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On generalization of Phillips Szãsz Type Operators

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Abstract:

In this paper, we give a popularization form Phillips-Sazas-Type operators symbolize by $M_{n,\nu}(f;x)$. We prove the convergence for this operators $as\ n \to \infty$. Also show that a Voronovskaja-type asymptotic formula for our operators. And obtain an error estimate in terms of modulus of continuity of the function being approximated.

Keywords and phrases: Linear positive operator, Simultaneous approximation, Voronovskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

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حول تعميم مؤثرات زازا من النمط فلبز

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الخلاصة :

في هذا البحث، نقدم تعميم للمؤثر من النمط مجموع تكامل Phillips-Sazas ونرمز له بالرمز $M_{n,\nu}(f;x)$. أولا سنثبت التقارب لهذا المؤثر عندما $n \to \infty$. كذلك سنوضح صيغة من النمط Voronovskaja. وأخيرا، نحصل على الخطأ المخمن في حدود مقياس الاستمرارية للدوال المقربة.

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1. Introduction:

O. Szãsz (1950) in [5], a generalized Bernstein's polynomials as :

$$L_n(f;x)=\sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), x\in R_0, n\in N:=\{1,2,\dots\}, \text{ where } q_{n,k}(x)=\frac{(nx)^k}{e^{nx}k!}, k\in N^0$$
$$:=N\cup\{0\}.$$

Kasana et. al. [1], A modification of the classical Szãsz operators in Summation-Integral type operators to approximate a space of integrable functions on $R_0 := [0,\infty)$ is given by:

$$R_n(f(t);x)=n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k}(t)f(t)dt.$$

Another a new modification of Summation-Integral Szãsz type operators in Phillips type operators is defined in [2] as:

$$S_n(f(t);x)=n\sum_{k=1}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k-1}(t)f(t)dt+f(0)q_{n,0}(x).$$

Also, Rempulska and Walczak [4], proposed a modification of the Szãsz operators and studied some direct results in ordinary approximation as:

$$D_{n,v}(f;x)=\sum_{k=0}^{\infty} q_{n,k}(x)\sum_{j=0}^{v}\frac{f^{(j)}(\frac{k}{n})}{j!}\left(x-\frac{k}{n}\right)^j, \qquad x\in R_0, v\in N^0,$$

for $f^{(j)}\in C_h^v[0,\infty)=\{f\in C_h[0,\infty): f^{(k)}\in C_h[0,\infty), k=1,\dots,v\,;\ v\in N^0\}$,

where $C_h[0,\infty)=\{f\in C[0,\infty): |f(t)|\le Ce^{ht} \text{ for some } C>0, h>0\}$.

A more recent (2011) [3] advanced better a modification of the [4], as:

$$R_{n,v}(f;x)=n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k}(t)\sum_{j=0}^{v}\frac{f^{(j)}(t)}{j!}(t-x)^j dt.$$

The purpose of this paper is study a new sequence of linear positive operators $M_{n,v}(f;x)$ for $f\in C_h^v[0,\infty)$ given as follows:

$$M_{n,v}(f;x)=n\sum_{k=r}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k-r}(t)\sum_{j=0}^{v}\frac{f^{(j)}(t)}{j!}(t-x)^j dt+f(0)\sum_{k=0}^{r-1} q_{n,k}(x). \quad (1)$$

The space $C_h[0,\infty)$ is normed by $\|f\|_{C_h}=\sup_{t\in[0,\infty)}|f(t)|\,e^{-ht}$, $h>0$.

Throughout this paper, we assume that C denotes a positive constant not necessarily the same at different occurrences, and $[h]$ denote the integer part of h.

2. Auxiliary Results:

Before we study the operator (1) we offer some results in the form of lemmas which we shall require to prove the main results of the paper.

