\left\|\Gamma_0 \Upsilon''(\chi)\right\| \le \frac{2}{\varepsilon+\sigma}$, when $\|\chi-\chi_0\| \le t-t_0 \le r$,

then $\Psi(t)$ in Eq. (30) is a majorant function for the nonlinear operator $P(\chi)$.

Proof: Rewrite Eqs. (4) and (30) in the form

$$t=\Phi(t),\ \Phi(t)=t+c_0\Psi(t), \tag{31}$$

$$\chi=S(\chi),\ S(\chi)=\chi-\Gamma_0(\chi), \tag{32}$$

where $c_0=-\frac{1}{\Psi'(t_0)}=\frac{1}{\varepsilon+\sigma}$. We need to show that Eqs (31) and (32) satisfy the majorizing conditions [8, Theorem 1, pp. 525]. In deed

$$\left\|S(\chi_0)-\chi_0\right\|=\left\|-\Gamma_0\Upsilon(\chi_0)\right\| \le \frac{\varepsilon\sigma}{\varepsilon+\sigma}=\Phi(t_0)-t_0, \tag{33}$$

and since $\|\chi-\chi_0\| \le t-t_0$, with utilizing the remark in [8, pp. 504] we have

$$\begin{aligned}\left\|S'(\chi)\right\|&=\left\|S'(\chi)-S'(\chi_0)\right\| \le \int_{\chi_0}^{\chi}\left\|S''(\chi)\right\|d\chi=\int_{\chi_0}^{\chi}\left\|\Gamma_0\Upsilon''(\chi)\right\|d\chi\\ &\le \int_{t_0}^{t} c_0\Psi''(\tau)d\tau=\int_{t_0}^{t}\frac{2}{\varepsilon+\sigma}d\tau=\frac{2}{\varepsilon+\sigma}(t-t_0)=\Phi''(t).\end{aligned} \tag{34}$$

Therefore $\Psi(t)$ is a majorant function of $\Upsilon(\chi)$.□

Theorem 2: Let the function $\alpha(t),\ g(t)\in C_{[t_0,T]}$, $\xi_0(t)\in C^1_{[t_0,T]}$ and the kernels $\omega_1(t,\tau),\omega_2(t,\tau)\in C^1_{[t_0,T]\times[t_0,T]}$ and $\left(\xi_0(t),\mu_0(t)\right)\in\Theta_0$, the if

1- The system (5) has a unique solution in the interval $[t_0,T]$; i.e., there exits Γ_0 and

$$\left\|\Gamma_0\right\| \le \sum_{j=1}^{\infty}\left(c_2\left(H_1+c_1H_2\right)\right)^j\frac{\left(T-H_3\right)^{j-1}}{(j-1)!}=\eta_2,$$

2- $\left\|\Delta\chi\right\| \le \frac{\varepsilon\sigma}{\varepsilon+\sigma}$,

3- $\left\|\Upsilon''(\chi)\right\| \le \eta_1$, 4- $r < \max\{\varepsilon + t_0, \sigma + t_0\}$,

where ε and σ as in Eq. (30). Then the system (2) has a unique solution χ^* in the closed ball Θ_0 and the sequence $\chi_m(t) = (\xi_m(t), \mu_m(t))$, $m \geq 0$ of successive approximations

$$\left.\begin{aligned} &\Delta\xi_m(t) - \int_{\mu_0(t)}^{t} \varpi(t,\tau)\cos(\xi_0(\tau))\Delta\xi_m(t)d\tau = \varphi_{m-1}(t), \\ &\Delta\mu_m(t) = \frac{1}{\nu(t)}\left[\Delta\xi_m(t) + \int_{\mu_0(t)}^{t} \omega_2(t,\tau)\cos(\xi_0(\tau))\Delta\xi_m(t)d\tau\right. \\ &\qquad \left. + \int_{\mu_{m-1}(t)}^{t} \omega_2(t,\tau)\sin(\xi_{m-1}(\tau))d\tau + \xi_{m-1}(t) - \beta(t)\right], \end{aligned}\right\}$$

where $\Delta\xi_m(t) = \xi_m(t) - \xi_{m-1}(t),\ \Delta\mu_m(t) = \mu_m(t) - \mu_{m-1}(t),\ m = 2,3,\ldots$, and χ_m converges to the solution χ^*. The rate of convergence is given by

$\left\|\chi^* - \chi_m\right\| \leq \left(\frac{2\varepsilon}{\varepsilon+\sigma}\right)\varepsilon,$ if ε the minimum zero of Eq. (30), or $\left\|\chi^* - \chi_m\right\| \leq \left(\frac{2\sigma}{\varepsilon+\sigma}\right)\sigma,$ if σ the minimum zero of Eq. (30).

