On Intuitionistic fuzzy d-ideal of d-algebra

حول مثالي d الضبابي البديهي في جبر d

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ABSTRACT :

we introduce the notion of intuitionistic fuzzy d-ideals of d-algebra and we investigate several interesting properties, and study some relation on intuitionistic fuzzy d-algebra . For the family of all intuitionistic fuzzy d-ideal of d-algebra we introduce the notion of equivalence and investigate some related properties .

Keywords: d-algebra , d-ideal , fuzzy d-ideal , intuitionistic fuzzy set , fuzzy set .

الخلاصة :

قدمنا في هذا البحث مفهوم مثالي d الضبابي البديهي مع تحقيق العديد من الخصائص المهمة ودرسنا بعض العلاقات على جبر d البديهي . وقدمنا مفهوم التكافؤ على عائلة من مثاليات d الضبابية البديهية وتحقيق بعض الخصائص ذات الصلة .

1. Introduction

BCK-algebra and BCI-algebra are two classes of abstract algebras introduced by Y. Imai and K. Iseki [4 ,11] . It is known that BCK-algebras is a proper subclass of BCI-algebras. A d-algebra is another useful generalization of BCK-algebra was introduced by J. Negger and H. S. Kim [2]. J. Negger , Y. B. Jun and H. S. Kim [3] discussed ideal theory in d-algebra. Zadeh introduced the concept of fuzzy set in 1965 [6] . In 1986 Atanassov introduced the concept of " intuitionistic fuzzy set [5] as a generalization of fuzzy set . In [9] Y. B. Jun, J. Neggers and H. S. Kim apply the ideal theory in fuzzy d-ideals of d-algebras . Y. B. Jun , H. S. Kim and D.S. Yoo in [10] introduced the notion of intuitionistic fuzzy d-algebra . In this paper we introduce the notion of intuitionistic fuzzy d-ideals of d-algebra and we investigate several interesting properties, and study some relation on intuitionistic fuzzy d-algebra . For the family of all intuitionistic fuzzy d-ideal of d-algebra we introduce the notion of equivalence and investigate some properties .

2. Background

Definition (2.1) : [2] A non-empty set X with a binary operation $*$ and a constant 0 is called a d-algebra if it's satisfying the following :

i. $x * x = 0$
ii. $0 * x = 0$
iii. $x * y = 0$ and $y * x = 0$ imply that $x = y$

$\forall\ x, y \in X$. We will refer to $x * y$ by xy, And $x \leq y$ if and only if $xy = 0$.

Definition (2.2) : **[3]** Let X be a d-algebra and $\phi \neq A \subseteq X$. Then A is called a d-subalgebra of X if $xy \in A$ whenever $x, y \in A$. And if $\phi \neq A \subseteq X$. Then A is called a BCK-ideal of X if it satisfies:
(D_0) $0 \in A$
(D_1) $xy \in A$ and $y \in A$ implies $x \in A$.

Definition (2.3):[3] In A d-algebra $(X;*,0)$ the $\phi \neq A \subseteq X$. A is called a d-ideal of X if it satisfies:
(D_1) $xy \in A$ and $y \in A$ then $x \in A$.
(D_2) $x \in A$ and $y \in X$ then $xy \in A$, i. e. $AX \subseteq A$.

Definition (2.4): [6] A fuzzy set μ in a non-empty set X is a function from X into the closed interval [0,1] of the real numbers. And If μ be a fuzzy set in X, for all t∈[0,1].The set μ_t={x∈X, μ(x)≥t} is called a ***level subset of*** μ.

Definition (2.5): [7] A fuzzy set μ in d-algebra X is called a fuzzy d-subalgebra of X if it satisfies $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in X$. And it's called a fuzzy BCK-ideal of X if it satisfies the following inequalities :
1) $\mu(0) \geq \mu(x)$, $\forall\, x \in X$
2) $\mu(x) \geq \min\{\mu(xy), \mu(y)\}$, $\forall\, x, y \in X$

Definition (2.6) : [9] Let μ be a fuzzy set in d-algebra X. Then, μ is called a fuzzy d-ideal of X if it satisfies :
(Fd_1) $\mu(x) \geq min\{\mu(xy), \mu(y)\}$.
(Fd_2) $\mu(xy) \geq \mu(x)$. $\forall\ x, y \in X$.

Definition (2.7) [10] : For any $r \in [0,1]$ and a fuzzy set υ in a non empty set of X, the set $U(\upsilon, r) = \{x : \upsilon(x) \geq r\}$ is called an upper r-level cut of υ , and the set $L(\upsilon(x), r) = \{x : \upsilon(x) \leq r\}$ is called a lower r-level cut of υ.