Lemma 1:[1] For the equation $q_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$; $x \in R_0$ and $m \in N^0$, we have:

(1) $\int_0^\infty q_{n,k}(t)\, t^m\, dt = \frac{(k+m)!}{k!\ n^{m+1}}$; (2) $\sum_{k=0}^{\infty} q_{n,k}(x) = 1$; (3) $\sum_{k=0}^{\infty} k q_{n,k}(x) = nx$;

(4) $\sum_{k=0}^{\infty} k^2 q_{n,k}(x) = n^2x^2 + nx$; (5) suppose that $\Phi_{n,m}(x) = \sum_{k=0}^{\infty} k^m q_{n,k}(x)$, then

$\Phi_{n,m+1}(x) = x\Phi'_{n,m}(x) + nx\Phi_{n,m}(x)$, and

$$\Phi_{n,m}(x) = (nx)^m + \frac{m(m-1)}{2}(nx)^{m-1} + terms\ in\ lower\ powers\ of\ x;\ \ m \geq 1.$$

Lemma 2: [1] There exist polynomials $Q_{i,j,r}(x)$ independent of n and k for sufficiently large n, such that:

$$x^r q_{n,k}^{(r)}(x) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\,(k-nx)^j\, Q_{i,j,r}(x)\, q_{n,k}(x).$$

3. The Convergence Theorem of $M_{n,v}(f;x)$:

The next theorem shows that the operators $M_{n,v}(f(t);x) \to f(x)$ as $n \to \infty$.

Theorem 1: For $v \in N^0$, $n \in N = \{1,2,3,\dots\}$ and $x \in R_0$, the following conditions are hold

(i) $M_{n,v}(1;x) = 1$;

(ii) $M_{n,v}(t;x) = x + \frac{2}{n}(1-r) \to x$ as $n \to \infty$;

(iii) $M_{n,v}(t^2;x) = x^2 + \frac{4x}{n}(1-r) + \frac{4}{n^2}(1-r)(2-r) \to x^2$ as $n \to \infty$;

$$(iv)\quad M_{n,v}(t^m;x) = x^m + \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\frac{(m-j+i)^2}{n}x^{m-1} + O(n^{-2}).$$

Therefore, $M_{n,v}(f(t);x) \to f(x)$ as $n \to \infty$.

Proof: Using **Lemma 1** and direct computation, we have :

$$(i)\ M_{n,v}(1;x) = n\sum_{k=r}^{\infty} q_{n,k}(x)\int_0^\infty q_{n,k-r}(t)\,dt + \sum_{k=0}^{r-1} q_{n,k}(x) = 1.$$

$$(ii) M_{n,v}(t;x) = n\sum_{k=r}^{\infty} q_{n,k}(x)\int_0^\infty q_{n,k-r}(t)\sum_{j=0}^{v}\frac{(t-x)^j}{j!}D^j\ t\,dt + 0$$

$$M_{n,v}(t;x) = n\sum_{k=r}^{\infty} q_{n,k}(x)\int_{0}^{\infty} q_{n,k-r}(t)(t+(t-x)+0)+0$$

$$= \frac{2}{n}\sum_{k=r}^{\infty} q_{n,k}(x)(k-r+1) - x\left(\sum_{k=r}^{\infty} q_{n,k}(x)\right)$$

$$= x + \frac{2}{n}(1-r) \to x \text{ as } n \to \infty.$$

Using the same technique we get the value of $M_{n,v}(t^2;x)$is followed immediately as:

$$(iii)\ M_{n,v}(t^2;x) = x^2 + \frac{4x}{n}(1-r) + \frac{4}{n^2}(1-r)(2-r) \ \to x^2 \text{ as } n \to \infty.$$

$$(iv)\ M_{n,v}(t^m;x) = n\sum_{k=r}^{\infty} q_{n,k}(x)\int_{0}^{\infty} q_{n,k-r}(t)\sum_{j=0}^{v}\frac{(t-x)^j}{j!}\ D^j t^m\ dt + 0$$

$$= n\sum_{k=r}^{\infty} q_{n,k}(x)\int_{0}^{\infty} q_{n,k-r}(t)\sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}\ (-x)^{j-i}\ t^{m-j+i}\ dt$$

$$= \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}(-x)^{j-i}\ n\sum_{k=r}^{\infty} q_{n,k}(x)\int_{0}^{\infty} q_{n,k-r}(t)\ t^{m-j+i}\ dt$$

$$= \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\left(x^m + \frac{(m-j+i)^2}{n}x^{m-1} + O(n^{-2})\right)$$

$$= x^m + \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\ \frac{(m-j+i)^2}{n}x^{m-1} + O(n^{-2})$$

Therefore, $M_{n,v}(f(t);x) \to f(x)$ as $n \to \infty$.

