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Common fixed point theorems in fuzzy normed spaces and intuitionistic fuzzy normed spaces

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Abstract

In this paper, we prove new common fixed point theorems for somemappings by using the conceptsR-weakly commuting, compatibleand reciprocally continuous mappings in fuzzy normed space and intuitionistic fuzzy normed space.

Keywords: fuzzy normed space, intuitionistic fuzzy normed space, common fixed point theorem.

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Common fixed point theorems in fuzzy normed spaces and intuitionistic fuzzy normed spaces

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الخلاصة

في هذا البحث نحن برهنا نظريات نقطة صامدة مشتركة جديدة لتطبيقات بأستخدام مفاهيم التطبيقات المستمرة بالتبادل,المتوافقة والانتقال الضعيف- R في الفضاء المعياري الضبابي والفضاء المعياري الضبابي الحدسي.

1. Introduction

The concept of fuzzy was introduced by Zadeh [13]. Atanassov [1] defined the notion of intuitionistic fuzzy set. Katsaras [5] introduced the concept of fuzzy normed space.In [9] Park and Saadati introduced the concept of intuitionistic fuzzy normed space. M.S. Chauhan,V.H. Badshah and Deepti Sharma [3] proved common fixed point theorem for four self-mappings in fuzzy normed space using compatible mappings. In this paper, we prove new common fixed point theorem for mappings by using the concepts R-weakly commuting, compatibleand reciprocally continuous mappings in fuzzy normed space and intuitionistic fuzzy normed space.

2. Preliminaries

Definition (2.1)[13]:

Let X be a non-empty set and I be the closed interval $I = [0,1]$ of real numbers. A fuzzy set H in X (or a fuzzy subset from X) is a function from X into $I = [0\,,1]$. If H is a fuzzy set in X then H is described as characteristic function which connects every $x \in X$ to real number $H(x)$ in the interval I. $H(x)$is the grade of membership function to x in$H$. Hcan be described completely as : $H = \{(x\,, H(x)) : x \in X,\ 0 \leq$ $Hx \leq 1$ or $H = Hxx : x \in X$

where$H(x)$ is called the membership function for the fuzzy set H.

The family of all fuzzy sets in X is denoted by I^X.

Definition (2.2)[1]:

Let X be a non-empty set. An intuitionistic fuzzy set (In short, IFS) B is an object having the form:

$B = \{\langle x, G_B(x), H_B(x)\rangle, x \in X\}$, where the functions $G_B : X \longrightarrow I$ and $H_B : X \longrightarrow I$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A (respectively) and $0 \leq G_B(x) + H_B(x) \leq 1$ for all $x \in X$. The family of all intuitionistic fuzzy sets denoted by IF(X).

Furthermore, we call:

$\pi_B(x) = 1 - G_B(x) - H_B(x), x \in X$, the intuitionistic index or hesitancy degree of x in B. It is obvious that $0 \leq \pi_B(x) \leq 1$ for all $x \in X$.

Definition (2.3)[12]:

Let $*$ be a binary operation on the set $I=[0,1]$, i.e$*$: $[0,1]\times[0,1]\rightarrow[0,1]$ is a function, then $*$ is said to be t-norm (triangular-norm) on the set I if $*$ satisfies the following axioms :

(1) $*$ is commutative and associative ;

(2) $a*1=a$ for all $a\in[0,1]$;

(3)if $b,c\in I$ such that $b\leq c$, then $a*b\leq a*c$ for all $a\in I$;

In addition, if $*$ is continuous then $*$ is called a continuous t-norm.

Definition (2.4)[12]:

Let $\lozenge$ be a binary operation on the set $I=[0,1]$, then $\lozenge$ is said to be t-conorm (triangular-conorm) on the set I if $\lozenge$ satisfies the following axioms:

(1) $\lozenge$ iscommutative and associative ;

(2) $a\lozenge 0=a$ for all $a\in[0,1]$;

(3)if $b,c\in I$ such that $b\leq c, then\ a\lozenge b\leq a\lozenge c$ for all $a\in I$;

In addition, If $\lozenge$ is continuous then $\lozenge$is called a continuous t-conorm.

Definition (2.5) [5]:

The 3-tuple $(X,G,*)$ is said to be a fuzzy normed space (In short, FNS) if X is a linear space over the field F, $*$ is a continuous t-norm and G is a fuzzy set in $X\times(0,\infty)(i.e.\ G:X\times(0,\infty)\rightarrow[0,1])$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$;

$(\boldsymbol{FN.1})$ $G(x,t)>0$;

$(\boldsymbol{FN.2})$ $G(x,t)=1$ if and only if $x=0$;

$(\boldsymbol{FN.3})$ $G(\alpha x,t)=G\left(x,\frac{t}{|\alpha|}\right)$for all $\alpha\in F|\{0\}$;

$(\boldsymbol{FN.4})$ $G(x+y,t+s)\geq G(x,t)*G(y,s)$;

$(\boldsymbol{FN.5})$ $G(x,\bullet):(0,\infty)\rightarrow[0,1]$is continuous ;

$(\boldsymbol{FN.6})$ $\lim_{t\rightarrow\infty}G(x,t)=1$ and$\lim_{t\rightarrow 0}G(x,t)=0$;

Furthermore, assume that $(X,G,*)$ satisfying the following conditions:

$(\boldsymbol{FN.7})$ $\alpha*\alpha=\alpha$, $\forall\alpha\in[0,1]$;

$$(\boldsymbol{FN.8})\ G(x,t)>0,\forall t>0\Longrightarrow x=0 .$$

Lemma(2.6)[10,4]:

Let $(X,G,*)$ be a fuzzy normed space.Then:

(1)$G(x,\bullet)$is non-decreasing with respect to t for each $x\in X$.

(2)$G(-x,t)=G(x,t)$, hence$G(x-y,t)=G(y-x,t)$for all

$x, y \in X$.

Definition (2.7) [2]:

Let $(X, G, *)$ be a fuzzy normed space, then:

(**1**)A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for each $\epsilon \in (0,1)$and $t > 0$, there exists $n_0 \in Z^+$ such that

$G(x_n - x, t) > 1 - \epsilon$for all $n \geq n_0$.

(or equivalently,$\lim_{n\to\infty} G(x_n - x, t) = 1$)

(**2**)A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in Z^+$ such that $G(x_n - x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. (or equivalently,$\lim_{n,m\to\infty} G(x_n - x_m, t) = 1$) .

Definition (2.8) [2]:

A fuzzy normed space $(X, G, *)$is said to be complete if every Cauchy sequence is convergent sequence.

Definition(2.9) [8]:

Let $(X, G, *)$ be a fuzzy normed linear space and $S: X \to X$ be a mapping . $S$is said to be continuous ,if for every $x \in X$, $x_n \to x$ implies

$$Sx_n \to Sx.$$

Lemma (2.10)[8]:

In fuzzy normed space$(X, G, *)$,if for $x \in X$, $G(x, kt) \geq G(x, t)$,for every $t > 0$ and some $0 < k < 1$,then $x = 0$.

Lemma (2.11)[8]:

A sequence $\{x_n\}$ in fuzzy normed space $(X, G, *)$ satisfying $G(x_{n+1} - x_n, kt) \geq G(x_n - x_{n-1}, t)$, for every $t > 0$ and some $0 < k < 1$,is a Cauchy sequence.

