Journal of College of Education for pure sciences(JCEPS)
Web Site: http://eps.utq.edu.iq/ Email: com@eps.utq.edu.iq
Volume 7, Number 4, December 2017

Compare Between Least Square Method and Petrov-Galerkin Method for Solving the First Order Linear Ordinary Delay Differential Equations

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Abstract:

The problem of find approximate solution for the first order linear ordinary delay differential equations by using two of the special cases of the weighted residual method which are the least squares method and Petrov-Galerkin method with compare between these methods for which the best approximate solution near to the exact solution, also numerical applications are considered in this work.

Keywords: Delay differential equation, weighted residual method, least squares method, Petrov-Galerkin method**.**

1. Introduction

Delay differential equations have a great importance in life problems and many applications in other sciences as mechanics, physics, engineering, economics, biology, etc. As an application in mathematic Nadia in [1] applied these equations in the theory of automatic control.

For this importance of delay differential equations many scientists and mathematicians worked on this field of applied mathematics by using several methods of solution like the method of steps and Laplace transformation method Gadeer in [2] presented these methods and Asmaa in [3] applied the collocation method.

In this work we use the least squares method and Petrov-Galerkin method to find the approximate solution for the first order linear ordinary delay differential equation illustrated these methods by examples and show which give the best approximate solution near to the exact solution.

This work consists seven sections. The form of ordinary delay differential equations considered in the first section, while in the second section the weighted residual method present that. The third and fourth sections illustrate the least squares method and Petrov-Galerkin method respectively. The steps to find the approximate solution gives in section five. Two examples as numerical applications and solving these examples by these two methods given in section six. Finally the conclusion for this work are presented in the last section

2. The Form of Ordinary Delay Differential Equations

The definition of the n-th order linear delay differential equations with one constant delay may be written as:

$$D(\kappa,\eta(\kappa),\eta(\kappa-\varsigma_1),\ \eta(\kappa-\varsigma_2),\ \ldots,\ \eta(\kappa-\varsigma_k),\ \eta'(\kappa),\ \eta'(\kappa-\varsigma_1),\ \eta'(\kappa-\varsigma_2),\ \ldots,\ \eta'(\kappa-\varsigma_k),\ \eta''(\kappa-\varsigma_1),\ \eta''(\kappa-\varsigma_2),\ \ldots,\ \eta''(\kappa-\varsigma_k),\ \eta^{(n)}(\kappa-\varsigma_1),\ \eta^{(n)}(\kappa-\varsigma_2),\ \ldots,\ \eta^{(n)}(\kappa-\varsigma_h) = \aleph(\kappa)$$

where D and $\aleph$ are given functions and ς_i, i = 1, …,h are known functions which are called the time lag.

The first order linear delay differential equation has the form

$$\mathring{a}\eta'(\kappa) + \ddot{a}\eta'(\kappa-\varsigma) + \dot{c}\eta(\kappa) + \hat{c}\eta(\kappa-\varsigma) = \aleph(\kappa)$$

where $\aleph(\kappa)$ is a given continuous function and the time lag ς is constant and $\mathring{a}$, $\ddot{a}$, $\dot{c}$ and $\hat{c}$ are constant coefficients.

3. Weighted Residual Method

This method can be described as the operator equation

$$O(u) = h \quad \text{in } \Omega$$

Such that O is an operator dependent on variable u, h is a famed function of the separate variables.

The solution z(m) is approximated in the weighted-residual method by the following form

$$\tilde{z}(m) \approx Z_N(m) = \sum_{j=1}^{N} r_j \Psi_j(m) + \Psi_0(m)$$

Substitution the approximate solution Z_N into the left-hand side of the operator equation gives a function

$h_N = O(Z_N)$

The difference $O(Z_N) - h_N$ called the residual of the approximation, is nonzero:

$$R = O(Z_N) - h_N = O(\sum_{j=1}^{N} r_j \Psi_j(m) + \Psi_0(m)) - h_N \neq 0$$

The concept of these methods is to require the residual to zero over the domain Ω, that is

$$\int_{\Omega} R(m) W_j dm = 0 \quad (j = 1, 2, \ldots, n)$$

Such that the number of weighted functions W_i is equal to the number of the constants r_j in Z_N, [4], [5].