Definition (2.8) [5] : An intuitionistic fuzzy set (IFS for short) A in a set X is an object having the form $A = \{< x, \alpha_A(x), \beta_A(x) >: x \in X\}$, such that $\alpha_A: X \to [0,1]$ and $\beta_A: X \to [0,1]$ denoted the degree of membership (namely $\alpha_A(x)$) and the degree of non membership (namely $\beta_A(x)$) for any elements $x \in X$ to the set A, and $0 \leq \alpha_A(x) + \beta_A(x) \leq 1, \forall\, x \in X$.

For the sake of simplicity, we shall use the notation $A = \{< x, \alpha_A, \beta_A >\}$ instead of $A = \{< x, \alpha_A(x), \beta_A(x) >: x \in X\}$.

Definition (2.9) [1] : Let f be a mapping from a set X to a set Y, if $B = \{< y, \alpha_B(y), \beta_B(y) >: y \in Y\}$ is an IFS in Y , then the pre-image of B under f denoted by $f^{-1}(B)$ is the IFS in X defined by :
$f^{-1}(B) = \{< x, f^{-1}(\alpha_B(x)), f^{-1}(\beta_B(x)) >: x \in X\}$
such that $f^{-1}(\alpha_B(x)) = \alpha_B(f(x))$ and also $f^{-1}(\beta_B(x)) = \beta_B(f(x))$
And if $A = \{< x, \alpha_A(x), \beta_A(x) >: x \in X\}$ is an IFS in X , then the image of A under f denoted by $f(A) = \left\{< y, f_{sup}(\alpha_A(y)), f_{inf}(\beta_A(y)) >: y \in Y\right\}$, where

$$f_{sup}(\alpha_A(y)) = \begin{cases} sup_{x \in f^{-1}(y)} \alpha_A(x) & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$, and

$$f_{inf}(\beta_A(y)) = \begin{cases} inf_{x \in f^{-1}(y)} \beta_A(x) & if\ f^{-1}(y) \neq \emptyset \\ 0 & otherwais \end{cases}$$ for each $y \in Y$.

Definition (2.10) [8] : Let X be a non empty set . IF , A is an IFS from X , then
(i) $\square A = \{< x, \alpha_A(x): x \in X >\} = \{< x, \alpha_A(x), 1 - \alpha_A(x) :\ x \in X >\} = \{< x, \alpha_A(x), \overline{\alpha_A}(x) >\}$
(ii) $\diamond\, A = \{< x, 1 - \beta_A(x) >:\ x \in X\} = \{< x, 1 - \beta_A(x), \beta_A(x): x \in X\} = \left\{< x, \overline{\beta_A}(x), \beta_A(x) >\right\}$

Definition (2.11) [10] : Let X be a d-algebra .
An IFS $A =< x, \alpha_A, \beta_A >$ in X is called an intuitionistic fuzzy d-algebra if satisfies $\alpha_A(xy) \geq \min\{\alpha_A(x), \alpha_A(y)$ and $\beta_A(xy) \leq \max\{\beta_A(x), \beta_A(y)\}$, for all $x, y \in X$.

Proposition(2.12)]10]:Every IFS d-algebra $A =< x, \alpha_A, \beta_A >$ of X satisfies the inequalities $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$.

Definition (2.13) [10]: An IFS $A =< \alpha_A, \beta_A >$ in X *is called* an intuitionistic fuzzy BCK-ideal of X if it satisfies the following inequalities :
(i) $\alpha_A(0) \geq \alpha_A(x)$, $\beta_A(0) \leq \beta_A(x)$
(ii) $\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\}$
(iii) $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\}$, for a $x, y \in X$

3. Intuitionistic fuzzy d-ideal

In what follows let X denoted a d-algebra

Definition(3.1) : An intuitionistic fuzzy d-ideal of X " shortly $IFd-ideal$ " is the $IFS\ A=<\alpha_A,\beta_A>$ in X with the following inequalities :

(IFd_1) $\alpha_A(x)\geq \min\{\alpha_A(xy),\alpha_A(y)\}$

(IFd_2) $\beta_A(x)\leq \max\{\beta_A(xy),\beta_A(y)\}$

(IFd_3) $\alpha_A(xy)\geq \alpha_A(x)$

(IFd_4) $\beta_A(xy)\leq \beta_A(x)$

for all $x,y\in X$

Example (3.2) : Let $X=\{0,p,q\}$ with the following table

*	0	p	q
0	0	0	0
p	q	0	q
q	p	p	0

Note that if $\alpha_A(x)=\begin{cases}0.9 & if x=0\\ 0.01 & if x=1,2\end{cases}$, $\beta_A(x)=\begin{cases}0.1 & if x=0\\ 0.5 & if x=1,2\end{cases}$,

then $A=<\alpha_A,\beta_A>$ is an $IFd-ideal$ of X.