Definition (2.12)[9]:

The 5-tuple $(X, G, H, *, \Diamond)$ is said to be an intuitionistic fuzzy normed space (In short, IFNS) if X is a linear space over field F,$*$ is a continuous t-norm , $\Diamond$ is a continuous t-conorm and G, H are fuzzy sets in $X \times (0, \infty)(i.e.\ G, H: X \times (0, \infty) \to [0,1])$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

$(\boldsymbol{IFN.1})$ $G(x, t) + H(x, t) \leq 1$;

$(\boldsymbol{IFN.2})$ $G(x, t) > 0$;

$(\boldsymbol{IFN.3})$ $G(x, t) = 1$ if and only if $x = 0$;

$(\boldsymbol{IFN.4})$ $G(\alpha x, t) = G\left(x, \frac{t}{|\alpha|}\right)$for each $F|\{0\}$;

$(\boldsymbol{IFN.5})$ $G(x+y,t+s) \geq G(x,t) * G(y,s)$;

$(\boldsymbol{IFN.6})$ $G(x,\bullet):\ (0,\infty) \to [0,1]$is continuous;

$(\boldsymbol{IFN.7})$ $\lim_{t\to\infty} G(x,t) = 1$ and$\lim_{t\to 0} G(x,t) = 0$;

$(\boldsymbol{IFN.8})$ $H(x,t) < 1$;

$(\boldsymbol{IFN.9})$ $H(x,t) = 0$ if and only if $x = 0$;

$(\boldsymbol{IFN.10})$ $H(\alpha x,t) = H\left(x,\frac{t}{|\alpha|}\right)$for each $F|\{0\}$;

$(\boldsymbol{IFN.11})$ $H(x+y,t+s) \leq H(x,t) \lozenge H(y,s)$

$(\boldsymbol{IFN.12})$ $H(x,\bullet):\ (0,\infty) \to [0,1]$is continuous .

$(\boldsymbol{IFN.13})$ $\lim_{t\to\infty} H(x,t) = 0$ and$\lim_{t\to 0} H(x,t) = 1$.

Furthermore, assume that $(X,G,H,*,\lozenge)$ satisfying the following conditions:

$(\boldsymbol{IFN.14})$ $\alpha * \alpha = \alpha$ and $\alpha \lozenge \alpha = \alpha$ $,\forall \alpha \in [0,1]$;

$(\boldsymbol{IFN.15})$ $G(x,t) > 0$ and $H(x,t) < 1, \forall\, t > 0 \implies x = 0$.

Lemma (2.13)[9]:

Let$(X,G,H,*,\lozenge)$ be an intuitionistic fuzzy normed space. Then for any t> 0:

$(\boldsymbol{i})$Every fuzzy normed space $(X,G,*)$ is an intuitionistic fuzzy normed space of the form $(X,G,1-G,*,\lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are dual.

$(\boldsymbol{ii})$$G(x,t)$and $H(x,t)$ are non-decreasing and non-increasing with respect to t, respectively.

$(\boldsymbol{iii})$ $G(x-y,t) = G(y-x,t)$and $H(x-y,t) = H(y-x,t)$for all $x,y \in X$.

Definition (2.14) [11]:

Let $(X,G,H,*,\lozenge)$ be an intuitionistic fuzzy normed space, then:

$(\boldsymbol{1})$ A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, if for each $\epsilon \in (0,1)$and $t > 0$, there exist $n_0 \in Z^+$ such that $G(x_n - x, t) > 1-\epsilon$ and $H(x_n - x, t<\epsilon$

for all $n \geq n_0$. (or equivalently$\lim_{n\to\infty} G(x_n - x,t) = 1$and $\lim_{n\to\infty} H(x_n - x,t) = 0$).

$(\boldsymbol{2})$ A sequence $\{x_n\}$ in X is said to be Cauchy , if for each $\epsilon \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that $G(x_n - x_m, t) > 1-\epsilon$ and $H(x_n - x_m, t) < \epsilon$ for all $n,m \geq n_0$. (or equivalently $\lim_{n,m\to\infty} G(x_n - x_m, t) = 1$ and
$\lim_{n,m\to\infty} H(x_n - x_m, t) = 0$).

Definition (1.15) [11]:

An intuitionistic fuzzy normed space $(X, G, H, *, \Diamond)$is said to be complete if every Cauchy sequence in $X$is convergent sequence.

Definition (2.16) [7]:

Let $(X, G, H, *, \Diamond)$ and $(Y, G, H, *, \Diamond)$ are two intuitionistic fuzzy normed space . A function $g : X \to Y$ is said to be continuous at a point $x_0 \in X$ if for any sequence $\{x_n\}$ in X converging to a point $x_0 \in X$, then the sequence $\{g(x_n)\}$ in Y convergence to a point $(x_0) \in Y$. If g is continuous at each point of X, then g is continuous on X.

Lemma (2.17)[6]:

Let $(X, G, H, *, \Diamond)$ be an intuitionistic fuzzy normed space and$\{z_n\}$ be a sequence in X such that

$G(z_{n+1} - z_n, kt) \geq G(z_n - z_{n-1}, t)$ $and H(z_{n+1} - z_n, kt) \leq H(z_n - z_{n-1}, t)$, $\forall\, t > 0$, $n \in N$ and $0 < k < 1$. Then the sequence $\{z_n\}$ is a Cauchy sequence in$(X, G, H, * ,\Diamond$.

Lemma (2.18):

Let $(X, G, H, *, \Diamond)$ be an intuitionistic fuzzy normed space, if there exists $k \in (0,1)$ such that

$G(z - w, kt) \geq G(z - w, t)$

And

$H(z - w, kt) \leq H(z - w, t)$

$\forall\, z, w \in X$, then$z = w$

Proof:

Since $G(z - w, kt) \geq G(z - w, t)$and $H(z - w, kt) \leq H(z - w, t)$

Then $G(z - w, t) \geq G(z - w, \frac{t}{k})$ and $H(z - w, t) \leq H(z - w, \frac{t}{k})$

$$G(z - w, \frac{t}{k}) = G(z - w, \frac{kt}{k^2}) \geq G(z - w, \frac{t}{k^2})$$

And

$$H(z - w, \frac{t}{k}) = H(z - w, \frac{kt}{k^2}) \leq H(z - w, \frac{t}{k^2})$$

By repeat the above operations, we get

$$G(z - w, t) \geq G(z - w, \frac{t}{k}) \geq G(z - w, \frac{t}{k^2}) \geq \cdots \geq G(z - w, \frac{t}{k^n}) \geq \cdots$$

And

$$H(z - w, t) \leq H(z - w, \frac{t}{k}) \leq H(z - w, \frac{t}{k^2}) \leq \cdots \leq H(z - w, \frac{t}{k^n}) \leq \cdots$$

Since $k \in (0,1)$ then for $n \in N$ which tends to 1 and 0 as $n \to \infty$, respectively

$\Longrightarrow G(z-w,t) = 1 \; and \; H(z-w,t) = 0 \, , \forall \, t > 0$

$\Longrightarrow z = w$. ■

3. Main Results

Definition (3.1):

Let F and T be self mappings of a fuzzy normed space $(Y, G,*)$, then a pair (F,T) is said to be compatible if $\lim_{n\to\infty} G(FTy_n - TFy_n \, ,t) = 1$ for all $t > 0$, whenever $\{y_n\}$ is a sequence in Y such that $\lim_{n\to\infty} Fy_n = \lim_{n\to\infty} Ty_n = v$ for some $v \in Y$.