4. Least Squares Method [6]

In this method determine the unknown constants c_j by minimize the integral for $\frac{\partial R}{\partial r_j} R$ in domain Ω to zero, that is mean

$$\int_{\Omega} \frac{\partial R}{\partial r_j} R \, dm = 0$$

Therefore the weight functions for the least squares method is the derivatives of the residual function with respect to the constants r_j

$$W_i = \frac{\partial R}{\partial r_j} \quad (j = 1, 2, \ldots, n)$$

5. Petrov-Galerkin Method [4]

The weight functions for the Petrov-Galerkin method is considered as $W_i \neq \phi_i$.

In this method also minimize the integral for $\phi_i R$ in domain Ω to zero, that is mean

$$\int_{\Omega} \phi_i R \, dm = 0$$

6. Steps to Find the Approximate Solution

To find the approximate solution for equation (1) which is form

$$\mathring{a}\eta'(\kappa) + \ddot{a}\eta'(\kappa - \varsigma) + \dot{c}\eta(\kappa) + \hat{c}\eta(\kappa - \varsigma) = \aleph(\kappa) \qquad (1)$$

where $\aleph(\kappa)$ is a given continuous function and the time lag ς is constant and $\mathring{a}$, $\ddot{a}$, $\dot{c}$ and $\hat{c}$ are constant coefficients, with the initial condition $\eta_0(\kappa)$ on the domain $c \le \kappa \le d$.

We apply the following steps

(1) Suppose that the solution $\tilde{\eta}(\kappa)$ is the approximate solution to the equation (1) which has the expression

$$\tilde{\eta}(\kappa) = \Psi_0(\kappa) + \sum_{j=1}^{N} r_j \Psi_j(\kappa)$$

where $\Psi_0(\kappa) = a + b\kappa$ which satisfy the initial condition.

$$\Psi_i(\kappa) = \kappa^i(1-\kappa)$$

Substitute the values of $\Psi_0(\kappa)$ and $\Psi_i(\kappa)$ in $\tilde{\eta}(\kappa)$.

(2) Substitute the values of equation (1) with respect to $\tilde{\eta}(\kappa)$ which gives the residual $R(\kappa)$.

(3) .Find $\int_c^d W_j R(\kappa) d\kappa = 0 \quad \text{for all} \quad j = 1, \dots, N$

(4) The result of step 4 is a system of equations for the unknown constants r_j (j =1,…,N), solved this system to find the values of r_j (j =1,…,N).

(5) Substitute the values of r_j (j =1,…,N) in $\tilde{\eta}(\kappa)$ get the approximate solution.

7. Numerical Applications

Example (1):

Consider the first order linear ordinary delay differential equation

$$s'(e) = s'(e-1) + e \qquad (*)$$

with the initial condition $s_0(e) = e + 1, -1 \le e \le 0$.

Solution:

To find the exact solution:

In the interval $0 \le e \le 1$

$$s_1'(e) = s_0'(e-1) + e = 1 + e$$

By integrate each sides from 0 to e we get

$$s_1(e) = \frac{e^2}{2} + e + 1$$

In the interval $1 \le e \le 2$

$$s_2'(e) = s_1'(e-1) + e = 2e$$

By integrate each sides from 1 to e we get

$$s_2(e) = e^2 + \frac{3}{2}$$

Similarly, we find the exact solution for the next intervals.

To find the approximate solution

Suppose $\tilde{s}(e) = \phi_0(e) + r_1\phi_1(e)$

where $\phi_0(e) = a + be$ and $\phi_1(e) = e(1-e) = e - e^2$

$\phi_0(-1) = 0 = a - b$, $\phi_0(0) = 1 = a \Rightarrow b = 1 \Rightarrow \phi_0(e) = 1 + e$

So we get that

$\tilde{s}(e) = \phi_0(e) + r_1\phi_1(e)$

$\tilde{s}(e) = 1 + e + r_1 e - r_1 e^2 = 1 + (1 + r_1)e - r_1 e^2$

$\tilde{s}(e) = 1 + r_1 - 2r_1 e$

$\tilde{s}(e-1) = 1 + r_1 - 2r_1 e + 2r_1 = 1 + 3r_1 - 2r_1 e$

Substitute these values in equation (*) we obtain

$R(e) = 1 + r_1 - 2r_1 e - 1 - 3r_1 + 2r_1 e - e = - e - 2r_1$

To find the approximate solution $\tilde{s}_1(e)$ in the interval [0,1]

