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Extending Application of Adomian Decomposition Method for Solving a Class of Volterra Integro-Differential Equations within Local Fractional Integral Operators

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Abstract:

In this paper, we consider the local fractional Adomian decomposition method for solving the second kind Volterra integro-differential equations within local fractional integral operators. This application maintains the efficiency and accuracy of the Adomian analytic method for solving local fractional integral equations. An illustrative examples are given to show the accuracy and reliability of the results.

Keywords: Adomain decomposition method; Local fractional operators; Local fractional Volterra integro-differential equations.

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توسيع تطبيق طريقة تحليل ادوميان لحل صنف من معادلات فولترا التكاملية التفاضلية ضمن المؤثرات التكاملية الكسرية المحلية

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الخلاصة:

في هذا البحث ندرس طريقة تحليل ادوميان الكسرية المحلية لحل معادلات فولترا التكاملية التفاضلية من النوع الثاني ذات المؤثر التكاملي الكسري المحلي. هذا التطبيق يؤكد كفاءة ودقة طريقة ادوميان التحليلية لحل المعادلات التكاملية الكسرية المحلية. قدمت امثلة توضيحية تبين دقة النتائج.

1. Introduction:

The theory of local fractional calculus is one of useful tools to process the fractal and continuously non differentiable functions [1-8]. It was successfully applied in local fractional Fokker Planck equation [1], the fractal heat conduction equation [2,8], fractal-time dynamical systems [4], fractal elasticity [5], local fractional diffusion equation [8], local fractional Laplace equation [7,9], local fractional integral equations [10,11,12], local fractional ordinary and partial differential equations [13,14,15], fractal wave equation [9,16].

In this work, we consider analysis solution to the local fractional Volterra integro-differential equation of the second kind. This paper is organized as follows: In section 2, the concept of local fractional calculus and integrals are given. In section 3, Adomian decomposition method is proposed based on local fractional integrals. An illustrative examples are shown in section 4. Conclusions are given in section 5.

2. Preliminary Definitions

Definition 1 [17]: The function $f(x)$ is local fractional continuous at $x = x_0$, if it is valid for

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \ ,0 < \alpha \le 1 \tag{2.1}$$

with $\left| x - x_0 \right| < \delta$, for $\varepsilon > 0$ and $\varepsilon \in R$. For $x \in (a,b)$; it is so called local fractional continuous on the interval (a,b), denoted by $f(x) \in C_{\alpha}(a,b)$.

Definition 2 [17]: The local fractional derivative of $f(x)$ at $x = x_0$ is defined as

$$D_x^{\alpha} f(x_0) = \frac{d^{\alpha}}{dx^{\alpha}} f(x)\Big|_{x=x_0} = f^{(\alpha)}(x) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}} \tag{2.2}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)\Delta(f(x) - f(x_0))$.

Local fractional derivative of high order is written in the form

Definition 3 [17]: A partition of the interval $[a,b]$ is denoted as (t_j, t_{j+1}), $j = 0,..., N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$. The local fractional integral of $f(x)$ in the interval $[a,b]$ is given by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha}. \tag{2.3}$$

Definition 4 [17]: In fractal space, the Mittage Leffler function, sine function, cosine function are, respectively defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\,,\ 0<\alpha\le 1 \tag{2.4}$$

$$\sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}\,,\ 0<\alpha\le 1 \tag{2.5}$$

$$\cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]}\,,\ 0<\alpha\le 1 \tag{2.6}$$

3. Analysis of the Method

The standard $k\alpha$ order local fractional Volterra integro-differential equation of the second kind is given by

$$u^{(k\alpha)}(x)=f(x)+\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^\alpha\,, \tag{3.1}$$

subject to initial conditions

$$u(0)=a_0\,,\ u^{(\alpha)}(0)=a_1\,,\ ,\dots,\ u^{((k-1)\alpha)}(0)=a_{k-1}\,,$$

where $u^{(k\alpha)}(x)=\dfrac{d^{k\alpha}u(x)}{dx^{k\alpha}}$ is linear local fractional derivative operator.

