Subclass of Multivalent Functions Defined by using Differential Operator

الصف الجزئي لدوال متعددة معرفة باستخدام مشتقة العامل

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Abstract:

In the present paper, we introduce a subclass $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ of multivalent analytic functions in the open unit disc U. We study coefficient inequalities, closure theorem , radii of starlikeness, convexity and close-to-convexity. We also obtain weighted mean, arithmetic mean and linear combination.

الخلاصة:

في هذا البحث ناقشنا الصف الجزئي $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ لدول متعددة في القرص المفتوح U ودرسنا متباينات المعامل ونظرية الانغلاق وانصاف الاقطار كذلك تطرقنا لمتوسط مرجح والوسط الحسابي والمجموع الخطي

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1- Introduction :

Let $\mathcal{A}_p$ denote the class of all functions of the form:

$$f(z)=z^p+\sum_{k=p+i}^{\infty}a_k z^k\ , \quad (\,p\in\mathbb{N}=\{1,2,3,\dots\}\,), \tag{1}$$

which are analytic and multivalent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.
Let $\mathcal{M}_p$ denote the subclass of $\mathcal{A}_p$ containing of functions of the form:

$$f(z)=z^p+\sum_{k=p+i}^{\infty}a_k z^k\ , \quad (a_k\geq 0, p\in\mathbb{N}=\{1,2,3,\dots\}\,), \tag{2}$$

which are analytic and multivalent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.
For the functions $f\in\mathcal{M}_p$ given by (2) and $g\in\mathcal{M}_{\mathrm{p}}$ defined by

$$g(z)=z^p+\sum_{k=p+i}^{\infty}b_k z^k\ , \quad (a_k\geq 0, p\in\mathbb{N}\,), \tag{3}$$

We define the convolution (or Hadamard product) of f and g by

$$(f*g)(z)=z^p+\sum_{k=p+i}^{\infty}a_k b_k z^k\ . \tag{4}$$

A function $f\in\mathcal{M}_p$ is said to be p-valently starlike of order α if it satisfies the inequality:[2]

$$Re\left\{\frac{z\acute{f}(z)}{f(z)}\right\}>\alpha \qquad (z\in U; 0\leq\alpha<p; p\in N). \tag{5}$$

We denote by $\mathcal{M}_p^*$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in \mathcal{M}_p$ is said to be p-valently convex of order α if it satisfies the inequality:[2]

$$Re\left\{1+\frac{z\acute{\acute{f}}(z)}{\acute{f}(z)}\right\} > \alpha \qquad (z \in U; 0 \le \alpha < p; p \in N). \tag{6}$$

We denote by $C(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [6] and Goodman [7])

$$f(z) \in C(p,\alpha) \Leftrightarrow \frac{z\acute{f}(z)}{p} \in \mathcal{M}_j^*(p,\alpha) \qquad (0 \le \alpha < p; p \in N). \tag{7}$$

A function $f \in \mathcal{M}_p$ is closed –to-convex of order α if

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha, \quad (z \in U; 0 \le \alpha < p) \tag{8}$$

Definition (1)[6] : Let $\gamma, \beta, m \in R, \gamma \ge 0, \beta \ge 0, m \ge 0, p \in N$ and

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.$$

Then, we define the linear operator

$$D_{p,m}^{\gamma,\beta}: \mathcal{A}_p \to \mathcal{A}_p \text{ by}$$

$$D_{p,m}^{\gamma,\beta} f(z) = z^p + \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m a_k z^k, z \in U. \tag{9}$$

Definition (2): Let g be a fixed function defined by (3). The function $f \in M_p$ given by (2) is said to be in the class $\mathcal{K}_\text{p}(\gamma,\beta,m,\lambda,A,B,\alpha)$ if and only if

$$\left|\frac{z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} - \left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}{\lambda z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} + (A+B)\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}\right| < \alpha, \tag{10}$$

where

$$0 < \lambda < 1, 0 < A < 1, 0 \le B < 1, 0 < \alpha < 1, \gamma, \beta, m \in R, \gamma \ge 0, \beta \ge 0, m \ge 0, p \in N,$$

and for each $f \in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$ we have

$$f^{(p)}(z) = \delta(p,q)z^{p-q} + \sum_{k=p+1}^{\infty} \delta(p,q)a_k z^k$$

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} 1 & (q=0) \\ p(p-1)\dots(p-q+1) & (q \ne 0) \end{cases}$$

Some of the following properties studied for other classes in [1],[2], [4] and [5].