(1) By using least squares method

$$W = \frac{\partial R}{\partial r_1} = -2$$

$$\int_0^1 WR(e)de = \int_0^1 (2e + 4r_1)de = e + 4r_1 e \Big|_0^1 = 1 + 4r_1$$

To find the value of r_1 we solve the following

$1 + 4r_1 = 0$

Thus $r_1 = -0.25$

So the approximate solution in the interval [0,1] is

$\tilde{s}_1(e) = 1 + 0.75\, e + 0.25\, e^2$

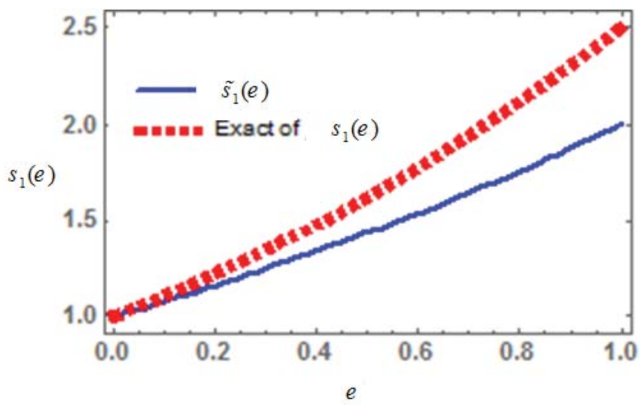

Figure 1: Exact and approximate solution in the interval [0,1] by using

least squares method for example (1)

(2) By using Petrov-Galerkin method

$W = -e$

$$\int_0^1 WR(e)de = \int_0^1 (e^2 + 2r_1 e)de = \frac{e^3}{3} + r_1 e^2 \Big|_0^1 = \frac{1}{3} + r_1$$

To find the value of r_1 we solve the following

$$\frac{1}{3} + r_1 = 0$$

Thus $r_1 = -0.33$

So the approximate solution in the interval [0,1] is

$\tilde{s}_1(e) = 1 + 0.67\, e + 0.33\, e^2$

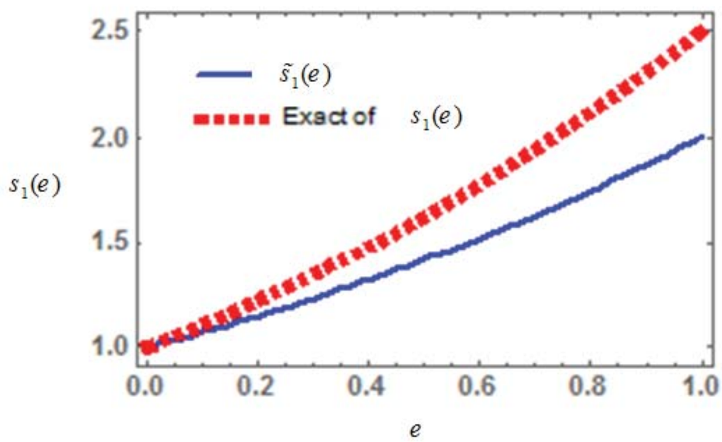

Figure 2: Exact and approximate solution in the interval [0,1] by using Petrov-Galerkin method for example (1)

As a similar way we find the approximate solution $\tilde{s}_2(e)$ in the interval [1,2] as follows

(1) By using least squares method

$$\int_1^2 WR(e)de = \int_1^2 (2e + 4r_1)de = e + 4r_1 e \Big|_1^2 = 1 + 4r_1$$

To find the value of r_1 we solve the following

$1 + 4r_1 = 0$

Thus $r_1 = -0.25$

So the approximate solution in the interval [1,2] is

$\tilde{s}_2(e) = 1 + 0.75\, e + 0.25\, e^2$

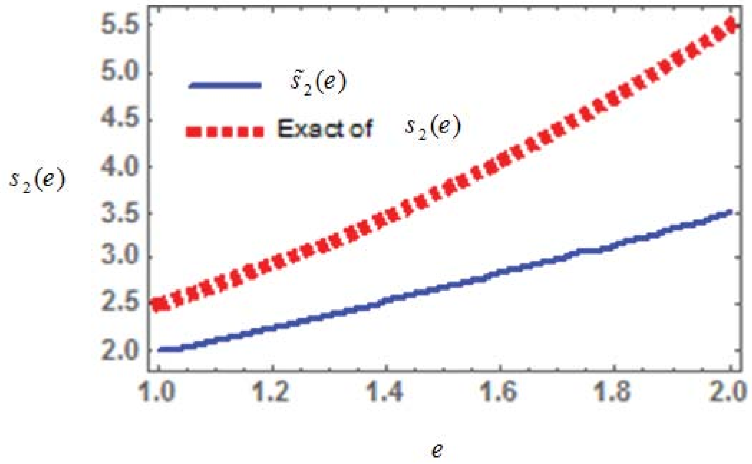

Figure 3: Exact and approximate solution in the interval [1,2] by using least squares method for example (1)