By integrating both sides of (3.1) leads to

$$L^{(-k\alpha)}\left(u^{(k\alpha)}(x)\right)=L^{(-k\alpha)}\left(f(x)\right)+L^{(-k\alpha)}\left(\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^\alpha\right) \tag{3.2}$$

Thus, we obtain

$$u(x)=a_0+a_1\frac{x^\alpha}{\Gamma(1+\alpha)}+\dots+a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}+L^{(-k\alpha)}\left(f(x)\right)+L^{(-k\alpha)}\left(\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^\alpha\right) \tag{3.3}$$

where the initial conditions $u(0), u^{(\alpha)}(0), u^{(2\alpha)}(0),\dots, u^{((k-1)\alpha)}(0)$ are used, and

$$L^{(-k\alpha)}(\cdot)={}_0I_x^{(k\alpha)}=\overbrace{{}_0I_x^{(\alpha)}\ {}_0I_x^{(\alpha)}\cdots{}_0I_x^{(\alpha)}}^{k\ time}(\cdot)\,. \tag{3.4}$$

Then, we use the decomposition series

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{3.5}$$

in both sides (3.3) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = a_0 + a_1\frac{x^{\alpha}}{\Gamma(1+\alpha)} + ... + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)} + L^{(-k\alpha)}(f(x)) + L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)\left(\sum_{n=0}^{\infty}u_n(t)\right)(dt)^{\alpha}\right) \quad (3.6)$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+.... = a_0 + a_1\frac{x^{\alpha}}{\Gamma(1+\alpha)} + ... + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)} + L^{(-k\alpha)}(f(x)) + L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_0(t)(dt)^{\alpha}\right) + L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_1(t)(dt)^{\alpha}\right) + \cdots \quad (3.7)$$

To determine the components $u_0(x), u_1(x), u_2(x), \cdots$ of the solution $u(x)$ we set the recurrence relations

$$u_0(x) = a_0 + a_1\frac{x^{\alpha}}{\Gamma(1+\alpha)} + ... + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)} + L^{(-k\alpha)}(f(x)) \quad (3.8)$$

$$u_{n+1}(x) = L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_n(t)(dt)^{\alpha}\right),\ n \geq 0. \quad (3.9)$$

It means that

$$u_0(x) = a_0 + a_1\frac{x^{\alpha}}{\Gamma(1+\alpha)} + ... + a_{k-1}\frac{x^{(k-1)\alpha}}{\Gamma(1+\alpha)} + L^{(-k\alpha)}(f(x)) \quad (3.10)$$

$$u_1(x) = L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_0(t)(dt)^{\alpha}\right). \quad (3.11)$$

$$u_2(x) = L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_1(t)(dt)^{\alpha}\right). \quad (3.12)$$

$$u_3(x) = L^{(-k\alpha)}\left(\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_2(t)(dt)^{\alpha}\right). \quad (3.13)$$

$\vdots$

4. An Illustrative Examples

In this section three examples for the local fractional Volterra integro-differential equation from the second kind is presented in order to demonstrate the simplicity and the efficiency of the above method.

Example 1. We consider the local fractional Volterra integro-differential equation

$$u^{(\alpha)}(x)=1-\frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)(dt)^\alpha \quad ,u(0)=0 \tag{4.1}$$

Let the solution in the series form

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{4.2}$$

Applying the integral operator $L^{(-\alpha)}$ defined by

$$L^{(-\alpha)}(\cdot)={}_0I_x^{(\alpha)}(\cdot)=\frac{1}{\Gamma(1+\alpha)}\int_0^x(\cdot)(dt)^\alpha \tag{4.3}$$

to both sides of (4.1), and using the given initial condition we obtain

$$u(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}-L^{(-\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)(dt)^\alpha\right) \tag{4.4}$$

Then substituting (4.2) in (4.4), we have that

$$\sum_{n=0}^{\infty}u_n(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}-L^{(-\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\sum_{n=0}^{\infty}u_n(t)\right)(dt)^\alpha\right) \tag{4.5}$$

From (3.8) and (3.9), we get

$$u_0(x)=\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.6}$$

$$u_1(x)=L^{(-\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x u_0(t)(dt)^\alpha\right)=-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \tag{4.7}$$

$$u_2(x)=L^{(-\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x u_1(t)(dt)^\alpha\right)=\frac{x^{5\alpha}}{\Gamma(1+5\alpha)} \tag{4.8}$$

$$u_3(x)=L^{(-\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x u_2(t)(dt)^\alpha\right)=-\frac{x^{7\alpha}}{\Gamma(1+7\alpha)} \tag{4.9}$$

and so on.