4. The *m*-th Order Moment for $M_{n,v}(f;x)$:

In this part, we define the m-th order moment for the operators $M_{n,v}(f;x)$ which is denoted by $TM_{n,v}(x)$. Then we prove a recurrence relation for this moment.

Definition 1: For $m \in N^0$, the m-th order moment $T_{n,m}(x)$ for the operators $M_{n,v}(f;x)$ is defined as:

$$TM_{n,v}(x) = M_{n,v}((t-x)^m; x)$$

$$= n\sum_{k=r}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k-r}(t) \sum_{j=0}^{v} \frac{(t-x)^j}{j!} D^j (t-x)^m dt$$

$$+ (-x)^m \sum_{k=0}^{r-1} q_{n,k}(x). \qquad (2)$$

Lemma 3: The moment for the operators $M_{n,v}(f;x)$ has the following formula

$$TM_{n,v}(x) = \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \left\{ x^i + \frac{i^2}{n} x^{i-1} + T.L.P.(x) + o(1) \right\} + (-x)^m \sum_{k=0}^{r-1} q_{n,k}(x).$$

Proof: By using **Theorem 1**, we have :

$$T_{n,m}(x) = n\sum_{k=r}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k-r}(t) \sum_{i=0}^{m} \binom{m}{i} t^i (-x)^{m-i} dt + (-x)^m \sum_{k=0}^{r-1} q_{n,k}(x)$$

$$= \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \left\{ n\sum_{k=r}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k-r}(t) t^i \, dt \right\} + (-x)^m \sum_{k=0}^{r-1} q_{n,k}(x)$$

Now,

$$= \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \left\{ x^i + \frac{i^2}{n} x^{i-1} + T.L.P.(x) + o(1) \right\} + (-x)^m \sum_{k=0}^{r-1} q_{n,k}(x)$$

Theorem 2: Suppose that $r \in N, f \in C_h^{\nu}[0,\infty)$ for some $h > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} M_{n,v}^{(r)}(f(t);x) = f^{(r)}(x). \qquad (3)$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (3) holds uniformly on $[a,b]$.

Proof: By Taylor's expansion of f , we get

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^r,$$ where $\varepsilon(t,x) \to 0$ as $t \to x$. Hence,

$$M_{n,v}^{(r)}(f(t);x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} M_{n,v}^{(r)}((t-x)^i;x) + M_{n,v}^{(r)}(\varepsilon(t,x)(t-x)^r;x) := J_1 + J_2.$$

By using **Theorem 1,** if $i < r$ we have $M_{n,v}^{(r)}(t^i;x) = 0$. Hence,

$$J_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} M_{n,v}^{(r)}((t-x)^i;x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{r} \binom{r}{j} (-x)^{r-j} M_{n,v}^{(r)}(t^j;x)$$

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= f :N; :T; / NL :N; JÆ :P N T; L B:N; :T; =OJ \ »

t L @N / @TN LJ ″ (GLN, ») M JÆ :T; – M JÆFN :P; ″ (r, ») :PF T;F / FL (R, FLr) &F :PÆT; :PF T;N @P

E :rÆT; :F T;N ″ (NFs, GLr) M JÆ :T; q ªL u E v

Next, making use of **Lemma 2** we have:

u Q ″ (tEE FQN, EÆFRr) JEE s +3EÆFÆN :T; + / TN ″ (GLN, ») M JÆ :T; G F JT F – M JÆFN :P; ″ (r, ») :PF T;F / FL (R, FLr) &F :PÆT; :PF T;N @P

Since :PÆT; \ r =O P\ Then for given P r there exists P r such that :PÆT; O Æ whenever r O PF T O For PF T R there exists a constant % P r such that :PÆT; :PF T;NQ %ADP