(2) By using Petrov-Galerkin method

$$\int_1^2 \mathrm{WR}(e)\mathrm{dt} = \int_1^2 (e^2 + 2\mathrm{r}_1 e)\,\mathrm{dt} = \frac{e^3}{3} + \mathrm{r}_1 e^2 \Big|_0^1 = \frac{7}{3} + 3\mathrm{r}_1$$

To find the value of r_1 we solve the following

$$\frac{7}{3} + 3\mathrm{r}_1 = 0$$

Thus $\mathrm{r}_1 = -0.77$

So the approximate solution in the interval [1,2] is

$$\tilde{s}_2(e) = 1 + 0.23\, e + 0.77\, e^2$$

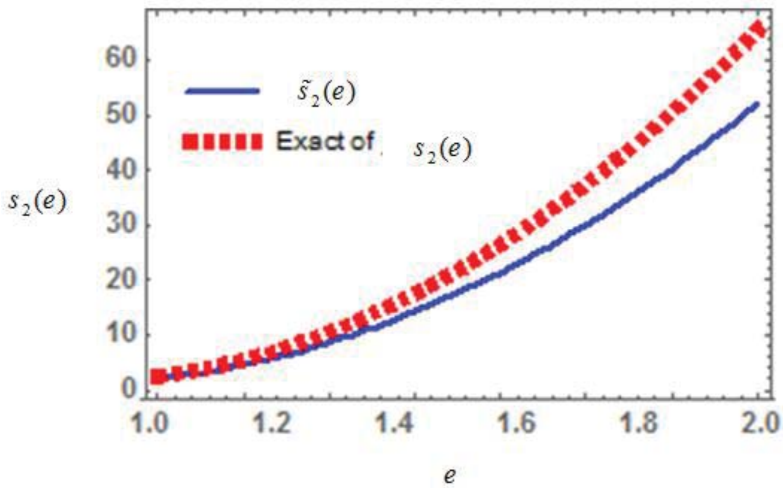

Figure 4: Exact and approximate solution in the interval [1,2] by using Petrov-Galerkin method for example (1)

Similarly we can apply that steps if we take the intervals [2,3], [3,4], [4,5], …

Example (2):

Consider the first order linear ordinary delay differential equation

$$s'(e) = s'(e-1) + e^2 \qquad (*)$$

with the initial condition $s_0(e) = e + 1, -1 \le e \le 0$.

Solution:

To find the exact solution:

In the interval $0 \le e \le 1$

$s_1'(e) = s_0'(e-1) + e^2 = 1 + e^2$

By integrate each sides from 0 to e we get

$s_1(e) = 0.33e^3 + e + 1$

In the interval $1 \le e \le 2$

$s_2'(e) = s_1'(e-1) + e^2 = 2 - 2e + 2e^2$

By integrate each sides from 1 to e we get

$s_2(e) = 0.66e^3 - e^2 + 2e + 1.33$

Similarly, we find the exact solution for the next intervals.

To find the approximate solution

Suppose $\tilde{s}(e) = \phi_0(e) + r_1\phi_1(e) + r_2\phi_2(e)$

where $\phi_0(e) = a + be$, $\phi_1(e) = e(1-e) = e - e^2$ and

$\phi_2(e) = e^2(1-e) = e^2 - e^3$

$\phi_0(-1) = 0 = a - b$, $\phi_0(0) = 1 = a \quad \Rightarrow \quad b = 1 \quad \Rightarrow \quad \phi_0(e) = 1 + e$

So we get that

$\tilde{s}(e) = \phi_0(e) + r_1\phi_1(e) + r_2\phi_2(e)$

$\tilde{s}(e) = 1 + e + r_1 e - r_1 e^2 + r_2 e^2 - r_2 e^3$

$\quad = 1 + (1 + r_1)t + (r_2 - r_1) e^2 - r_2 e^3$

$\tilde{s}'(e) = 1 + 2(r_2 - r_1) e - 3r_2 e^2$

$\tilde{s}'(e-1) = 1 - r_2 + 2r_1 - (2r_1 + 4r_2)e + 3r_2 e^2$

Substitute these values in equation (*) we obtain

$R(e) = r_2 - 2r_1 + e - e^2 + (3r_1 + 4r_2)e - (r_1 + 2r_2)e^2 - r_2 e^3$

To find the approximate solution $\tilde{s}_1(e)$ in the interval [0,1]