Proposition(3.3) *: If* $IFS\ A=<\alpha_A,\beta_A>$ is an $IFd-ideal$ of X , then $\alpha_A(0)\geq\alpha_A(x)$, $\beta_A(0)\leq\beta_A(x)$ for all $x\in X$.

proof : since $\alpha_A(xx)\geq\alpha_A(x)$ so $\alpha_A(0)\geq\alpha_A(x)$ and since $\beta_A(xx)\leq\beta_A(x)$ so $\beta_A(0)\leq\beta_A(x)$, for all $x\in X$.

Lemma (3.4) : Let an $IFS\ A=<\alpha_A,\beta_A>$ in X be an $IFd-ideal$ of X, If $ab\leq c$, then $\alpha_A(a)\geq\min\{\alpha_A(b),\alpha_A(c)\}$, $\beta_A(a)\leq\max\{\beta_A(b),\beta_A(c)\}$.

proof : Let $a,b,c\in X$ such that $ab\leq c$. Then $(ab)c=0$

$\alpha_A(a)\geq\min\{\alpha_A(ab),\alpha_A(b)\}\geq\min\{\min\{\alpha_A((ab)c),\alpha_A(c)\},\alpha_A(b)\}$

$\geq\min\{\min\{\alpha_A(0),\alpha_A(c)\},\alpha_A(b)\}\geq\min\{\alpha_A(c),\alpha_A(b)\}$.

$\beta_A(a)\leq\max\{\beta_A(ab),\beta_A(b)\}\leq\max\{\max\{\beta_A((ab)c),\beta_A(c)\},\beta_A(b)\}$

$\leq\max\{\max\{\beta_A(0),\beta_A(c)\},\beta_A(b)\}=\max\{\beta_A(c)\},\beta_A(b)\}$.

Lemma (3.5) : Let $A=<\alpha_A,\beta_A>$ be an $IFd-ideal$ of X If $x\leq y$ in X , then $\alpha_A(x)\geq\alpha_A(y)$, $\beta_A(x)\leq\beta_A(y)$.

proof : Let $x,y\in X$ such that $x\leq y$. Then $xy=0$, and

$\alpha_A(x)\geq\min\{\alpha_A(xy),\alpha_A(y)\}=\min\{\alpha_A(0),\alpha_A(y)\}=\alpha_A(y)$

$\beta_A(x)\leq\max\{\beta_A(xy),\beta_A(y)\}=\max\{\beta_A(0),\beta_A(y)\}=\beta_A(y)$.

This complete the proof .

Theorem (3.6) : If $A=<\alpha_A,\beta_A>$ is an $IFd-ideal$ of X then for any $x,c_1,c_2,\dots,c_n\in X$, such that $\left(\dots\left((xc_1)c_2\right)\dots\right)c_n=0$ implies

$\alpha_A(x)\geq\min\{\alpha_A(c_1),\alpha_A(c_2),\dots,\alpha_A(c_n)\}$, and $\beta_A(x)\leq\max\{\beta_A(c_1),\beta_A(c_2),\dots,\beta_A(c_n)\}$

proof *: By using inducation on n and lemma 3.4 and 3.5 .*

Theorem (3.7) : *Every* $IFd-ideal$ is an intuitionistic fuzzy d-algebra

proof : Let $A =< \alpha_A, \beta_A >$ be an $IFd-ideal$, *so* $\alpha_A(xy) \geq \alpha_A(x)$, and $\beta_A(xy) \leq \beta_A(x)$, *then*

$$\alpha_A(xy) \geq \alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\} \geq \min\{\alpha_A(x), \alpha_A(y)\}$$

$\beta_A(xy) \leq \beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\} \leq \max\{\beta_A(x), \beta_A(y)\}$.