Hence,

u Q L ″ (N, ELr) @N A E (tEE FQN, EÆFRr) +3EÆFÆN :T; + / TN M ″ (tEE FQN, EÆFRr) JEE s ″ (GLN, ») M JÆ :T; G F JT F L – (PF T O) M JÆFN :P; PF T N @P

E – (PF T R) M JÆFs :P; % ADP @P q

w E x

″ (N, ELr) @N A E (tEE FQN, EÆFRr) +3EÆFÆN :T; + / TN L / :T; Æ T – :rÆ» ;

Now, using Cauchy-Schwarz inequality for integration and then for summation, we are led to:

w Q % ″ (tEE FQN, EÆFRr) JE m JtF ″ (GLN, ») M JÆ :T; + G/J F Tp q tF – s/t

H LJ ″ (GLN, ») M JÆ :T; – (PF T O) M JÆFN :P; :PF T;tN @PM – s/t

We have:

$= n^{2j}\left(O\left(n^{-j}\right) + O(n^{-s})\right) = O\left(n^{j}\right)$, for any $s > 0$.

Since,

$$n\sum_{k=r}^{\infty} q_{n,pk}(x)\int_{|t-x|<\delta} q_{n,pk-r}(t)(t-x)^{2r}\,dt = O(n^{-r}).$$

Hence,

$$J_5 = C\varepsilon \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O\left(n^{\frac{j}{2}}\right) O\left(n^{-\frac{r}{2}}\right) = \varepsilon\, O\left(n^{\frac{2i+j}{2}-\frac{r}{2}}\right) = \varepsilon O(1).$$

Next, again using Cauchy-Schwarz inequality for integration and then for summation, we have:

$$J_6 \le C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left(\sum_{k=r}^{\infty} q_{n,k}(x)(k-nx)^{2j}\right)^{\frac{1}{2}} \left(n\sum_{k=r}^{\infty} q_{n,k}(x) \int\limits_{|t-x|\ge\delta} q_{n,k-r}(t)e^{2ht}\,dt\right)^{\frac{1}{2}}.$$

Using of Taylor's expansion, Cauchy-Schwarz inequality for integration and then for summation and **Lemma 3** , we have:

$$= O(n^{-s})\,;\ \ s \ge 0, \text{where } I \coloneqq n\sum_{k=r}^{\infty} q_{n,k}(x)\int_{|t-x|\ge\delta} q_{n,k-r}(t)e^{2ht}\,dt.$$

Therefore,

$$= C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O\left(n^{\frac{j}{2}}\right) O(n^{-s}) = o(1), \qquad \text{for } s > \frac{r}{2}.$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $J_3 \to 0$ as $n \to \infty$. Also $J_4 \to 0$ as $n \to \infty$ and hence $J_2 = o(1)$. Combining the estimates J_1 and J_2, (3) is immediate. ■

The next theorem is a Voronovskaja-type asymptotic formula for the operators $M_{n,v}^{(r)}(f(t);x),\ r \in N$

Theorem 3: Let $f \in C_h^{\gamma}[0,\infty)$ for some $h > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left\{M_{n,v}^{(r)}(f(t);x) - f^{(r)}(x)\right\} = (r+1)f^{(r+1)}(x) + xf^{(r+2)}(x).$$

(4). Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (4) holds uniformly on $[a,b]$.

Proof: By using Taylor's expansion of f, we have:

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2}, \text{where } \varepsilon(t,x) \to 0\ \ as\ \ t \to x.$$

Then,

$$M_{n,v}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} M_{n,v}^{(r)}\big((t-x)^i;x\big) + M_{n,v}^{(r)}(\varepsilon(t,x)(t-x)^{r+2};x).$$