Definition (3.2):

Let F and T be self mappings of a fuzzy normed space $(Y, G,*)$, then a pair (F,T) is said to be reciprocally continuous if $FTw_n \to Fw$ and $TFw_n \to Tw$ whenever $\{w_n\}$ is a sequence such that $Fw_n \to w$ and $Tw_n \to w$ for some $w \in Y$ as $n \to \infty$

Definition (3.3)[8]:

Let F and T be self mappings of a fuzzy normed space $(Y, G,*)$, then a pair (F,T) is said to R −weakly commuting if given $x \in Y$,there exist $R > 0$ such that $G(FTx - TFx, t) \geq G(Fx - Tx, \frac{t}{R})$.

Definition (3.4):

Let F and T be self mappings of a fuzzy normed space $(Y, G, H,*,\Diamond)$, then a pair (F,T) is said to be compatible if $\lim_{n\to\infty} G(FTy_n - TFy_n \, ,t) = 1$ and $\lim_{n\to\infty} H(FTy_n - TFy_n \, ,t) = 0$ for all $t > 0$, whenever $\{y_n\}$ is a sequence in Y such that $\lim_{n\to\infty} Fy_n = \lim_{n\to\infty} Ty_n = v$ for some $v \in Y$.

Definition (3.5):

Let F and T be self mappings of an intuitionistic fuzzy normed space $(Y, G, H,*$ $,\Diamond$.then a pair F,T is said to be reciprocally continuous if $FTwn \to Fw$ and $TFwn \to Tw$ whenever $\{w_n\}$ is a sequence such that $Fw_n \to w$ and $Tw_n \to w$ for some $w \in Y$ as $n \to \infty$.

Definition (3.6):

A pair (F,S) of self mappings of intuitionistic fuzzy normed space $(Y, G, H,*,\Diamond)$ is said to be R −weakly commuting if there exists a positive real number R such that $G(FSz - SFz, t) \geq G\left(Fz - Sz, \frac{t}{R}\right)$ and $H(FSz - SFz, t) \leq H\left(Fz - Sz, \frac{t}{R}\right)$ for every $z \in Y$ and $t > 0$.

Proposition (3.7):

Let $(Y, G,*)$ be a complete fuzzy normed space with $(1+q)*k = k+q*k$ for all $q, k \in [0,1]$ and let F, D, P and Q be four self mappings of Y satisfying

(i) $F(Y) \subset Q(Y)$ and $D(Y) \subset P(Y)$

(ii) there exists a constant $r \in (0,1)$ such that

$$\big(1+bG(Pz-Qw,rt)\big)*G(Fz-Dw,rt)$$
$$\geq b\big(G(Fz-Pz,rt)*G(Dw-Qw,rt)*G(Dw-Pz,rt)\big)$$
$$+G(Qw-Pz,t)*G(Fz-Pz,t)*G(Dw-Qw,t)*G(Dw-Pz,\beta t)$$
$$*G(Fz-Qw,(2-\beta)t)$$

For all $, w \in Y$ $,0 \leq b \leq 1$ and $\beta \in (0,2)$ and $t > 0$. If the pairs (F,P) and (D,Q) are R −weakly commuting ,then one continuity of the mappings in compatible pair (F,P) or (D,Q) implies their reciprocal continuity.

Proof

Let F and P be compatible and P be continuous we will show that F and P are reciprocally continuous.let $\{x_n\}$ be a sequence such that $Fx_n \to x$ and $Px_n \to x$ for some $x \in Y$ as $n \to \infty$.Since P is continuous

$\Longrightarrow PFx_n \to Px$ and $PPx_n \to Px$ as $n \to \infty$

And since (F,P) is compatible, we have

$\lim_{n\to\infty} G(FPx_n - PFx_n, t) = 1$ $,\forall\, t > 0$

$\Longrightarrow \lim_{n\to\infty} G(FPx_n - Px, t) = 1$ $,\forall\, t > 0$

$\Longrightarrow$ $FPx_n \to Px$ as $n \to \infty$

Since $F(Y) \subset Q(Y)$

For each n, there exists $\{y_n\}$ such that $F(Px_n) = Q(y_n)$ i.e. $FPx_n = Qy_n$, thus we have

$PPx_n \to Px$ as $n \to \infty$, $PFx_n \to Px$ as $n \to \infty$, $FPx_n \to Px$ as $n \to \infty$ and $Qy_n \to Px$ as $n \to \infty$.

From (ii) and choose $\beta = 1$, we have

$$\big(1+bG(PPx_n-Qy_n,rt)\big)*G(FPx_n-Dy_n,rt)$$
$$\geq b\big(G(FPx_n-PPx_n,rt)*G(Dy_n-Qy_n,rt)*G(Dy_n-PPx_n,rt)\big)$$
$$+G(Qy_n-PPx_n,t)*G(FPx_n-PPx_n,t)*G(Dy_n-Qy_n,t)$$
$$*G(Dy_n-PPx_n,t)*G(FPx_n-Qy_n,t)$$

Taking $n \to \infty$

$$\Longrightarrow \left(1+bG(Px-Px,rt)\right)*G(Px-Dy_n,rt)$$
$$\geq b\left(G(Px-Px,rt)*G(Dy_n-Px,rt)*G(Dy_n-Px,rt)\right)$$
$$+G(Px-Px,t)*G(Px-Px,t)*G(Dy_n-Px,t)*G(Dy_n-Px,t)$$
$$*G(Px-Px,t)$$

$$\Longrightarrow G(Px-Dy_n,rt)\geq G(Dy_n-Px,t)$$

By lemma (2.10), we have $Dy_n \to Px$ as $n \to \infty$

Again by (ii) and we choose $\beta = 1$, we get

$$\left(1+bG(Px-Qy_n,rt)\right)*G(Fx-Dy_n,rt)$$
$$\geq b\left(G(Fx-Px,rt)*G(Dy_n-Qy_n,rt)*G(Dy_n-Px,rt)\right)$$
$$*G(Qy_n-Px,t)*G(Fx-Px,t)*G(Dy_n-Qy_n,t)$$
$$*G(Dy_n-Px,t)*G(Fx-Qy_n,t)$$

Taking $n \to \infty$

$$\Longrightarrow \left(1+bG(Px-Px,rt)\right)*G(Fx-Px,rt)$$
$$\geq b\left(G(Fx-Px,rt)*G(Px-Px,rt)*G(Px-Px,rt)\right)$$
$$+G(Px-Px,t)*G(Fx-Px,t)*G(Px-Px,t)*G(Px-Px,t)$$
$$*G(Fx-Qy_n,t)$$

$$\Longrightarrow G(Fx-Px,rt)\geq G(Fx-Px,t)$$

By lemma (2.10)

$$\Longrightarrow Fx=Px$$

$\Longrightarrow PFx_n \to PxandFPx_n \to Px = Fx$ as $n \to \infty$

$\Longrightarrow F$ and$P$ are reciprocally continuous on Y.