2- Coefficient Inequalities:

Theorem (1): Let $f \in \mathcal{M}_p$. Then $f \in \mathcal{K}_\text{p}(\gamma,\beta,m,\lambda,A,B,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)a_k b_k \le \alpha p!\,(\lambda+A+B). \tag{11}$$

Where

$$0 < \lambda < 1, 0 < A < 1, 0 \le B < 1, 0 < \alpha < 1, \gamma, \beta, m \in R, \gamma \ge 0, \beta \ge 0, m \ge 0, p \in N.$$

The result is sharp for the function

$$f(z) = z^p + \frac{\alpha p!\,(\lambda+A+B)}{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k} z^k.$$

Proof : Suppose that the inequality (11) holds true and $|z| = 1$. Then we have

$$\left|\frac{z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} - \left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}{\lambda z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} + (A+B)\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}\right|$$

$$= \left|z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} - \left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}\right| - \alpha\left|\lambda z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} + (A+B)\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}\right|$$

$$= \left|\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}(k-p)\delta(k,p-1)\, a_k b_k z^{k-p+1}\right| - \alpha\left|p!\,(\lambda+A+B)z + \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[\left(\lambda(k-p+1)+(A+B)\right)\right]\delta(k,p-1)a_k b_k z^{k-p+1}\right|$$

$$\leq \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}(k-p)\delta(k,p-1)\, a_k b_k |z|^{k-p+1} - \alpha p!\,(\lambda+A+B)|z| - z\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[\left(\lambda(k-p+1)+(A+B)\right)\right]\delta(k,p-1)a_k b_k |z|^{k-p+1}$$

$$= \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\left(\lambda(k-p+1)+(a+B)\right)\right]\delta(k,p-1) - \alpha p!\,(\lambda+A+B) \leq 0$$

by hypothesis.
Hence by maximum modulus principle, $f \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$.
Conversely : suppose that $f \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then from (10), we have

$$\left|\frac{z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} - \left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}{\lambda z\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)} + (A+B)\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p-1)}}\right|$$

$$= \left|\frac{\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}(k-p)\delta(k,p-1)\, a_k b_k z^{k-p+1}}{p!\,(\lambda+A+B)z + \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[\left(\lambda(k-p+1)+(A+B)\right)\right]\delta(k,p-1)a_k b_k z^{k-p+1}}\right| < \alpha.$$

Since $Re(z) \leq |z|$ for all z ($z \in U$), we get

$$Re\left(\frac{\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}(k-p)\delta(k,p-1)\, a_k b_k z^{k-p+1}}{p!\,(\lambda+A+B)z + \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[\left(\lambda(k-p+1)+(A+B)\right)\right]\delta(k,p-1)a_k b_k z^{k-p+1}}\right) < \alpha. \qquad (13)$$

We choose the value of z on the real axis so that $\left(D_{p,m}^{\gamma,\beta}(f*g)(z)\right)^{(p)}$ is real.
Letting $z \to 1^-$. Through real values, we obtain inequality (11).
Finally, sharpness follows if we have

$$f(z)=z^p+\frac{\alpha p!\,(\lambda+A+B)}{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}z^k$$
$$k=p+1,p+2,\dots \qquad (14)$$

Corollary (1): Let $f\in\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then

$$a_n \le \frac{\alpha p!\,(\lambda+A+B)}{p!\,(\lambda+A+B)z+\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)a_kb_kz^{k-p+1}},$$
$$k=p+1,p+2,\dots \qquad (15)$$

3- Closure Theorem:

Theorem (2): Let the functions f_i defined by

$$f_i(z)=z^p+\sum_{k=p+1}^{\infty}a_{k,i}z^k, \qquad \left(a_{k,i}\ge 0\,,p\in\mathbb{N},i=1,2,\dots,\ell\right),$$

be in the class $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ for every $\mathrm{i}=1,2,\dots,\ell$. Then the function h defined by

$$h(z)=z^p+\sum_{k=p+1}^{\infty}e_kz^k, \quad (e_k\ge 0\,,p\in\mathbb{N}),$$

also belongs to class $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$, where

$$e_n=\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}\,, \qquad n=p+1,p+2,\dots\ .$$

Proof: Since $f_i\in\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$, we have

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)a_{k,i}b_k$$
$$\le \alpha p!\,(\lambda+A+B).$$

for every $i=1,2,\dots,\ell$. Hence

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(a+B)\big)\right]\delta(k,p-1)e_kb_k$$
$$=\sum_{k=j+p}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(a+B)\big)\right]\delta(k,p-1)b_k\left(\frac{1}{\ell}\sum_{i=1}^{\ell}a_{k,i}\right)$$
$$=\frac{1}{\ell}\sum_{i=1}^{\ell}\left(\sum_{k=j+p}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(a+B)\big)\right]\delta(k,p-1)b_k\,a_{k,i}\right)$$
$$\le \alpha p!\,(\lambda+A+B).$$

Therefore, by Theorem (1), we have $h\in\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$.
This completes the proof of the theorem.

4-Radii of Starlikeness, Convexity and Close-to-convexity.

Using the inequality (5), (6) and (8) and Theorem (1), we can compute the radii of starlikeness, convexity and close-to-convex.

Theorem (3): If $f \in \mathcal{K}_p(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then f is p-valently starlike of order $\rho, (0 \le \rho < p)$ in the disk $|z| < r = r_1(\gamma,\beta,m,\lambda,A,B,\alpha)$, where

$r_1(\gamma,\beta,m,\lambda,A,B,\alpha,\rho)$

$$= \inf_n \left\{ \frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{(k-p)\alpha p!\,(\lambda+A+B)} \right\}^{\frac{1}{k-p}}.$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right| \le p-\rho, \qquad \text{for } |z| < r_1. \tag{16}$$

But

$$\left|\frac{zf'(z)}{f(z)}-p\right| = \left|\frac{zf'(z)-pf(z)}{f(z)}\right| = \left|\frac{-\sum_{k=p+1}^{\infty} n a_k z^{k-p}}{z^p-\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| \le \frac{\sum_{k=p+1}^{\infty}(k-p)a_k\,|z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k\,|z|^{k-p}}.$$

Thus , (16) will be satisfied if

$$\frac{\sum_{k=p+1}^{\infty}(k-p)a_k\,|z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k\,|z|^{k-p}} \le p-\rho,$$

or if

$$\sum_{n=1}^{\infty} \frac{(k-p)}{(p-\rho)} a_k|z|^{k-p} \le 1. \tag{17}$$

Since $f \in \mathcal{K}_p(\gamma,\beta,m,\lambda,A,B,\alpha)$, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{\alpha p!\,(\lambda+A+B)} a_k \le 1 .$$

Hence, (17) will be true if

$$\frac{(k-p)}{(p-\rho)}|z|^{k-p} \le \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{\alpha p!\,(\lambda+A+B)},$$

or equivalently

$$|z| \le \left\{ \frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{(k-p)\alpha p!\,(\lambda+A+B)} \right\}^{\frac{1}{k-p}}, n \ge 1 ,$$

Setting $|z| = r_1$we get the desired result.

Theorem(4): Let $f \in \mathcal{K}_p(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then f is p-valently convex of order ρ , $(0 \le \rho < p)$ in $|z| < r = r_2(\gamma,\beta,m,\lambda,A,B,\alpha,\rho)$, where

$r_2(\gamma,\beta,m,\lambda,A,B,\alpha,\rho)$

$$= \inf_n \left\{ \frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{k(k-p)\alpha p!\,(\lambda+A+B)} \right\}^{\frac{1}{k-p}},$$

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| \le p-\rho, \text{for} |z|<r_2. \tag{18}$$