From **Theorem 2,** we get:

$$M_{n,v}^{(r)}(\varepsilon(t;x)(t-x)^{r+2};x) \to 0 \; as \;\; n \to \infty.$$

We have:

$$M_{n,v}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} M_{n,v}^{(r)}((t-x)^i;x)$$

$$= \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \left\{ \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \, M_{n,v}^{(r)}(t^j;x) \right\}$$

$$= \frac{f^{(r)}(x)}{r!} M_{n,v}^{(r)}(t^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!} \left((r+1)(-x)M_{n,v}^{(r)}(t^r;x) + M_{n,v}^{(r)}(t^{r+1};x) \right) + \frac{f^{(r+2)}(x)}{(r+2)!}$$

$$\times \left(\tfrac{(r+2)(r+1)}{2} (-x)^2 M_{n,v}^{(r)}(t^r;x) + (r+2)(-x)M_{n,v}^{(r)}(t^{r+1};x) + M_{n,v}^{(r)}(t^{r+2};x) \right).$$

$$= M_{n,v}^{(r)}(t^r;x) \left\{ \frac{f^{(r)}(x)}{r!} + \frac{f^{(r+1)}(x)}{(r+1)!}(r+1)(-x) + \frac{f^{(r+2)}(x)}{(r+2)!} \frac{(r+2)(r+1)}{2} (-x)^2 \right\}$$

$$+ M_{n,v}^{(r)}(t^{r+1};x) \left\{ \frac{f^{(r+1)}(x)}{(r+1)!} + \frac{f^{(r+2)}(x)}{(r+2)!}(r+2)(-x) \right\} + M_{n,v}^{(r)}(t^{r+2};x) \left\{ \frac{f^{(r+2)}(x)}{(r+2)!} \right\}.$$

Since,

$$M_{n,v}(t^r;x) = x^r + \frac{r^2}{n} x^{r-1} + o(1) + T.L.P.(x).$$

$$M_{n,v}^{(r)}(t^r;x) = r!$$

$$M_{n,v}^{(r)}(t^{r+1};x) = x(r+1)! + \frac{(r+1)^2}{n} r!$$

$$M_{n,v}^{(r)}(t^{r+2};x) = x^2 \frac{(r+2)^2}{2} + \frac{(r+2)^2}{n} (r+1)! \, x$$

we obtain:

$$n\left\{ M_{n,v}^{(r)}(f(t);x) - f^{(r)}(x) \right\} =$$

$$= n \frac{f^{(r)}(x)}{r!} r! + n \frac{f^{(r+1)}(x)}{(r+1)!} \left((r+1)(-x)r! + x(r+1)! + \frac{(r+1)^2}{n} r! \right)$$

$$+n\frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2}(-x)^2r!\right)+(r+2)(-x)x(r+1)!+(r+2)(-x)\frac{(r+1)^2}{n}r!$$

$$+x^2\frac{(r+2)^2}{2}+\frac{(r+2)^2}{n}(r+1)!\,x\Big)-nf^{(r)}(x).$$

$$=(r+1)f^{(r+1)}(x)+xf^{(r+2)}(x).$$

Finally, we give an estimate of the degree of approximation by the operators $M_{n,v}^{(r)}(f(t);x)$

Theorem 4: Let $f\in C_h^{v}[0,\infty)$ for some $h>0$ and $r\le q\le r+2$. If $f^{(q)}$ exists and continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then for sufficiently large n,

$$\left\|M_{n,v}^{(r)}(f(t);x)-f^{(r)}(x)\right\|_{C[a,b]}\le C_1n^{-1}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}$$

$$+C_2n^{\frac{-1}{2}}\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right)+O(n^{-2}),$$

where C_1,C_2 are constants independent of f and $n,\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$, and $\|.\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof: By our hypothesis

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t,x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$.

For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we get:

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q.$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}}{i!}(t-x)^i.$$

Now,

$$M_{n,v}^{(r)}(f(t);x)-f^{(r)}(x)=\left(\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}M_{n,v}^{(r)}\big((t-x)^i;x\big)-f^{(r)}(x)\right)$$

$$+M_{n,v}^{(r)}\left(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t);x\right)+M_{n,v}^{(r)}\big(h(t,x)\big(1-\chi(t)\big);x\big):=\textstyle\sum_1+\sum_2+\sum_3.$$

By using **Theorem 1**, we get:

∑1 L ″ B:E; :T; / ÆL ″ 1 p :F T; FF E @N / @TN LTI (ÆLN, FLN)

E ″ @ I A ″ @ F A :F s; FF E :I F FE E; / J TI Fs E 1 :J Ft ;M F B:N; :T; (FLr, ÆLr)

Consequently,

! ^s !%≻Æ? Q %s J Fs e″ .B:E; .%≻Æ? iE 1 :J Ft ;Æ ≻Æ?