Similarly, if the pair (D,Q) is compatible and Q is continuous then D and Q are reciprocally continuous on Y. ■

Theorem (3.8):

Let $(Y,G,*)$ be a complete fuzzy normed space with $(1+q)*k=k+q*k$for all $q,k\in[0,1]$and let F,D,P and Q be four self mappings of Y satisfying

(i) $F(Y)\subset Q(Y)$and$D(Y)\subset P(Y)$

(ii)there exists a constant $r\in(0,1)$ such that

$$\left(1+bG(Pz-Qw,rt)\right)*G(Fz-Dw,rt)$$
$$\geq b\left(G(Fz-Pz,rt)*G(Dw-Qw,rt)*G(Dw-Pz,rt)\right)$$
$$+G(Qw-Pz,t)*G(Fz-Pz,t)*G(Dw-Qw,t)*G(Dw-Pz,\beta t)$$
$$*G(Fz-Qw,(2-\beta)t)$$

For all $,w \in Y$,$0 \le b \le 1$ and $\beta \in (0,2)$ and $t > 0$. If the pairs (F,P) and (D,Q) are R −weakly commuting and one of the mappings in compatible pair (F,P) or (D,Q) is continuous,then $F, D, P and\ Q$ have a unique common fixed point.

Proof

Since $F(Y) \subset Q(Y)$, then for any point $z_0 \in Y$ there exists a point $z_1 \in Y$ such that $Fz_0 = Qz_1$ .And since $D(Y) \subset P(Y)$,for this point $z_1 \in Y$ there exists $z_2 \in Y$ such that $Dz_1 = Pz_2$.

We can define a sequence $\{w_n\}$ in Y such that for $n = 0,1,2, \dots w_{2n} = Fz_{2n} = Qz_{2n+1}$

And

$w_{2n+1} = Dz_{2n+1} = Pz_{2n+2}$

From(ii), for all $t > 0$ and $\beta = 1 - h$ with $h \in (0,1]$ and let $z = z_{2n+2}$,$w = z_{2n+1}$ in(ii), then we get

$$\begin{aligned}
&\big(1 + bG(Pz_{2n+2} - Qz_{2n+1}, rt)\big) * G(Fz_{2n+2} - Dz_{2n+1}, rt) \\
&\qquad \ge b\big(G(Fz_{2n+2} - Pz_{2n+2}, rt) * G(Dz_{2n+1} - Qz_{2n+1}, rt) \\
&\qquad * G(Dz_{2n+1} - Pz_{2n+2}, rt)\big) + G(Qz_{2n+1} - Pz_{2n+2}, t) \\
&\qquad * G(Fz_{2n+2} - Pz_{2n+2}, t) * G(Dz_{2n+1} - Qz_{2n+1}, t) \\
&\qquad * G(Dz_{2n+1} - Pz_{2n+2}, (1-h)t) * G(Fz_{2n+2} - Qz_{2n+1}, (1+h)t)
\end{aligned}$$

Since $w_{2n} = Fz_{2n} = Qz_{2n+1}$and $w_{2n+1} = Dz_{2n+1} = Pz_{2n+2}$, we get

$$\begin{aligned}
&\big(1 + bG(w_{2n} - w_{2n+1}, rt)\big) * G(w_{2n+1} - w_{2n+2}, rt) \\
&\qquad \ge b\big(G(w_{2n+2} - w_{2n+1}, rt) * G(w_{2n+1} - w_{2n}, rt) \\
&\qquad * G(w_{2n+1} - w_{2n+1}, rt)\big) + G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) \\
&\qquad * G(w_{2n+1} - w_{2n}, t) * G(w_{2n+1} - w_{2n+1}, (1-h)t) \\
&\qquad * G(w_{2n+2} - w_{2n}, (1+h)t) \\
&\qquad \ge b\big(G(w_{2n+2} - w_{2n+1}, rt) * G(w_{2n+1} - w_{2n}, rt)\big) \\
&\qquad + G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) * G(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow G(w_{2n+1} - w_{2n+2}, rt) \\
&\qquad \ge G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) * G(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

Put $h = 1$

$$\Longrightarrow G(w_{2n+1} - w_{2n+2}, rt) \ge G(w_{2n} - w_{2n+1}, t) * G(w_{2n+1} - w_{2n+2}, t)$$

In the similar way, we also get

$$\Longrightarrow G(w_{2n+2} - w_{2n+3}, rt) \ge G(w_{2n+1} - w_{2n+2}, t) * G(w_{2n+2} - w_{2n+3}, t)$$

In general, for $m = 1,2,3, \dots$

$G(w_{m+1}-w_{m+2},rt) \geq G(w_m - w_{m+1},t) * G(w_{m+1}-w_{m+2},t)$

It follows that for $m = 1,2,3,\ldots$ and $p = 1,2,3,\ldots$

$$G(w_{m+1}-w_{m+2},rt) \geq G(w_m - w_{m+1},t) * G\left(w_{m+1}-w_{m+2},\frac{t}{r^p}\right)$$

As $p \to \infty$

$\Longrightarrow G(w_{m+1}-w_{m+2},rt) \geq G(w_m - w_{m+1},t)$

Then from lemma (2.11) we get $\{w_m\}$ is a Cauchy sequence in $(Y,G,*)$.

Since $(Y,G,*)$ is a complete space

$\Longrightarrow \exists x \in Y$ such that $w_m \to x$ as $n \to \infty$

$\Longrightarrow Fz_{2n} \to x as n \to \infty,$

$Qz_{2n+1} \to x as n \to \infty,$

$Dz_{2n+1} \to x as n \to \infty$ and $Pz_{2n+2} \to x$ as $n \to \infty$.

Now, assume that (F,P) is a compatible pair and P is continuous. then by Proposition(3.7) we get F and P are reciprocally continuous thus $PFz_n \to Px$ and $FPz_n \to Fx$ as $n \to \infty$.

Since (F,P) is a compatible pair

$\Longrightarrow \lim_{n\to\infty} G(FP\,z_n - PFz_n,t) = 1$

$\Longrightarrow G(Fx - Px,t) = 1$

$\Longrightarrow Fx = Px$

Since $F(Y) \subset Q(Y)$, there exists a point $y \in Y$ such that $Fx = Qy$

$\Longrightarrow Fx = Qy = Px$

From (ii) and take $\beta = 1$, we get

$$\begin{aligned}\big(1 + bG(Px - Qy,rt)\big) * G(Fx - Dy,rt) \\ &\geq b\big(G(Fx - Px,rt) * G(Dy - Qy,rt) * G(Dy - Px,rt)\big) \\ &+ G(Qy - Px,t) * G(Fx - Px,t) * G(Dy - Qy,t) * G(Dy - Px,t) \\ &* G(Fx - Qy,t)\end{aligned}$$

$\Longrightarrow G(Fx - Dy,rt) \geq G(Fx - Dy,t)\,, \forall t > 0$

Then by lemma (2.10)

$\Longrightarrow Fx = Dy$

$\Longrightarrow Fx = Dy = Px = Qy$

Since F and P are R −weakly commuting mappings, then there exists $R > 0$ such that