Proof: Since the first equation of the system (13) is a linear integral equation of the second kind, so it has a unique solution in term of $\Delta\xi(t)$, provided that $1+\phi(t) \neq 0$ and $\omega_2(t,\mu_0(t)) \neq 0 \quad \forall t \in [t_0,T]$ and $\varpi(t,\tau)$ which is defined in Eq. (15) is a continuous function, then $\Delta\mu(t)$ can be uniquely determined from the second equation of (13). Hence the existence of Γ_0 is achieved. Now, to prove that Γ_0 is bounded we need to find the resolvent kernel $\Gamma_0(t,\tau)$ of the first equation in system (13). Consider the integral operator U from $C[t_0,T] \to C[t_0,T]$ is given by

$$Z = U(\Delta\chi), \quad Z(t) = \int_{\mu_0(t)}^{t} \gamma(t,\tau)\Delta\xi(\tau)d\tau, \tag{35}$$

where $\gamma(t,\tau) = \varpi(t,\tau)\cos(x_0(\tau))$ and $\varpi(t,\tau)$ is define in Eq. (15). According to Eq. (35), the first equation in system (13) can be represented as

$$\Delta\xi - U\left(\Delta\xi\right) = \varphi_0(t). \tag{36}$$

The solution of Eq. (36) is written in terms of φ_0 by the formula

$$\Delta\xi^* = \varphi_0 + B(\varphi_0), \tag{37}$$

where B is an integral operator and can be written as a series in powers of U [8, Theorem 1, pp. 378]

$$B(\varphi_0) = U(\varphi_0) + U^2(\varphi_0) + \cdots + U^n(\varphi_0) + \cdots, \tag{38}$$

and it is famed that the powers of U are also integral operator. In deed

$$Z_n = U^n, \quad Z_n(t) = \int_{\mu_0(t)}^{t} \gamma^{(n)}(t,\tau)\Delta\xi(\tau)\,d\tau, \quad (n = 1,2,\ldots) \quad , \tag{39}$$

where $\gamma^{(n)}$ is the iterated kernel. Substituting Eq. (39) into (37) we get an expression for the solution of Eq. (36)

$$\Delta x^* = \varphi_0(t) + \sum_{j=1}^{\infty} \int_{\mu_0(t)}^{t} \gamma^{(n)}(t,\tau)\varphi_0(t)d\tau. \tag{40}$$

Next, we state that the series in Eq. (40) is convergent uniformly for all $t \in [t_0, T]$. Since

$$|\gamma(t,\tau)| = |\varpi(t,\tau)\cos(\xi_0(\tau))| \le |\xi(t,\tau)| \le \left|\frac{\omega_1(t,\tau)}{1+\phi(t)}\right| + \left|\frac{\phi(t)\omega_2(t,\tau)}{1+\phi(t)}\right|$$
$$\le c_2 H_1 + c_1 c_2 H_2. \tag{41}$$

Let $M = c_2 H_1 + c_1 c_2 H_2$, then by mathematical induction we obtain

$|\gamma^{(n)}(t,\tau)| \le \int_{\mu_0(t)}^{t} |\gamma(t,u)\gamma^{(n-1)}(t,\tau)| du \le \frac{M^n (T - H_3)^{(n-1)}}{(n-1)!}, (n = 1,2,\ldots)$, then

$\|U^n\| \le \max_{t\in[t_0,T]} \int_{\mu_0(t)}^{t} |\gamma^{(n)}(t,\tau)| du \le \frac{M^n (T-H_3)^{(n-1)}}{(n-1)!}$. Therefore the nth root test of the sequence implies

$$\sqrt[n]{\|U^n\|} \le \frac{M\left(T-H_3\right)^{1-\frac{1}{n}}}{\sqrt[n]{(n-1)!}} \underset{n\to\infty}{\to} 0.$$

(42)

As a result $\rho = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\|U^n\|}} = \infty$, and the first equation of the system (13) has no characteristic values. Since the series in Eq. (40) converges uniformly. Eq. (37) can be expressed in term of resolvent kernel of the first equation of (13)

$$\Delta\xi^* = \varphi_0(t) + \int\limits_{\mu_0(t)}^{t} \Gamma_0(t,\tau)\varphi_0(\tau)\,d\tau\,,$$