(1) By using least squares method

$$W_1=\frac{\partial R}{\partial r_1}=-2+3e-e^2$$

$$W_2=\frac{\partial R}{\partial r_2}=1+4e-2e^2-e^3$$

$$\int_0^1 W_1R(e)de=\frac{31}{30}r_1-\frac{23}{15}r_2-\frac{2}{15}$$

$$\int_0^1 W_2R(e)de=\frac{-23}{15}r_1+\frac{947}{210}r_2+\frac{11}{30}$$

To find the values of r_1 and r_2 we solve the following system

$$\frac{31}{30}r_1-\frac{23}{15}r_2=\frac{2}{15}$$

$$\frac{-23}{15}r_1+\frac{947}{210}r_2=-\frac{11}{30}$$

Thus $r_1 = 0.016$ and $r_2 = -0.075$

So the approximate solution in the interval [0,1] is

$\tilde{s}_1(e) = 1 + 0.75\,e + 0.25\,e^2$

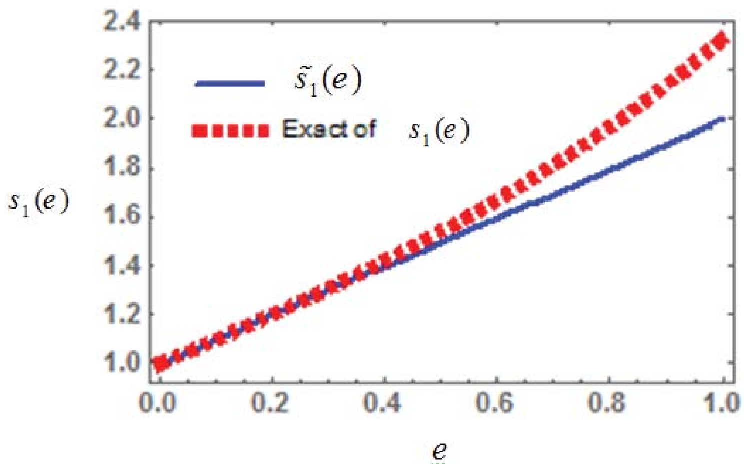

Figure 5: Exact and approximate solution in the interval [0,1] by using least squares method for example (2)

(2) By using Petrov-Galerkin method

$$W_1=-e$$

$$W_2=-e^2$$

$$\int_0^1 W_1R(e)de=\frac{-1}{4}r_1+\frac{17}{15}r_2+\frac{1}{12}$$

$$\int_0^1 W_2 R(e)de = \frac{7}{60}r_1 - \frac{23}{30}r_2 - \frac{1}{20}$$

To find the values of r_1 and r_2 we solve the following system

$$\frac{-1}{4}r_1 + \frac{17}{15}r_2 = -\frac{1}{12}$$

$$\frac{7}{60}r_1 - \frac{23}{30}r_2 = \frac{1}{20}$$

Thus $r_1 = 0.121$ and $r_2 = -0.046$

So the approximate solution in the interval [0,1] is

$\tilde{s}_1(e) = 1 + 1.121\, e - 0.167\, e^2 + 0.046\, e^3$

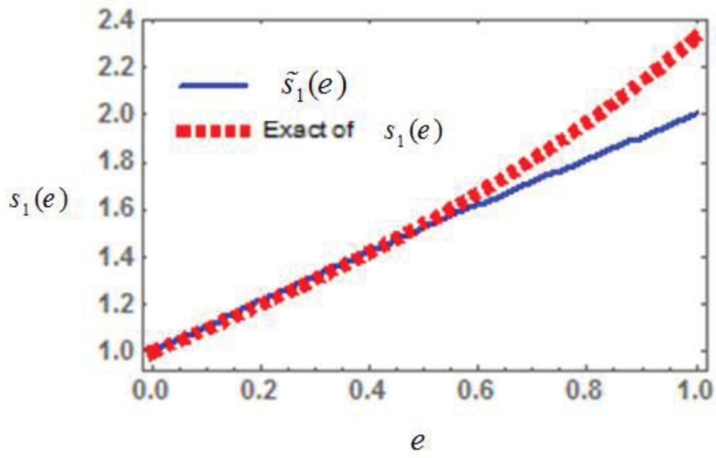

Figure 6: Exact and approximate solution in the interval [0,1] by using Petrov-Galerkin method for example (2)