The converse of this theorem is not true in general and the next example showing that

Example (3.8) : Let $X = \{0, r, s, t\}$ with the following table :

*	0	r	s	t
0	0	0	0	0
r	r	0	0	r
s	s	r	0	s
t	t	t	t	0

Let $A =< \alpha_A, \beta_A >$ be an IFS in X , *define*

$$\alpha_A(x) = \begin{cases} 0.7 & if x = 0, r, t \\ 0.3 & if x = s \end{cases}, \beta_A(x) = \begin{cases} 0.2 & if x = 0, r, t \\ 0.5 & if x = s \end{cases},$$

It's clear that $A =< \alpha_A, \beta_A >$ is an intuitionistic fuzzy d-algebra ,*but* $\beta_A(s) = 0.5 > 0.2 = \max\{\beta_A(sr), \beta_A(r)\}$, *so* $A =< \alpha_A, \beta_A >$ *is not* $IFd-ideal$.

Theorem (3.9) : Every $IFd-ideal$ in X is an intuitionistic fuzzy BCK-ideal

proof : It's clear .

The converse of This theorem is not true in general and the next example showing that

Example (3.10) : Let $X = \{0, m, n, p, q\}$ with the following cayley table

*	0	m	n	p	q
0	0	0	0	0	0
m	m	0	0	m	0
n	n	n	0	0	n
p	p	p	p	0	p
q	q	p	p	p	0

Let $A =< \alpha_A, \beta_A >$ be an IFS in X , *define*

$$\alpha_A(x) = \begin{cases} 0.5 & if x \neq p \\ 0.03 & if x = p \end{cases}, \quad \beta_A(x) = \begin{cases} 0.05 & if x \neq p \\ 0.3 & if x = p \end{cases},$$

Then $A =< \alpha_A, \beta_A >$ in X *is* an intuitionistic fuzzy BCK-ideal of X *but it is not* an $IFd-ideal$, *since* $\alpha_A(q * p) = \alpha_A(p) = 0.03 \leq \alpha_A(q) = 0.5$, and $\beta_A(q * p) = \beta_A(p) = 0.3 \geq \beta_A(q) = 0.05$.

Theorem (3.11) : If $\{A_i , i \in \Lambda\}$ is an arbitrary family of $IFd-ideal$ of d-algebra, then $\bigcap A_i$ is an $IFd-ideal$ of d-algebra, when $\bigcap A_i = \{< x, \wedge \alpha_{A_i}(x) , \vee \beta_{A_i}(x) \,| x \in X\}$

Proof : Since $\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\}$ and $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\}$ for all $x, y \in X$. Now for all $i \in \Lambda$

$\wedge \alpha_{A_i}(x) \geq \wedge\{\min\{\alpha_{A_i}(xy), \alpha_{A_i}(y)\}\} \geq \{\min\{\wedge \alpha_{A_i}(xy) , \wedge \alpha_{A_i}(y)\}\}$, and

$\vee \beta_{A_i}(x) \leq \vee\{\max\{\beta_{A_i}(xy), \beta_{A_i}(y)\}\} \leq \{\max\{\vee \beta_{A_i}(xy) , \vee \beta_{A_i}(y)\}\}$

And since $\alpha_A(xy) \geq \alpha_A(x)$, $\beta_A(xy) \leq \beta_A(x)$ for all $i \in \Lambda$. Then we have

$\wedge \alpha_{A_i}(xy) \geq \wedge \alpha_{A_i}(x)$, and $\vee \beta_{A_i}(xy) \leq \vee \beta_{A_i}(x)$ for all $x, y \in X$ and for all $i \in \Lambda$. Hence $\cap A_i = \{< x, \wedge \alpha_{A_i}(x) \ , \vee \beta_{A_i}(x) \,|x \in X\}$ is an $IFd - ideal$.

Lemma (3.12) : An $IFS \ A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X if and only if the fuzzy set α_A and $\overline{\beta_A}$ are a fuzzy d-ideal .

proof : Let $A =< \alpha_A, \beta_A >$ be an $IFd - ideal$ of X. It is easy to show α_A is a fuzzy d-ideal of X . For any $x, y \in X$ we have

$$\overline{\beta_A}(x) = 1 - \beta_A(x) \geq 1 - \max\{\beta_A(xy), \beta_A(y)\}$$
$$= \min\{1 - \beta_A(xy), 1 - \beta_A(y)\}$$
$$= \min\{\overline{\beta_A}(xy), \overline{\beta_A}(y)\}$$

And $\overline{\beta_A}(xy) = 1 - \beta_A(xy) \geq 1 - \beta_A(x) = \overline{\beta_A}(x)$

Hence $\overline{\beta_A}$ is a fuzzy d-ideal of X .

Conversely , let α_A and $\overline{\beta_A}$ are fuzzy d-ideal of X . For any $x, y \in X$ we get

$$\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\}$$
$$1 - \beta_A(x) = \overline{\beta_A}(x) = (x) \geq \min\{\overline{\beta_A}(xy), \overline{\beta_A}(y)\}$$
$$= \min\{1 - \beta_A(xy), 1 - \beta_A(y)\}$$
$$= 1 - \max\{\beta_A(xy), \beta_A(y)\}$$

That is $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\}$.