(43)

where

$$\Gamma_0(t,\tau) = \sum_{j=1}^{\infty}\gamma^j(t,\tau)\,.$$

(44)

Since the series in Eq. (44) is convergent we obtain

$$\|\Gamma_0\| = \|B(\varphi_0)\| \le \sum_{j=1}^{\infty}\|U^j\| \le M^j\,\frac{\left(T-H_3\right)^{j-1}}{(j-1)!} \le \eta_2\,.$$

(45)

To evaluate the validity of second condition, let us describe operator equation

$$\Upsilon(\chi) = 0\,,$$

(46)

as in Eq. (32) and it's successive approximation is

$$\chi_{n+1} = S(\chi_n),\ \ n = 0,1,2,\ldots$$

(47)

For the initial condition χ_0 we have

$$S\left(\chi_0\right) = \chi_0 - \Gamma_0\Upsilon\left(\chi_0\right),$$

(48)

then from the first condition of (Theorem 1) we have

$$\|\Gamma_0\Upsilon\left(\chi_0\right)\| = \|S\left(\chi_0\right) - \chi_0\| = \|\chi_1 - \chi_0\| = \|\Delta\chi\| \le \frac{\varepsilon\sigma}{\varepsilon+\sigma}\,.$$

Moreover, we need to show that $\left\|\Upsilon''(\chi)\right\| \le \eta_1$ for all $\chi \in \Theta_0$ where η_1 defined in Eq. (29). It is known that the second derivative $\Upsilon''(\chi_0)(\chi,\bar{\chi})$ of the nonlinear operator $\Upsilon(\chi)$ is expressed by 3-dimensional array $\Upsilon''(\chi_0)\chi\bar{\chi} = (D_1 \quad D_2)\begin{pmatrix}\bar{\xi}\\ \bar{\mu}\end{pmatrix}\begin{pmatrix}\xi\\ \mu\end{pmatrix}$, where

$$D_1 = \begin{pmatrix} \left.\dfrac{\partial^2\Upsilon_1}{\partial\xi^2}\right|_{\chi_0} & \left.\dfrac{\partial^2\Upsilon_1}{\partial\mu\partial\xi}\right|_{\chi_0} \\ \left.\dfrac{\partial^2\Upsilon_1}{\partial\xi\partial\mu}\right|_{\chi_0} & \left.\dfrac{\partial^2\Upsilon_1}{\partial\mu^2}\right|_{\chi_0} \end{pmatrix}, \text{ and } D_2 = \begin{pmatrix} \left.\dfrac{\partial^2\Upsilon_2}{\partial\xi^2}\right|_{\chi_0} & \left.\dfrac{\partial^2\Upsilon_2}{\partial\mu\partial\xi}\right|_{\chi_0} \\ \left.\dfrac{\partial^2\Upsilon_2}{\partial\xi\partial\mu}\right|_{\chi_0} & \left.\dfrac{\partial^2\Upsilon_2}{\partial\mu^2}\right|_{\chi_0} \end{pmatrix}.$$ then the norms of every components of D_1 and D_2 has the estimate

$$\left\|\frac{\partial^2\Upsilon_1}{\partial\xi^2}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|-\int_{\mu_0(t)}^{t}\omega_1(t,\tau)\xi(\tau)\bar{\xi}(\tau)\sin(\xi_0(\tau))\,d\tau\right| \le H_1(T-H_3),$$

$$\left\|\frac{\partial^2\Upsilon_1}{\partial\xi\partial\mu}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|\omega_1(t,\mu_0(t))\xi(\mu_0(t))\cos(\xi_0(\mu_0(t)))\bar{\xi}(t)\right| \le H_1,$$

$$\left\|\frac{\partial^2\Upsilon_1}{\partial\mu\partial\xi}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|\omega_1(t,\mu_0(t))\bar{\xi}(\mu_0(t))\cos(\xi_0(\mu_0(t)))\xi(t)\right| \le H_1,$$

$$\left\|\frac{\partial^2\Upsilon_1}{\partial\mu^2}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|\left[\omega_{1\tau}'(t,\mu_0(t))\sin(\xi_0(\mu_0(t))) + \omega_1(t,\mu_0(t))\cos(\xi_0(\mu_0(t)))\xi_0'(\mu_0(t))\right]\mu(t)\bar{\mu}(t)\right|$$
$$\le H_1' + H_1H_3',$$