$G(FPx - PFx,t) \geq G\left(Fx - Px,\frac{t}{R}\right)$

Since $Fx = Px$

$\Longrightarrow G\left(Fx - Px, \frac{t}{R}\right) = 1$

$\Longrightarrow G(FPx - PFx, t) \geq 1$

$\Longrightarrow G(FPx - PFx, t) = 1$

$\Longrightarrow FPx = PFx$

And since $Fx = Px$

$\Longrightarrow FFx = FPx$ and $PFx = PPx$

$\therefore FFx = FPx = PFx = PPx$

Similarly, since D and Q are $R-$weakly commuting mappings, then there exists $R > 0$ such that

$G(DQy - QDy, t) \geq G\left(Dy - Qy, \frac{t}{R}\right)$

Since $Dy = Qy$

$\Longrightarrow G\left(Dy - Qy, \frac{t}{R}\right) = 1$

$\Longrightarrow G(DQy - QDy, t) \geq 1$

$\Longrightarrow G(DQy - QDy, t) = 1$

$\Longrightarrow DQy = QDy$

And since $Dy = Qy$

$\Longrightarrow QDy = QQy$ and $DDy = DQy$

$\Longrightarrow DDy = QDy = DQy = QQy$

Again from (ii) and choose $\beta = 1$, we have

$$\begin{aligned}\big(1 + bG(PFx - Qy, rt)\big) * G(FFx - Dy, rt) \\ \geq b\big(G(FFx - PFx, rt) * G(Dy - Qy, rt) * G(Dy - PFx, rt)\big) \\ + G(Qy - PFx, t) * G(FFx - PFx, t) * G(Dy - Qy, t) \\ * G(Dy - PFx, t) * G(FFx - Qy, t)\end{aligned}$$

$\Longrightarrow G(FFx - Fx, rt) \geq G(FFx - Fx, t)$

Then by lemma (2.10), we get

$FFx = Fx$

$\Longrightarrow Fx = PFx$

$\therefore Fx$ is a common fixed point of F and P.

Again by (ii) and choose $\beta = 1$,we get

$$\big(1+bG(Px-QDy,rt)\big)*G(Fx-DDy,rt)$$
$$\geq b\big(G(Fx-Px,rt)*G(DDy-QDy,rt)*G(DDy-Px,rt)\big)$$
$$+G(QDy-Px,t)*G(Fx-Px,t)*G(DDy-QDy,t)$$
$$*G(DDy-Px,t)*G(Fx-QDy,t)$$

$\Longrightarrow G(Dy-QDy,rt)\geq G(Dy-QDy,t)$

Then by lemma (2.10), we get

$QDy=Dy=DDy$

Then Dy is a common fixed point of D and Q.

Since $Dy=Fx$

$\Longrightarrow Fx$is a common fixed point of F,D,P and Q.

For the uniqueness,

Suppose that Fv is a common fixed point of $F,D,P$and Q and $Fv\neq Fx$.

From (ii) and choose $\beta=1$, we get

$$\big(1+bG(PFx-QFv,rt)\big)*G(FFx-DFv,rt)$$
$$\geq b\big(G(FFx-PFx,rt)*G(DFv-QFv,rt)*G(DFv-PFx,rt)\big)$$
$$+G(QFv-PFx,t)*G(FFx-PFx,t)*G(DFv-QFv,t)$$
$$*G(DFv-PFx,t)*G(FFx-QFv,t)$$

$\Longrightarrow G(Fx-Fv,rt)\geq G(Fx-Fv,t)$

Then by lemma (2.10) we get

$Fx=Fv$

Therefore Fx is a unique common fixed point of $F,D,P$and Q. ■

<u>Proposition (3.9):</u>

Let $(Y,G,H,*,\lozenge)$ be a complete intuitionistic fuzzy normed space with $(1+q)*k=k+q*k$ and $(1+q)\lozenge k=k+q\lozenge k$ for all $q,k\in[0,1]$and let F,D,P and Q be four self mappings of Y satisfying

(i) $F(Y)\subset Q(Y)$and$D(Y)\subset P(Y)$

(ii)there a constant $r\in(0,1)$ such that

$$\big(1+bG(Pz-Qw,rt)\big)*G(Fz-Dw,rt)$$
$$\geq b\big(G(Fz-Pz,rt)*G(Dw-Qw,rt)*G(Dw-Pz,rt)\big)$$
$$+G(Qw-Pz,t)*G(Fz-Pz,t)*G(Dw-Qw,t)*G(Dw-Pz,\beta t)$$
$$*G(Fz-Qw,(2-\beta)t)$$

And

$$\left(1 + bH(Pz - Qw, rt)\right) \lozenge H(Fz - Dw, rt)$$
$$\leq b\left(H(Fz - Pz, rt) \lozenge H(Dw - Qw, rt) \lozenge H(Dw - Pz, rt)\right)$$
$$+ H(Qw - Pz, t) \lozenge H(Fz - Pz, t) \lozenge H(Dw - Qw, t)$$
$$\lozenge H(Dw - Pz, \beta t) \lozenge H(Fz - Qw, (2 - \beta)t)$$

For all , $w \in Y$,$0 \leq b \leq 1$ and $\beta \in (0,2)$ and $t > 0$. If the pairs (F, P) and (D, Q) are R −weakly commuting ,then one continuity of the mappings in compatible pair (F, P) or (D, Q) implies their reciprocal continuity.

Proof

Let F and P are compatible and P is continuous we will show that F and P are reciprocally continuous .let $\{x_n\}$ be a sequence such that $Fx_n \to x$ and $Px_n \to x$ for some $x \in Y$ as $n \to \infty$.Since P is continuous

$\Longrightarrow PFx_n \to Px$ and $PPx_n \to Px$ as $n \to \infty$

And since (F, P) is compatible, we have

$$\lim_{n\to\infty} G(FPx_n - PFx_n, t) = 1$$

And

$\lim_{n\to\infty} H(FPx_n - PFx_n, t) = 0$,$\forall\, t > 0$

$$\Longrightarrow \lim_{n\to\infty} G(FPx_n - Px, t) = 1$$

And

$\lim_{n\to\infty} H(FPx_n - Px, t) = 0$,$\forall\, t > 0$

$\Longrightarrow\ FPx_n \to Px$ as $n \to \infty$

Since $F(Y) \subset Q(Y)$

For each n, there exists $\{y_n\}$ such that $F(Px_n) = Q(y_n)$ i.e. $FPx_n = Qy_n$, thus we have

$PPx_n \to Px$ as $n \to \infty$, $PFx_n \to Px$ as $n \to \infty$, $FPx_n \to Px$ as $n \to \infty$ and $Qy_n \to Px$ as $n \to \infty$.