But

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| = \left|\frac{zf''(z)+(1-p)f'(z)}{f'(z)}\right| = \left|\frac{-\sum_{k=p+1}^{\infty} k(k-p)\, a_k z^{k-p}}{1-\sum_{k=p+1}^{\infty} k\, a_k z^{k-p}}\right|$$

$$\le \frac{\sum_{k=p+1}^{\infty} k(k-p)\, a_k |z|^{k-p}}{1-\sum_{k=p+1}^{\infty} k\, a_k |z|^{k-p}}.$$

Thus, (18) will be satisfied if

$$\frac{\sum_{k=p+1}^{\infty} k(k-p)\, a_k |z|^{k-p}}{1-\sum_{k=p+1}^{\infty} k\, a_k |z|^{k-p}} \le \; p-\beta$$

or if

$$\sum_{k=p+1}^{\infty} \left(\frac{k(k-p)a_k|z|^{k-p}}{(p-\rho)}\right) a_k |z|^{k-p} \le 1. \tag{19}$$

Since $f \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{\alpha p!\,(\lambda+A+B)} a_k \le 1\,.$$

Hence, (19) will be true if

$$\frac{k(k-p)a_k|z|^{k-p}}{(p-\rho)} \le \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{\alpha p!\,(\lambda+A+B)},$$

or equivalently

$$|z| \le \left\{\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{k(k-p)\alpha p!\,(\lambda+A+B)}\right\}^{\frac{1}{k-p}},$$

Setting $|z| = r_1$we get the desired result.

Theorem (5): Let a function $f \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then f is p-valently close -to-convex of order $\rho, (0 \le \rho < p)$ in the disk $|z| < r = r_3(\gamma,\beta,m,\lambda,A,B,\alpha)$, where

$r_3(\gamma,\beta,m,\lambda,A,B,\alpha,\rho)$

$$= \inf_n \left\{\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k}{k\alpha p!\,(\lambda+A+B)}\right\}^{\frac{1}{k-p}}.$$

Proof: It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le p-\rho, \qquad \text{for } |z|<r_3. \tag{20}$$

We have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le \sum_{k=p+1}^{\infty} k a_k \;|z|^{k-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \rho,$$

if

$$\sum_{k=p+1}^{\infty} \frac{k a_k |z|^{k-p}}{p-\rho} \leq 1. \tag{21}$$

Since $f \in \mathcal{K}_{\mathrm{p}}(\gamma, \beta, m, \lambda, A, B, \alpha)$, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p) - \alpha\big(\lambda(k-p+1) + (A+B)\big)\right]\delta(k, p-1) b_k}{\alpha p!\,(\lambda + A + B)} a_k \leq 1 .$$

Hence, (21) will be true if

$$\frac{k|z|^{k-p}}{p-\rho} \leq \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p) - \alpha\big(\lambda(k-p+1) + (A+B)\big)\right]\delta(k, p-1) b_k}{\alpha p!\,(\lambda + A + B)},$$

or equivalently

$$|z| \leq \left\{\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p) - \alpha\big(\lambda(k-p+1) + (A+B)\big)\right]\delta(k, p-1) b_k}{k\alpha p!\,(\lambda + A + B)}\right\}^{\frac{1}{k-p}}, n$$

$$\geq 1 ,$$

Setting $|z| = r_3$we get the desired result.

5- weighted Mean and Arithmetic Mean.

Definition (3): Let f_1 and f_2 be in the class $\mathcal{K}_{\mathrm{p}}(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then the weighted mean w_q of f_1 and f_2 is given by

$$w_q(z) = \frac{1}{2}[(1-q)f_1(z) + (1+q)f_2(z)], \qquad 0 < q < 1.$$

Theorem (6): Let f_1 and f_2 be in the class $\mathcal{K}_{\mathrm{p}}(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then the weighted mean w_q of f_1 and f_2 is also in the class $\mathcal{K}_{\mathrm{p}}(\gamma, \beta, m, \lambda, A, B, \alpha)$.