$$\left\|\frac{\partial^2\Upsilon_2}{\partial\xi^2}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|-\int_{\mu_0(t)}^{t}\omega_2(t,\tau)\xi(\tau)\bar{\xi}(\tau)\sin(\xi_0(\tau))\,d\tau\right| \le H_2(T-H_3),$$

$$\left\|\frac{\partial^2\Upsilon_2}{\partial\xi\partial\mu}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|\omega_2(t,\mu_0(t))\xi(\mu_0(t))\cos(\xi_0(\mu_0(t)))\bar{\xi}(t)\right| \le H_2,$$

$$\left\|\frac{\partial^2\Upsilon_2}{\partial\mu\partial\xi}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|-\omega_2(t,\mu_0(t))\bar{\xi}(\mu_0(t))\cos(\xi_0(\mu_0(t)))\xi(t)\right| \le H_1,$$

$$\left\|\frac{\partial^2\Upsilon_2}{\partial\mu^2}\right\| = \max_{\|\chi\|\le 1,\|\bar{\chi}\|\le 1}\left|\left[\omega_{2\tau}'(t,\mu_0(t))\sin(\xi_0(\mu_0(t))) + \omega_2(t,\mu_0(t))\cos(\xi_0(\mu_0(t)))\xi_0'(\mu_0(t))\right]\mu(t)\bar{\mu}(t)\right|$$
$$\le H_2' + H_2H_3'.$$

Therefore, all second derivatives are exist and bounded. i.e.

$$\left\|\Upsilon''(\chi)\right\| \le \eta_1 . \tag{49}$$

Since $\Psi(t)$ majorizes the operator $\Upsilon(\chi)$, and utilizing the second condition of (Theorem 1) we get

$$\|\Gamma_0 \Upsilon''(\chi)\| \le \frac{2}{\varepsilon+\sigma}. \tag{50}$$

Let us consider the discriminate of the equation $\Psi(t)=0$

$$D=\varepsilon^2-2\varepsilon\sigma+\sigma^2=(\varepsilon-\sigma)^2,$$

and the two roots of $\Psi(t)=0$ are $r_1=\min\{\varepsilon+t_0,\sigma+t_0\}$ and $r_2=\max\{\varepsilon+t_0,\sigma+t_0\}$. Therefore, when $r_1<r<r_2$ implies

$$\Psi(t)\le 0, \tag{51}$$

then under the assumption of the fourth condition; i.e. $\min\{\varepsilon+t_0,\sigma+t_0\}$ is the unique solution of $\Psi(t)=0$ in $[t_0,T]$ and the condition in Eq. (51) [7, Theorem 4, pp.530] implies that χ^* is the unique solution of operator equation (4) [8, Theorem 6, pp.532] and

$$\|\chi^*-\chi_0\| \le t^*-t_0, \tag{52}$$

where t^* is a unique solution of $\Psi(t)=0$ in $[t_0,r]$.

As for the rate of convergence, let us write the equation $\Psi(t)=0$ in a same form as in Eq. (31), then its successive approximation is

$$t_{m+1}=\Phi(t_m),\ m=0,1,2,\dots \tag{53}$$

To establish the difference between t^* and successive approximation t_m

$$t^*-t_m=\Phi(t^*)-\Phi(t_{m-1})=\Phi'(\tilde{t}_m)(t^*-t_{m-1}), \tag{54}$$

where, $\tilde{t}_m\in(t_{m-1},t^*)$ and

$$\Phi'(t)=1+c_0\Psi'(t)=\frac{2}{\varepsilon+\sigma}(t-t_0), \tag{55}$$

therefore, in case $\varepsilon + t_0$ is the minimum root of Eq. (30)

$$\Phi'(\tilde{t}_m) = \frac{2}{\varepsilon+\sigma}\left(\tilde{t}_m - t_0\right) \le \frac{2}{\varepsilon+\sigma}\left(t^* - t_0\right) = \frac{2\varepsilon}{\varepsilon+\sigma},$$

then

$$t^* - t_m \le \frac{2\varepsilon}{\varepsilon+\sigma}\left(t^* - t_{m-1}\right),$$
$$t^* - t_{m-1} \le \frac{2\varepsilon}{\varepsilon+\sigma}\left(t^* - t_{m-2}\right),$$
$$\vdots$$
$$t^* - t_1 \le \frac{2\varepsilon}{\varepsilon+\sigma}\left(t^* - t_0\right),$$

consequently, $t^* - t_m \le \left(\frac{2\varepsilon}{\varepsilon+\sigma}\right)^m \varepsilon$, it implies $\left\|\chi^* - \chi_m\right\| \le \left(t^* - t_m\right) = \left(\frac{2\varepsilon}{\varepsilon+\sigma}\right)^m \varepsilon$.