Now , $\alpha_A(xy) \geq \alpha_A(x)$ and $\overline{\beta_A}(xy) \geq \overline{\beta_A}(x)$ so $1 - \beta_A(xy) \geq 1 - \beta_A(x)$ and then $\overline{\beta_A}(xy) \leq \overline{\beta_A}(x)$

Hence $A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X.

Theorem (3.13) : Let $A =< \alpha_A, \beta_A >$ be an IFS in X. Then $A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X if and only if $\square A = \{< \alpha_A, \overline{\alpha_A} >\}$ and $\diamond A = \{< \overline{\beta_A}, \beta_A >\}$ are $IFd - ideal$ of X.

proof : If $A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X, then $\alpha_A = \overline{\overline{\alpha_A}}$ and β_A are fuzzy d-ideal of X from Lemma (3.12) , Hence $\square A = \{< \alpha_A, \overline{\alpha_A} >\}$ and $\diamond A = \{< \overline{\beta_A}, \beta_A >\}$ are $IFd - ideal$ of X. Conversely , if $\square A = \{< \alpha_A, \overline{\alpha_A} >\}$ and $\diamond A = \{< \overline{\beta_A}, \beta_A >\}$ are $IFd - ideal$ of X, then the fuzzy set α_A and $\overline{\beta_A}$ are a fuzzy d-ideal of X . Hence $A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X .

Theorem (3.14) An IFS $\mathrm{A} =< \alpha_A, \beta_A >$ is an $IFd - ideal$ of X if and only if for all $s, t \in [0,1]$ the sets $U(\alpha_A, t)$ and $L(\beta_A(x), s)$ are either empty or d-ideal of X.

proof : Let $A =< \alpha_A, \beta_A > \ IFd - ideal$ of X and $U(\alpha_A, t), L(\beta_A, s)$ non empty set for any $s, t \in [0,1]$. Let $x, y \in X$ such that $xy \in U(\alpha_A, t)$ and $y \in U(\alpha_A, t)$, so $\alpha_A(xy) \geq t$, and $\alpha_A(y) \geq t$ then $\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\} \geq t$, so that $x \in U(\alpha_A, t)$. And let $x \in U(\alpha_A, t)$ and $y \in X$. Then $\alpha_A(x) \geq t$ and $\alpha_A(xy) \geq \alpha_A(x) \geq t$, so $xy \in U(\alpha_A, t)$. Hence $U(\alpha_A, t)$ is a d-ideal in X.

Now, If $x, y \in X$ such that $xy \in L(\beta_A, s)$ and $y \in L(\beta_A, s)$, then $\beta_A(xy) \leq s$ and $\beta_A(y) \leq s$, we get $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\} \leq s$, so that $x \in L(\beta_A, s)$. And let $x \in L(\beta_A, s)$ and $y \in X$. Then $\beta_A(x) \leq s$ and $\beta_A(xy) \leq \beta_A(x) \leq s$, so $xy \in L(\beta_A, s)$. Hence $L(\beta_A, s)$ is a d-ideal in X.

Conversely , suppose that for any $s, t \in [0,1]$, the set $U(\alpha_A, t)$ and $L(\beta_A, s)$ are either empty or d-ideal of X . For all $x \in X$, let $\alpha_A(x) = t$ and $\beta_A(x) = s$. we get $x \in U(\alpha_A, t) \cap L(\beta_A, s)$ and so $U(\alpha_A, t), L(\beta_A, s)$ are non empy set. Since $U(\alpha_A, t)$ and $L(\beta_A, s)$ are d-ideal , if there exist $a, b \in X$ such that $\alpha_A(a) < \min\{\alpha_A(ab), \alpha_A(b)\}$, then by taken $t_0 = \frac{1}{2}(\alpha_A(a) + \min\{\alpha_A(ab), \alpha_A(b)\})$, we have $\alpha_A(a) < t_0 < \min\{\alpha_A(ab), \alpha_A(b)\}$. Hence $a \notin U(\alpha_A, t_0)$, $ab \in U(\alpha_A, t_0)$ and $b \in U(\alpha_A, t_0)$. Thus $U(\alpha_A, t_0)$ is not a d-ideal of X , we get a contradiction .