From (ii) and choose $\beta = 1$, we have

$$\left(1 + bG(PPx_n - Qy_n, rt)\right) * G(FPx_n - Dy_n, rt)$$
$$\geq b\left(G(FPx_n - PPx_n, rt) * G(Dy_n - Qy_n, rt) * G(Dy_n - PPx_n, rt)\right)$$
$$+ G(Qy_n - PPx_n, t) * G(FPx_n - PPx_n, t) * G(Dy_n - Qy_n, t)$$
$$* G(Dy_n - PPx_n, t) * G(FPx_n - Qy_n, t)$$

And

$$\big(1 + bH(PPx_n - Qy_n, rt)\big) \lozenge H(FPx_n - Dy_n, rt)$$
$$\leq b\big(H(FPx_n - PPx_n, rt) \lozenge H(Dy_n - Qy_n, rt) \lozenge H(Dy_n - PPx_n, rt)\big)$$
$$+ H(Qy_n - PPx_n, t) \lozenge H(FPx_n - PPx_n, t) \lozenge H(Dy_n - Qy_n, t)$$
$$\lozenge H(Dy_n - PPx_n, t) \lozenge H(FPx_n - Qy_n, t)$$

Taking $n \to \infty$

$$\Longrightarrow \big(1 + bG(Px - Px, rt)\big) * G(Px - Dy_n, rt)$$
$$\geq b\big(G(Px - Px, rt) * G(Dy_n - Px, rt) * G(Dy_n - Px, rt)\big)$$
$$+ G(Px - Px, t) * G(Px - Px, t) * G(Dy_n - Px, t) * G(Dy_n - Px, t)$$
$$* G(Px - Px, t)$$

And

$$\big(1 + bH(Px - Px, rt)\big) \lozenge H(Px - Dy_n, rt)$$
$$\leq b\big(H(Px - Px, rt) \lozenge H(Dy_n - Px, rt) \lozenge H(Dy_n - Px, rt)\big)$$
$$+ H(Px - Px, t) \lozenge H(Px - Px, t) \lozenge H(Dy_n - Px, t)$$
$$\lozenge H(Dy_n - Px, t) \lozenge H(Px - Px, t)$$

$$\Longrightarrow G(Px - Dy_n, rt) \geq G(Dy_n - Px, t)$$

And

$$H(Px - Dy_n, rt) \leq H(Dy_n - Px, t)$$

By lemma (2.18) ,we have $Dy_n \to Px$ as $n \to \infty$

Again by (ii) and we choose $\beta = 1$, we get

$$\big(1 + bG(Px - Qy_n, rt)\big) * G(Fx - Dy_n, rt)$$
$$\geq b\big(G(Fx - Px, rt) * G(Dy_n - Qy_n, rt) * G(Dy_n - Px, rt)\big)$$
$$+ G(Qy_n - Px, t) * G(Fx - Px, t) * G(Dy_n - Qy_n, t)$$
$$* G(Dy_n - Px, t) * G(Fx - Qy_n, t)$$

And

$$\big(1 + bH(Px - Qy_n, rt)\big) \lozenge H(Fx - Dy_n, rt)$$
$$\leq b\big(H(Fx - Px, rt) \lozenge H(Dy_n - Qy_n, rt) \lozenge H(Dy_n - Px, rt)\big)$$
$$+ H(Qy_n - Px, t) \lozenge H(Fx - Px, t) \lozenge H(Dy_n - Qy_n, t)$$
$$\lozenge H(Dy_n - Px, t) \lozenge H(Fx - Qy_n, t)$$

Taking $n \to \infty$

$$\Longrightarrow \big(1 + bG(Px - Px, rt)\big) * G(Fx - Px, rt)$$
$$\geq b\big(G(Fx - Px, rt) * G(Px - Px, rt) * G(Px - Px, rt)\big)$$
$$+ G(Px - Px, t) * G(Fx - Px, t) * G(Px - Px, t) * G(Px - Px, t)$$
$$* G(Fx - Qy_n, t)$$

And

$$\big(1 + bH(Px - Px, rt)\big) \lozenge H(Fx - Px, rt)$$
$$\leq b\big(H(Fx - Px, rt) \lozenge H(Px - Px, rt) \lozenge H(Px - Px, rt)\big)$$
$$+ H(Px - Px, t) \lozenge H(Fx - Px, t) \lozenge H(Px - Px, t) \lozenge H(Px - Px, t)$$
$$\lozenge H(Fx - Qy_n, t)$$

$$\Longrightarrow G(Fx - Px, rt) \geq G(Fx - Px, t)$$

And

$$H(Fx - Px, rt) \leq H(Fx - Px, t)$$

By lemma (2.18)

$$\Longrightarrow Fx = Px$$

$\Longrightarrow PFx_n \to Px$ and $FPx_n \to Px = Fx$ as $n \to \infty$

$\Longrightarrow F$ and P are reciprocally continuous on Y.

Similarly, if the pair (D, Q) is compatible and Q is continuous then D and Q are reciprocally continuous on Y. ■

Theorem (3.10):

Let $(Y, G, H, *, \lozenge)$ be a complete intuitionistic fuzzy normed space with $(1 + q) * k = k + q * k$ and $(1 + q) \lozenge k = k + q \lozenge k$ for all $q, k \in [0,1]$ and let F, D, P and Q be four self mappings of Y satisfying

(i) $F(Y) \subset Q(Y)$ and $D(Y) \subset P(Y)$

(ii) there a constant $r \in (0,1)$ such that

$$\big(1 + bG(Pz - Qw, rt)\big) * G(Fz - Dw, rt)$$
$$\geq b\big(G(Fz - Pz, rt) * G(Dw - Qw, rt) * G(Dw - Pz, rt)\big)$$
$$+ G(Qw - Pz, t) * G(Fz - Pz, t) * G(Dw - Qw, t) * G(Dw - Pz, \beta t)$$
$$* G(Fz - Qw, (2 - \beta)t)$$

And

$$\big(1 + bH(Pz - Qw, rt)\big) \lozenge H(Fz - Dw, rt) \leq b\big(H(Fz - Pz, rt) \lozenge H(Dw - Qw, rt) \lozenge H(Dw - Pz, rt)\big) + H(Qw - Pz, t) \lozenge H(Fz - Pz, t) \lozenge H(Dw - Qw, t) \lozenge H(Dw - Pz, \beta t) \lozenge H(Fz - Qw, (2 - \beta)t)$$

For all $z, w \in Y$,$0 \leq b \leq 1$ and $\beta \in (0,2)$ and $t > 0$. If the pairs (F, P) and (D, Q) are point wise R −weakly commuting and one of the mappings in compatible pair (F, P) or (D, Q) is continuous ,then $F, D, P and\ Q$ have a unique common fixed point.

Proof

Since $F(Y) \subset Q(Y)$, then for any point $z_0 \in Y$ there exists a point $z_1 \in Y$ such that $Fz_0 = Qz_1$ .And since $D(Y) \subset P(Y)$,for this point $z_1 \in Y$ there exists $z_2 \in Y$ such that $Dz_1 = Pz_2$.