As a similar way we find the approximate solution $\tilde{s}_2(e)$ in the interval [1,2] as follows

(1) By using least squares method

$$\int_1^2 W_1 R(e)de = \frac{1}{30}r_1 - \frac{1}{5}r_2 - \frac{2}{15}$$

$$\int_1^2 W_2 R(e)de = \frac{-1}{5}r_1 + \frac{619}{70}r_2 + \frac{27}{10}$$

To find the values of r_1 and r_2 we solve the following system

$$\frac{1}{30}r_1 - \frac{1}{5}r_2 = \frac{2}{15}$$

$$\frac{-1}{5}r_1 + \frac{619}{70}r_2 = -\frac{27}{10}$$

Thus $r_1 = 2.508$ and $r_2 = -0.248$

So the approximate solution in the interval [1,2] is

$\tilde{s}_2(e) = 1 + 3.508\, e - 2.26\, e^2 + 0.248\, e^3$

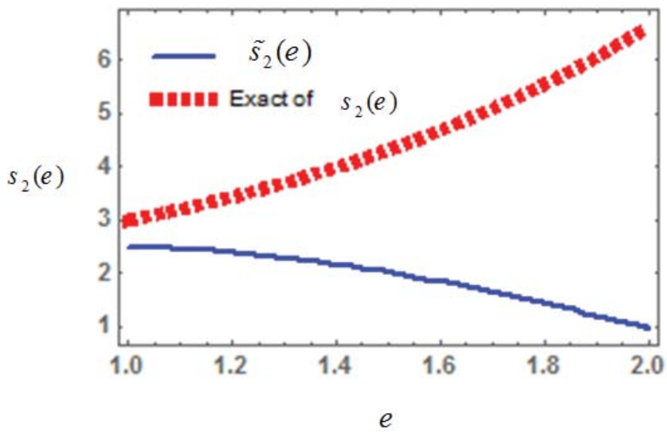

Figure 7: Exact and approximate solution in the interval [1,2] by using least squares method for example (2)

(2) By using Petrov-Galerkin method

$$\int_1^2 W_1 R(e)\,de = \frac{1}{4}r_1 - \frac{43}{15}r_2 - \frac{17}{12}$$

$$\int_1^2 W_2 R(e)\,de = \frac{23}{60}r_1 + \frac{167}{30}r_2 + \frac{49}{20}$$

To find the values of r_1 and r_2 we solve the following system

$$\frac{1}{4}r_1 - \frac{43}{15}r_2 = \frac{17}{12}$$

$$\frac{23}{60}r_1 + \frac{167}{30}r_2 = -\frac{49}{20}$$

Thus $r_1 = 0.346$ and $r_2 = -0.463$

So the approximate solution in the interval [1,2] is

$\tilde{s}_2(e) = 1 + 1.346\, e + 0.117\, e^2 + 0.463\, e^3$

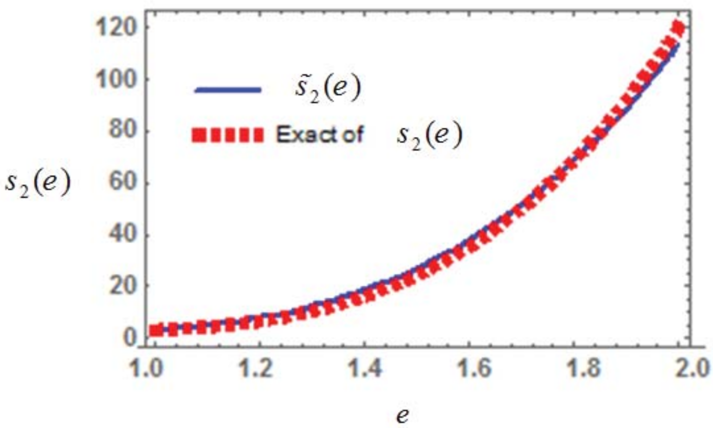

Figure 8: Exact and approximate solution in the interval [1,2] by using Petrov-Galerkin method for example (2)

Similarly we can apply that steps if we take the intervals [2,3], [3,4], [4,5], …

8. Conclusion

From this work we can conclude that the Petrov-Galerkin method gives best approximate solution to the first order linear ordinary delay differential equation near to the exact solution than the least squares method since when we applied numerically these two methods we saw that the least squares method gives near approximate to the exact solution for the first interval [0,1] while in the second interval [1,2] obtained divergent or equal to the approximate solution in the interval [0,1].

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