Now, let $\alpha_A(ab) < \alpha_A(a)$. Then taking $t_0 = \frac{1}{2}(\alpha_A(ab) + \alpha_A(a))$, we have $\alpha_A(ab) < t_0 < \alpha_A(a)$. Hence $a \in U(\alpha_A, t_0)$ and $b \in X$, but $ab \notin U(\alpha_A, t_0)$. So $U(\alpha_A, t_0)$ is not d-ideal we get a contradiction .

Finally , suppose that $a, b \in X$ such that $\beta_A(a) > \max\{\beta_A(ab), \beta_A(b)\}$.

put $s_0 = \frac{1}{2}(\beta_A(a) + \max\{\beta_A(ab), \beta_A(b)\})$. So $\max\{\beta_A(ab), \beta_A(b)\} < s_0 < \beta_A(a)$, there are $ab \in L(\beta_A, s_0)$, $b \in L(\beta_A, s_0)$, but $a \notin L(\beta_A, s_0)$, a contradiction . And let $a, b \in X$ such that $\beta_A(ab) > \beta_A(a)$. Then taking $s_0 = \frac{1}{2}(\beta_A(ab) + \beta_A(a))$. We have
$\beta_A(a) < s_0 < \beta_A(ab)$, therefore $a \in L(\beta_A, s_0)$, and $b \in X$, but $ab \notin L(\beta_A, s_0)$ which is a contradiction. The proof is completed .

Theorem (3.15) : If an $IFS\ A =< \alpha_A, \beta_A >$ is an $IFd-ideal$ of X, then the sets $X_\alpha = \{x \in X: \alpha_A(x) = \alpha_A(0)\}$ and $X_\beta = \{x \in X: \beta_A(x) = \beta_A(0)\}$ are d-ideal of X.
proof : Let $x, y \in X$, let $xy \in X_\alpha$ and $y \in X_\alpha$. Then $\alpha_A(xy) = \alpha_A(0) = \alpha_A(y)$, so $\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\} = \alpha_A(0)$, by proposition (3.3) we get $\alpha_A(x) = \alpha_A(0)$, thus $x \in X_\alpha$. Let $x \in X_\alpha$ and $y \in X$. Then $\alpha_A(x) = \alpha_A(0)$ and so $\alpha_A(xy) \geq \alpha_A(x) = \alpha_A(0)$, so by using proposition (3.3) we get $\alpha_A(xy) = \alpha_A(0)$. Then $xy \in X_\alpha$. Thus X_α is a d-ideal .
Now let $xy \in X_\beta$ and $y \in X_\beta$. Then $\beta_A(xy) = \beta_A(0) = \beta_A(y)$, so $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\} = \beta_A(0)$, by using proposition (3.3) we get $\beta_A(x) = \beta_A(0)$, and we get $x \in X_\beta$. Let $x \in X_\beta$, and $y \in X$. Then $\beta_A(x) = \beta_A(0)$ and so $\beta_A(xy) \leq \beta_A(x) = \beta_A(0)$, so by using proposition (3.3) we get $\beta_A(xy) = \beta_A(0)$. Then $xy \in X_\beta$. Thus X_β is a d-ideal .

Theorem (3.16) : Any d-ideal of X can be realized as both $\alpha - level$ d-ideal and $\beta - level$ d-ideal of some $IFd - ideal$ of X .
proof : Let J be a d-ideal of X and let α_A and β_A be a fuzzy sets in X define by
$\alpha_A(x) = \begin{cases} t & if x \in J \\ 0 & otherwise \end{cases}$, $\beta_A(x) = \begin{cases} s & if x \in J \\ 1 & otherwise \end{cases}$
for any $x \in X$ where t and s are fixed number in (0,1) such that $t + s < 1$. Let $x, y \in X$, and let $xy \in J$ and $y \in J$, then $x \in J$. Hence $\alpha_A(x) \geq \min\{\alpha_A(xy), \alpha_A(y)\}$ *and* $\beta_A(x) \leq \max\{\beta_A(xy), \beta_A(y)\}$. And if $x \in J$, and $, y \in X$ then $xy \in J$, so $\alpha_A(xy) \geq \alpha_A(x)$ *and* $\beta_A(xy) \leq \beta_A(x)$. Thus $A =< \alpha_A, \beta_A >$ is an $IFd - ideal$ ideal of X. It is clear that $U(\alpha_A, t) = S = L(\beta_A, s)$ and the proof is completed .

Theorem (3.17) : Let $\{J_t: t \in \mathrm{I}\}$ be a family of d-ideal of X such that
i)$X = \bigcup_{t \in I} J_t$
ii) $s > t$ if and only if $J_s \subset J_t$ for all $s, t \in \mathrm{I}$.