We can define a sequence $\{w_n\}$ in Y such that for $n = 0,1,2, \dots w_{2n} = Fz_{2n} = Qz_{2n+1}$

And

$w_{2n+1} = Dz_{2n+1} = Pz_{2n+2}$

From(ii), for all $t > 0$ and $\beta = 1 - h$ with $h \in (0,1]$ and let $z = z_{2n+2}$,$w = z_{2n+1}$ in(ii), then we get

$$\big(1 + bG(Pz_{2n+2} - Qz_{2n+1}, rt)\big) * G(Fz_{2n+2} - Dz_{2n+1}, rt) \geq b\big(G(Fz_{2n+2} - Pz_{2n+2}, rt) * G(Dz_{2n+1} - Qz_{2n+1}, rt) * G(Dz_{2n+1} - Pz_{2n+2}, rt)\big) + G(Qz_{2n+1} - Pz_{2n+2}, t) * G(Fz_{2n+2} - Pz_{2n+2}, t) * G(Dz_{2n+1} - Qz_{2n+1}, t) * G(Dz_{2n+1} - Pz_{2n+2}, (1 - h)t) * G(Fz_{2n+2} - Qz_{2n+1}, (1 + h)t)$$

And

$$\big(1 + bH(Pz_{2n+2} - Qz_{2n+1}, rt)\big) \lozenge H(Fz_{2n+2} - Dz_{2n+1}, rt) \leq b\big(H(Fz_{2n+2} - Pz_{2n+2}, rt) \lozenge H(Dz_{2n+1} - Qz_{2n+1}, rt) \lozenge H(Dz_{2n+1} - Pz_{2n+2}, rt)\big) + H(Qz_{2n+1} - Pz_{2n+2}, t) \lozenge H(Fz_{2n+2} - Pz_{2n+2}, t) \lozenge H(Dz_{2n+1} - Qz_{2n+1}, t) \lozenge H(Dz_{2n+1} - Pz_{2n+2}, (1 - h)t) \lozenge H(Fz_{2n+2} - Qz_{2n+1}, (1 + h)t)$$

Since $w_{2n} = Fz_{2n} = Qz_{2n+1}$ and $w_{2n+1} = Dz_{2n+1} = Pz_{2n+2}$, we get

$$\begin{aligned}
\big(1 + bG(w_{2n} - w_{2n+1}, rt)\big) * G(w_{2n+1} - w_{2n+2}, rt)
&\geq b\big(G(w_{2n+2} - w_{2n+1}, rt) * G(w_{2n+1} - w_{2n}, rt) \\
&* G(w_{2n+1} - w_{2n+1}, rt)\big) + G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) \\
&* G(w_{2n+1} - w_{2n}, t) * G(w_{2n+1} - w_{2n+1}, (1-h)t) \\
&* G(w_{2n+2} - w_{2n}, (1+h)t) \\
&\geq b\big(G(w_{2n+2} - w_{2n+1}, rt) * G(w_{2n+1} - w_{2n}, rt)\big) \\
&+ G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) * G(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

And

$$\begin{aligned}
\big(1 + bH(w_{2n} - w_{2n+1}, rt)\big) \lozenge H(w_{2n+1} - w_{2n+2}, rt)
&\leq b\big(H(w_{2n+2} - w_{2n+1}, rt) \lozenge H(w_{2n+1} - w_{2n}, rt) \\
&\lozenge H(w_{2n+1} - w_{2n+1}, rt)\big) + H(w_{2n} - w_{2n+1}, t) \\
&\lozenge H(w_{2n+2} - w_{2n+1}, t) \lozenge H(w_{2n+1} - w_{2n}, t) \\
&\lozenge H(w_{2n+1} - w_{2n+1}, (1-h)t) \lozenge H(w_{2n+2} - w_{2n}, (1+h)t) \\
&\leq b\big(H(w_{2n+2} - w_{2n+1}, rt) \lozenge H(w_{2n+1} - w_{2n}, rt)\big) \\
&+ H(w_{2n} - w_{2n+1}, t) \lozenge H(w_{2n+2} - w_{2n+1}, t) * G(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow G(w_{2n+1} - w_{2n+2}, rt)
&\geq G(w_{2n} - w_{2n+1}, t) * G(w_{2n+2} - w_{2n+1}, t) * G(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

And

$$\begin{aligned}
H(w_{2n+1} - w_{2n+2}, rt)
&\leq H(w_{2n} - w_{2n+1}, t) \lozenge H(w_{2n+2} - w_{2n+1}, t) \lozenge H(w_{2n} - w_{2n+1}, ht)
\end{aligned}$$

Letting $h = 1$

$$\Longrightarrow G(w_{2n+1} - w_{2n+2}, rt) \geq G(w_{2n} - w_{2n+1}, t) * G(w_{2n+1} - w_{2n+2}, t)$$

And

$$H(w_{2n+1} - w_{2n+2}, rt) \leq H(w_{2n} - w_{2n+1}, t) \lozenge H(w_{2n+1} - w_{2n+2}, t)$$

By the same way, we also get

$$\Longrightarrow G(w_{2n+2} - w_{2n+3}, rt) \geq G(w_{2n+1} - w_{2n+2}, t) * G(w_{2n+2} - w_{2n+3}, t)$$

And

$$H(w_{2n+2} - w_{2n+3}, rt) \leq H(w_{2n+1} - w_{2n+2}, t) \lozenge H(w_{2n+2} - w_{2n+3}, t)$$

In general, for $m = 1,2,3, \dots$

$$G(w_{m+1} - w_{m+2}, rt) \geq G(w_m - w_{m+1}, t) * G(w_{m+1} - w_{m+2}, t)$$

And

$H(w_{m+1} - w_{m+2}, rt) \leq H(w_m - w_{m+1}, t) \lozenge H(w_{m+1} - w_{m+2}, t)$

It follows that for $m = 1,2,3, \dots$ and $p = 1,2,3, \dots$

$$G(w_{m+1} - w_{m+2}, rt) \geq G(w_m - w_{m+1}, t) * G\left(w_{m+1} - w_{m+2}, \frac{t}{r^p}\right)$$

And

$$H(w_{m+1} - w_{m+2}, rt) \leq H(w_m - w_{m+1}, t) \lozenge H\left(w_{m+1} - w_{m+2}, \frac{t}{r^p}\right)$$

As $p \to \infty$

$\Longrightarrow G(w_{m+1} - w_{m+2}, rt) \geq G(w_m - w_{m+1}, t)$

And

$H(w_{m+1} - w_{m+2}, rt) \leq H(w_m - w_{m+1}, t)$

Then from lemma (2.17) we get $\{w_m\}$ is a Cauchy sequence in $(Y, G, H, *, \lozenge)$.

Since $(Y, G, H, *, \lozenge)$ is a complete space

$\Longrightarrow \exists x \in Y$ such that $w_m \to x$ as $n \to \infty$

$\Longrightarrow Fz_{2n} \to x as n \to \infty,$

$Qz_{2n+1} \to x as n \to \infty,$

$Dz_{2n+1} \to x as n \to \infty$ and $Pz_{2n+2} \to x$ as $n \to \infty$.

Now, assume that (F, P) is a compatible pair and P is continuous. then by Proposition(3.9) we get F and P are reciprocally continuous thus $PFz_n \to Px$ and $FPz_n \to Fx$ as $n \to \infty$.