This gives the solution in a series form

$$u(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{x^{7\alpha}}{\Gamma(1+7\alpha)}\cdots$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]} \tag{4.10}$$

and hence the exact solution is given by

$$u(x)=\sin_{\alpha}(x^{\alpha}) \tag{4.11}$$

Example 2. We consider the local fractional Volterra integro-differential equation

$$u^{(2\alpha)}(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha} \quad ,u(0)=1,\, u^{(\alpha)}(0)=1 \tag{4.12}$$

Let the solution in the series form

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{4.13}$$

Applying the integral operator $L^{(-2\alpha)}$ to both sides of (4.12), and using the given initial condition we obtain

$$u(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+L^{(-2\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha}\right) \tag{4.14}$$

Then substituting (4.13) in (4.14), we have that

$$\sum_{n=0}^{\infty}u_n(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+L^{(-2\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}\left(\sum_{n=1}^{\infty}u_n(t)\right)(dt)^{\alpha}\right)$$

$$u_0(x)=a_0+a_1\frac{x^{\alpha}}{\Gamma(1+\alpha)}+L^{(-k\alpha)}\big(f(x)\big) \tag{4.15}$$

$$u_{n+1}(x)=L^{(-k\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u_n(t)(dt)^{\alpha}\right),\ n\geq 0. \tag{4.16}$$

Therefore,

$$u_0(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} \tag{4.17}$$

$$u_1(x)=L^{(-2\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u_0(t)(dt)^{\alpha}\right)$$

$$=\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\frac{x^{6\alpha}}{\Gamma(1+6\alpha)}+\frac{x^{7\alpha}}{\Gamma(1+7\alpha)} \tag{4.18}$$

and so on. This gives the solution in a series form

$$u(x)=1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \tag{4.19}$$

Hence, the exact solution

$$u(x) = E_\alpha(x^\alpha). \tag{4.20}$$

Example 3. We consider the local fractional Volterra integro-differential equation

$$\begin{aligned} &u^{(4\alpha)}(x) = -1 + \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha \\ &u(0) = -1,\ u^{(\alpha)}(0) = 1,\ u^{(2\alpha)}(0) = 1,\ u^{(3\alpha)}(0) = -1 \end{aligned} \tag{4.21}$$

Applying Equations (3.8) and (3.9), we arrive at the following iteration formula:

$$u_0(x) = -1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + L^{(-4\alpha)}\big(f(x)\big) \tag{4.22}$$

$$u_{n+1}(x) = -L^{(-k\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} u_n(t)(dt)^\alpha\right),\ n \ge 0. \tag{4.23}$$

$$u_0(x) = -1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)} \tag{4.24}$$

$$\begin{aligned} u_1(x) &= -L^{(-4\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} u_0(t)(dt)^\alpha\right) \\ &= \frac{x^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{x^{8\alpha}}{\Gamma(1+8\alpha)} \cdots\cdots \end{aligned} \tag{4.25}$$

and so on.

This gives the solution in a series form

$$\begin{aligned} u(x) &= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{x^{5\alpha}}{\Gamma(1+5\alpha)} \cdots\right) - \left(1 - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots\right) \\ &= \sin_\alpha(x^\alpha) - \cos_\alpha(x^\alpha). \end{aligned} \tag{4.26}$$

5. Conclusions

In this work, we discuss the methodology of the Adomian decomposition method to handling a class of local fractional Volterra integro-differential equation. Based on local fractional integral operator. We give an illustrative examples to elaborate the accuracy and reliable results

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