Then an IFS $A =< \alpha_A, \beta_A >$ in X define by $\alpha_A(x) = \sup\ \{J_t: t \in \mathrm{I}\}$; $\beta_A(x) = \inf\ \{J_t: t \in \mathrm{I}\}$, for all $x \in$ X is an $IFd - ideal$ of X .
proof : It is clear that $U(\alpha_A, t)$ and $L(\beta_A, s)$ are d-ideal of X (by theorem (3.14)), let $t \in [0, \alpha_A(0)]$, and $s \in [\beta_A(0), 1]$. we need to prove That $U(\alpha_A, t)$ is a d-ideal of X we discuses two cases :
1) $t = \sup\{i \in \mathrm{I} : \mathrm{i} < t\}$
2) $t \neq \sup\{i \in \mathrm{I} : \mathrm{i} < t\}$
In the first case we can show that $x \in U(\alpha_A, t) \Leftrightarrow x \in J_i$ for all $\mathrm{i} < t \Leftrightarrow x \in \bigcap_{\mathrm{i}<t} J_i$. Thus $U(\alpha_A, t) = \bigcap_{\mathrm{i}<t} J_i$ is a d-ideal of X .
In the second case, let $U(\alpha_A, t) = \bigcup_{\mathrm{i}\geq t} J_i$, if $x \in \bigcup_{\mathrm{i}\geq t} J_i$ then $x \in J_i$ for some $i \geq t$. So we get $\alpha_A(x) \geq i \geq t$, so that $x \in U(\alpha_A, t)$. Thus $\bigcup_{\mathrm{i}\geq t} J_i \subset U(\alpha_A, t)$. Now let $x \notin \bigcup_{i\geq t} J_i$. So $x \notin J_i$ for all $i \geq t$. But in this case $t \neq \sup\{i \in \mathrm{I} : \mathrm{i} < t\}$, so there exist $\epsilon > 0$ such that $(t - \epsilon, t) \cap \mathrm{I} = \emptyset$. Hence $x \notin J_i$ for all $i > t - \epsilon$, Since if $x \in J_i$, then $i \leq t - \epsilon$. Thus $\alpha_A(x) \leq t - \epsilon \leq t$ and so $x \notin U(\alpha_A, t)$. Therefore $U(\alpha_A, t) \subset \bigcup_{\mathrm{i}\geq t} J_i$. Thus $U(\alpha_A, t) = \bigcup_{\mathrm{i}\geq t} J_i$ which is a d-ideal of X .
Now we need prove That $L(\beta_A, s)$ is a d-ideal of X , and also we have two cases :
i) $s = \inf\{j \in \mathrm{I} : \mathrm{s} < j\}$
ii) $s \neq \inf\{j \in \mathrm{I} : \mathrm{s} < j\}$
In the first case we have $x \in L(\beta_A, s) \Leftrightarrow x \in J_j$ for any $\mathrm{s} < j \Leftrightarrow x \in \bigcap_{\mathrm{s}<j} J_j$. So $L(\beta_A, s) = \bigcap_{\mathrm{s}<j} J_j$ is a d-ideal of X .

In the case (ii) , there exist $\epsilon > 0$ such that $(s, s+\epsilon,) \cap \mathrm{I} = \emptyset$ we need to prove that $L(\beta_A, s) = \bigcup_{s \geq j} J_j$, if $x \in \bigcup_{s \geq j} J_j$, so $x \in J_j$ for some $j \leq s$. then $\beta_A(x) \leq j \leq s$, and we get $x \in L(\beta_A, s)$. Therefore $\bigcup_{s \geq j} J_j \subset L(\beta_A, s)$.

In the Converse side if $x \notin \bigcup_{s \geq j} J_j$. Then $x \notin J_j$ for any $j \leq s$, we get $x \notin J_j$ for all $j > s+\epsilon$, which means that if $x \in J_j$, then $j \geq s+\epsilon$. Hence $\beta_A(x) \geq s+\epsilon \geq s$ so $x \notin L(\beta_A, s)$. Thus $L(\beta_A, s) \subset \bigcup_{s \geq j} J_j$. So $L(\beta_A, s) = \bigcup_{s \geq r} I_r$ which is a d-ideal of X . The proof is completed .