Proof: By Definition (3), we have

$$w_q(z) = \frac{1}{2}[(1-q)f_1(z) + (1+q)f_2(z)]$$

$$w_q(z) = \frac{1}{2}\left[(1-q)\left(z^p + \sum_{k=p+1}^{\infty} a_{k,1} z^k\right) + (1+q)\left(z^p + \sum_{k=p+1}^{\infty} a_{k,2} z^k\right)\right]$$

$$2z^p + \sum_{k=p+1}^{\infty} \frac{1}{2}\left[(1-q)a_{k,1} + (1+q)a_{k,2}\right] a^k$$

Since f_1 and f_2 are in the class $\mathcal{K}_{\mathrm{p}}(\gamma, \beta, m, \lambda, A, B, \alpha)$. So by Theorem (1), we get

$$\sum_{k=p+1}^{\infty} \left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m \left[(k-p) - \alpha\big(\lambda(k-p+1) + (A+B)\big)\right]\delta(k, p-1) a_{k,1} b_k$$
$$\leq \alpha p!\,(\lambda + A + B),$$

and

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)a_{k,2}b_k \leq \alpha p!\,(\lambda+A+B).$$

Hence

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1) \times\left(\frac{1}{2}\left[(1-q)a_{k,1}+(1+q)a_{k,2}\right]\right)b_k$$

$$=\frac{1}{2}(1-q)\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)\,a_{k,1}b_k$$

$$+\frac{1}{2}(1+q)\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)\,a_{k,2}b_k$$

$$\leq\frac{1}{2}(1-q)\big(\alpha p!\,(\lambda+A+B)\big)+\frac{1}{2}(1+q)\big(\alpha p!\,(\lambda+A+B)\big)=\alpha p!\,(\lambda+A+B)$$

There for $w_q \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. The proof is complete.

Theorem (7): Let $f_1, f_2, \dots f_l$ defined by

$$f_i(z)=z^p+\sum_{k=p+1}^{\infty}a_{k,i}z^k\,,\big(a_{n,i}\geq 0, i=1,2,3,\dots l, k\geq p+1\big), \tag{22}$$

be in the class $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then the arithmetic mean of $f_i(z),(i=1,2,3,\dots,l)$ defined by

$$h(z)=\frac{1}{l}\sum_{i=1}^{\infty}f_i\,(z), \tag{23}$$

also in the class $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$.

Proof: By (22) and (23) we can write

$$h(z)=\frac{1}{l}\sum_{i=1}^{\infty}\left(z^p+\sum_{k=p+1}^{\infty}a_{k,i}z^k\right)$$
$$=z^p+\sum_{k=p+1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l}a_{k,i}\right)z^k.$$

Since $f_i \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ for every $(i=1,2,3,\dots,l)$ so by using Theorem (1) we prove that,

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)\left(\frac{1}{l}\sum_{i=1}^{l}a_{k,i}\right)b_k$$

$$=\frac{1}{l}\sum_{i=1}^{l}\sum_{k=p+1}^{\infty}\left(\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^{m}\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)\right)a_{k,i}\,b_k$$

$$\leq\frac{1}{l}\sum_{i=1}^{l}\alpha p!\,(\lambda+A+B)=\alpha p!\,(\lambda+A+B).$$

There for $h \in \mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. The proof is complete.

6- Convex Linear Combination:

Theorem (8): The class $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ is closed under convex linear combinations.

Proof: Let f and g be the arbitrary elements of $\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$ Then for every $(0<t<1)$, we show that $(1-t)f(z)+tg(z)\in\mathcal{K}_{\mathrm{p}}(\gamma,\beta,m,\lambda,A,B,\alpha)$. Thus , we have

$$(1-t)f(z)+tg(z)=z^p-\sum_{n=\mathrm{p}+1}^{\infty}[(1-t)\,a_n+t\,a_{n,2}]z^n.$$

Therefore

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k\left[(1-t)a_{n,1}+t\,a_{n,2}\right]$$

$$=(1-t)\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k a_{n,1}$$

$$+t\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\left[(k-p)-\alpha\big(\lambda(k-p+1)+(A+B)\big)\right]\delta(k,p-1)b_k a_{n,2}$$

$$\leq(1+t)\alpha p!\,(\lambda+A+B)+t\alpha p!\,(\lambda+A+B)=\alpha p!\,(\lambda+A+B).$$

This completes the proof.

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