To estimate ^t we proceed as follows:

^t Q / :N; JÆ F +B:M; : ; F B:M; :T; + / MŁ PF TM :P; TG

Q æB:M; k := F Æ E ;o / MŁ / :N; JÆ mFs E PF T G PF TM Tq

Q æB:M; k := F Æ E ;o / MŁ NJ ″ ZM:N; JÆ :T;Z− MJÆFN :P; ″ :PF T;F / ÆL &F : PF TM (GLN, r, FLr)

E Fs PF TMEs ;@PE ″ MJÆ :T; :TM E Fs TMEs ;iÆ P r (NFs, GLr)

Now, for O L rÆÆÆ we have:

J ″ MJÆ :T; G F JT − MJÆFN :P; ″ :PF T;F / ÆL &F PF TO @P (GLN, r, FLr)

Q ″ @ O E Am ″ MJÆ :T; :G F JT; tF q LJ ″ MJÆ :T; − MJÆFN :P; :PF T; tO @ÆM (ÆLr, GLN, GLN, r)

L ″ @ O E A 1 1J F t p 1 @J FO t A L 1 1J :FF O; t p ≻Æ? :w; (ÆLr)

Therefore, by using **Lemma 2**, :w; we get:

″ @ O E A J ″ ZM:N; JÆ :T;Z− MJÆFN :P; :PF T; O @PQ L ″ @ O E A tÆE FQN T−≻Æ? ÆÆRr +3ÆÆÆN :T; + / TN M (ÆLr, GLN, r, ÆLr)

$$\times\left(\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\left(n\sum_{k=r}^{\infty} q_{n,k}(x)\ |k-nx|^j \int_0^{\infty} q_{n,k-r}(t)|t-x|^s\ dt\right)\right)$$

$$=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O\left(n^{\frac{(j-s)}{2}}\right)=O\left(n^{\frac{(r-s)}{2}}\right), \text{uniformly on } [a,b], \qquad (6)$$

since $\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup\limits_{x\in[a,b]} \dfrac{|Q_{i,j,r}(x)|}{x^r}=M(x)$, but fixed.

Choosing $\delta=n^{\frac{-1}{2}}$ and applying (6), we are led to:

$$\|\Sigma_2\|_{C[a,b]}\leq \frac{\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right)}{v!}\left[O\left(n^{\frac{(r-q)}{2}}\right)+n^{\frac{1}{2}}O\left(n^{\frac{(r-q-1)}{2}}\right)+O(n^{-m})\right],$$

for any $m>0$,

$$\leq C_2 n^{\frac{-(r-q)}{2}}\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right).$$

Since $t\in[0,\infty)\backslash(a-\eta,b+\eta)$, we choose $\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$.

Thus,

$$|\Sigma_3|\leq \sum_{i=0}^{s}\binom{s}{i} n\sum_{k=r}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i|k-nx|^j\ \frac{|Q_{i,j,r}(x)|}{x^r}q_{n,k}(x)\int_{|t-x|\geq\delta} q_{n,k-r}(t)|h(t,x)|\ dt$$

$$+\sum_{k=0}^{r-1} q_{n,k}(x)\ |h(0,x)|.$$

For $|t-x|\leq\delta$, we find a constant $C>0$ such that $|h(t,x)|\leq Ce^{ht}$.

Finally using Schwarz inequality for integration and then for summation , we get:

$$|\Sigma_3|=O(n^{-s}), s>0 \text{ uniformly on } [a,b].$$

Combining the estimates of $\sum_1$, $\sum_2$, $\sum_3$, the required result is immediate.

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