In the same manner, if σ the minimum root of Eq. (30) we have $\left\|\chi^* - \chi_m\right\| \le \left(t^* - t_m\right) = \left(\frac{2\varepsilon}{\varepsilon+\sigma}\right)^m \sigma$. □

5. Numerical result

Consider

$$\xi(t) - \int_{\mu(t)}^{t} t\sin\left(\xi(\tau)\right)d\tau = t + t\sin\left(\frac{9}{10}t\right) - \frac{9}{10}t^2\cos\left(\frac{9}{10}t\right) - t\sin(t) + t^2\cos(t),$$
$$\xi(t) + \int_{\mu(t)}^{t} t^2\tau^2\sin\left(\xi(\tau)\right)d\tau = t + \frac{81}{100}t^4\cos\left(\frac{9}{10}t\right) - 2t^2\cos\left(\frac{9}{10}t\right) - \frac{9}{5}t^3\sin(\frac{9}{10}t)$$
$$-t^4\cos(t) + 2t^2\cos(t) + 2t^3\sin(t),\ \ t \in (0,1]. \tag{56}$$

The exact solution is

$$\xi^*(t) = t,$$
$$\mu^*(t) = \frac{9}{10}t.$$

$$n = 2, l = 5, h = 0.5, \xi_0 = \frac{t}{2}, \mu_0 = \frac{8}{10}t$$

Table 1: the numerical result of Eq. (56)

m	Error of ξ	Error of μ

1	0.01224	6.44130E-004
2	6.72201E-004	8.45722E-006
3	3.62961E-005	1.10663E-007
4	1.96159E-006	1.44809E-009
5	1.06007E-007	1.89487E-011
6	5.72880E-009	2.48246E-013

$$n=2, l=5, h=0.5, \xi_0=\frac{t^2}{4}, \mu_0=\frac{5}{10}t$$

Table 2: the numerical result of Eq. (56)

m	Error of ξ	Error of μ
1	0.03839	0.01727
2	0.00485	0.00167
3	5.87048E-004	1.57273E-004
4	7.14562E-005	1.48333E-005
5	8.69208E-006	1.39864E-006
6	1.05741E-006	1.31881E-007
11	2.81707E-011	9.81992E-013

We observe that by applying the MNM for the system of nonlinear Volterra type equations (56), the numerical and exact solutions are almost coincide with a small number of m.

Table 1 shows that only six iterations are needed for $\xi_m(t)$ and $\mu_m(t)$ to be very close to $\xi^*(t)$ and $\mu^*(t)$ respectively, while in Table 2 more iterations are needed to reasonable approximate solution when the initial guess is chosen to be far from the exact solution. Notations used here are: n is the number of partitions on $[t_0,T]$, l is the number of subpartitions on $\left(\mu_0(t_i),t_i\right)$ and $\left(\mu_{m-1}(t_i),t_i\right)$, $i=1,2,\ldots,n$,where m is the number of iterations and

$$\text{error of } \xi = \max_{t\in(0,1]}\left|\xi_m(t)-\xi^*\right|,$$

$$\text{error of } \mu = \max_{t\in(0,1]}\left|\mu_m(t)-\mu^*\right|,$$

6. Conclusion

The MNM is applied to solve the system of nonlinear Volterra integral equation with trigonometric function. We have sated a new idea by introducing a subgrid of collocation points τ_i^k, $i=1,2,\ldots,n;\ k=1,2,\ldots,l$ that are included in $\left(\mu_0(t_i),t_i\right)$ and $\left(\mu_{m-1}(t_i),t_i\right)$. Gauss-Legendre QF is used for each subgrid intervals. A numerical example revealed that the accuracy of the MNM can be achieved by a few numbers of iterations. From the Examples 1-3, we observe that by applying the NKM for the system of nonlinear Volterra type equations, the numerical and exact solutions are almost coincide for a small number of iteration m.

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