Since (F, P) is a compatible pair

$\Longrightarrow \lim_{n\to\infty} G(FP\, z_n - PFz_n, t) = 1$

And

$\lim_{n\to\infty} H(FP\, z_n - PFz_n, t) = 0$

$\Longrightarrow G(Fx - Px, t) = 1$

And

$H(Fx - Px, t) = 0$

$\Longrightarrow Fx = Px$

Since $F(Y) \subset Q(Y)$, there exists a point $y \in Y$ such that $Fx = Qy$

$\Longrightarrow Fx = Qy = Px$

From (ii) and take $\beta = 1$, we get

$$\big(1 + bG(Px - Qy, rt)\big) * G(Fx - Dy, rt)$$
$$\geq b\big(G(Fx - Px, rt) * G(Dy - Qy, rt) * G(Dy - Px, rt)\big)$$
$$+ G(Qy - Px, t) * G(Fx - Px, t) * G(Dy - Qy, t) * G(Dy - Px, t)$$
$$* G(Fx - Qy, t)$$

And $\big(1 + bH(Px - Qy, rt)\big) \lozenge H(Fx - Dy, rt) \leq b\big(H(Fx - Px, rt) \lozenge H(Dy -$ $Qy,rt \lozenge HDy - Px,rt + HQy - Px,t \lozenge HFx - Px,t \lozenge HDy - Qy,t \lozenge HDy - Px,t \lozenge H(Fx - Qy,t)$

$\Longrightarrow G(Fx - Dy, rt) \geq G(Fx - Dy, t)$

And

$H(Fx - Dy, rt) \leq H(Fx - Dy, t), \forall t > 0$

Then by lemma (2.18)

$\Longrightarrow Fx = Dy$

$\Longrightarrow Fx = Dy = Px = Qy$

Since F and P are R −weakly commuting mappings, then there exists $R > 0$ such that

$G(FPx - PFx, t) \geq G\left(Fx - Px, \frac{t}{R}\right)$ and $H(FPx - PFx, t) \leq H\left(Fx - Px, \frac{t}{R}\right)$

Since $Fx = Px$

$\Longrightarrow G\left(Fx - Px, \frac{t}{R}\right) = 1$ and $H\left(Fx - Px, \frac{t}{R}\right) = 0$

$\Longrightarrow G(FPx - PFx, t) \geq 1$ and $H(FPx - PFx, t) \leq 0$

$\Longrightarrow G(FPx - PFx, t) = 1$ and $H(FPx - PFx, t) = 0$

$\Longrightarrow FPx = PFx$

And since $Fx = Px$

$\Longrightarrow FFx = FPx$ and $PFx = PPx$

$\therefore FFx = FPx = PFx = PPx$

Similarly, since D and Q are point wise R −weakly commuting mappings, then there exists $R > 0$ such that

$$G(DQy - QDy, t) \geq G\left(Dy - Qy, \frac{t}{R}\right)$$

and

$$H(DQy - QDy, t) \leq H\left(Dy - Qy, \frac{t}{R}\right)$$

Since $Dy = Qy$

$\Longrightarrow G\left(Dy - Qy, \frac{t}{R}\right) = 1$ and $H\left(Dy - Qy, \frac{t}{R}\right) = 0$

$\Longrightarrow G(DQy - QDy, t) \geq 1$ and $H(DQy - QDy, t) \leq 0$

$\Longrightarrow G(DQy - QDy, t) = 1$ and $H(DQy - QDy, t) = 0$

$\Longrightarrow DQy = QDy$

And since $Dy = Qy$

$\Longrightarrow QDy = QQy$ and $DDy = DQy$

$\Longrightarrow DDy = QDy = DQy = QQy$

Again from (ii) and choose $\beta = 1$, we have

$$\big(1 + bG(PFx - Qy, rt)\big) * G(FFx - Dy, rt)$$
$$\geq b\big(G(FFx - PFx, rt) * G(Dy - Qy, rt) * G(Dy - PFx, rt)\big)$$
$$+ G(Qy - PFx, t) * G(FFx - PFx, t) * G(Dy - Qy, t)$$
$$* G(Dy - PFx, t) * G(FFx - Qy, t)$$

And

$$\big(1 + bH(PFx - Qy, rt)\big) \lozenge H(FFx - Dy, rt)$$
$$\leq b\big(H(FFx - PFx, rt) \lozenge H(Dy - Qy, rt) \lozenge H(Dy - PFx, rt)\big)$$
$$+ H(Qy - PFx, t) \lozenge H(FFx - PFx, t) \lozenge H(Dy - Qy, t)$$
$$\lozenge H(Dy - PFx, t) \lozenge H(FFx - Qy, t)$$

$\Longrightarrow G(FFx - Fx, rt) \geq G(FFx - Fx, t)$ and $H(FFx - Fx, rt) \leq H(FFx - Fx, t)$

Then by lemma (2.18), we get

$FFx = Fx$

$\Longrightarrow Fx = PFx$

$\therefore Fx$ is a common fixed point of F and P.

Again by (ii) and choose $\beta = 1$,we get

$$\big(1 + bG(Px - QDy, rt)\big) * G(Fx - DDy, rt)$$
$$\geq b\big(G(Fx - Px, rt) * G(DDy - QDy, rt) * G(DDy - Px, rt)\big)$$
$$+ G(QDy - Px, t) * G(Fx - Px, t) * G(DDy - QDy, t)$$
$$* G(DDy - Px, t) * G(Fx - QDy, t)$$

And

$$\big(1 + bH(Px - QDy, rt)\big) \lozenge H(Fx - DDy, rt)$$
$$\leq b\big(H(Fx - Px, rt) \lozenge H(DDy - QDy, rt) \lozenge H(DDy - Px, rt)\big)$$
$$+ H(QDy - Px, t) \lozenge H(Fx - Px, t) \lozenge H(DDy - QDy, t)$$
$$\lozenge H(DDy - Px, t) \lozenge H(Fx - QDy, t)$$

$\Longrightarrow G(Dy - QDy, rt) \geq G(Dy - QDy, t)andH(Dy - QDy, rt) \leq H(Dy - QDy, t)$

Then by lemma (2.18), we get

$QDy = Dy = DDy$

Then Dy is a common fixed point of D and Q.

Since $Dy = Fx$

$\Longrightarrow Fx$is a common fixed point of F, D, P and Q.

For the uniqueness,

Suppose that Fv is a common fixed point of $F, D, P$and Q and $Fv \neq Fx$.

From (ii) and choose $\beta = 1$, we get

$$\left(1 + bG(PFx - QFv, rt)\right) * G(FFx - DFv, rt)$$
$$\geq b\left(G(FFx - PFx, rt) * G(DFv - QFv, rt) * G(DFv - PFx, rt)\right)$$
$$+ G(QFv - PFx, t) * G(FFx - PFx, t) * G(DFv - QFv, t)$$
$$* G(DFv - PFx, t) * G(FFx - QFv, t)$$

And

$$\left(1 + bH(PFx - QFv, rt)\right) \lozenge H(FFx - DFv, rt)$$
$$\leq b\left(H(FFx - PFx, rt) \lozenge H(DFv - QFv, rt) \lozenge H(DFv - PFx, rt)\right)$$
$$+ H(QFv - PFx, t) \lozenge H(FFx - PFx, t) \lozenge H(DFv - QFv, t)$$
$$\lozenge H(DFv - PFx, t) \lozenge H(FFx - QFv, t)$$

$\Longrightarrow G(Fx - Fv, rt) \geq G(Fx - Fv, t)andH(Fx - Fv, rt) \leq H(Fx - Fv, t)$

Then by lemma (2.18) we get

$Fx = Fv$

Therefore Fx is a unique common fixed point of F,D,Pand Q.■

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