Theorem (3.18) : If f be a d-homorphism function from d-algebra X into a d-algebra Y and D an $IFd-ideal$ of Y . Then $f^{-1}(D)$ is an $IFd-ideal$ in X .

proof: For any $x, y \in X$, we have

$\alpha_{f^{-1}(D)}(xy) = \alpha_D(f(xy)) = \alpha_D(f(x)f(y)) \geq \alpha_D(f(x)) = \alpha_{f^{-1}(D)}(x)$ and

$\beta_{f^{-1}(D)}(xy) = \beta_D\big(f(xy)\big) = \beta_D(f(x)f(y)) \leq \beta_D\big(f(x)\big) = \beta_{f^{-1}(D)}(x)$, and we have

$$\begin{aligned}\min\{\alpha_{f^{-1}(D)}(xy), \alpha_{f^{-1}(D)}(y)\} &= \min\{\alpha_D(f(xy)), \alpha_D(f(y))\}\\ &= \min\{\alpha_D(f(x)f(y)), \alpha_D(f(y))\}\\ &\leq \alpha_D(f(x)) = \alpha_{f^{-1}(D)}(x)\end{aligned}$$

and
$$\begin{aligned}\max\{\beta_{f^{-1}(d)}(xy), \beta_{f^{-1}(D)}(y)\} &= \max\{\beta_D(f(xy)), \beta_D(f(y))\}\\ &= \max\{\beta_D(f(x)f(y)), \beta_D(f(y))\}\\ &\geq \beta_D(f(x)) = \beta_{f^{-1}(D)}(x)\end{aligned}$$

Hence $f^{-1}(D)$ is an $IFd-ideal$ of X .

Theorem (3.19) : : Let f be a d-homorphism function, from d-algebra X into d-algebra Y and let $A =< \alpha_A, \beta_A >$ be an IFS in Y. If $f^{-1}(A) =< \alpha_{f^{-1}(A)}, \beta_{f^{-1}(A)} >$ is an $IFd-ideal$ of X, then $A =< \alpha_A, \beta_A >$ is an $IFd-ideal$ of Y .

proof : Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follow that

$$\begin{aligned}\alpha_A(x) = \alpha_A(f(a)) = \alpha_{f^{-1}(A)}(a) &\geq \min\{\alpha_{f^{-1}(A)}(ab), \alpha_{f^{-1}(A)}(b)\}\\ &= \min\{\alpha_a(f(ab)), \alpha_A(f(b))\}\\ &= \min\{\alpha_a(f(a)f(b)), \alpha_A(f(b))\}\\ &= \min\{\alpha_a(xy), \alpha_A(y)\}\end{aligned}$$

$$\begin{aligned}\beta_A(x) = \beta_A(f(a)) = \beta_{f^{-1}(A)}(a) &\leq \max\{\beta_{f^{-1}(A)}(ab), \beta_{f^{-1}(A)}(b)\}\\ &= \max\{\beta_A(f(ab)), \beta_A(f(b))\}\\ &= \max\{\beta_A(f(a)f(b)), \beta_A(f(b))\}\\ &= \max\{\beta_A(xy), \beta_A(y)\} .\end{aligned}$$

Now

$$\alpha_A(xy) = \alpha_A(f(a)f(b)) = \alpha_A(f(ab)) = \alpha_{f^{-1}(A)}(ab) \geq \alpha_{f^{-1}(A)}(a) = \alpha_a(f(a)) = \alpha_a(x)$$

$$\beta_A(xy) = \beta_A(f(a)f(b)) = \beta_A(f(ab)) = \beta_{f^{-1}(A)}(ab) \leq \beta_{f^{-1}(A)}(a) = \beta_A(f(a)) = \beta_A(x)$$

And the proof is completed.

For any family of $IFd-ideal$ of X shortly $IF(X)$. IF $A =< \alpha_A, \beta_A >$ and $B =< \alpha_B, \beta_B >$ are IFS in $IF(X)$, we can define binary relations U^t and L^t on $IF(X)$ and $t \in [0,1]$ as follows :

$(A, B) \in U^t \Leftrightarrow U(\alpha_A, t) = U(\alpha_B, t)$ and $(A, B) \in L^t \Leftrightarrow L(\beta_A, t) = L(\beta_B, t)$. It is clear that U^t and L^t are equivalence relation on $IF(X)$.

If $A =< \alpha_A, \beta_A > \in IF(X)$, so $[A]_{U^t}$ (respectively $[A]_{L^t}$) is the equivalence class of A modulo U^t (respectively L^t), and denoted by $IF(X)/U^t$ (respectively $IF(X)/L^t$) . so $IF(X)/U^t = \{[A]_{U^t} : A =< \alpha_A, \beta_A > \in IF(X)\}$, (respectively $IF(X)/L^t = \{[A]_{L^t} : A =< \alpha_A, \beta_A > \in IF(X)\}$) .

We will denote for a family of all d-ideal of X by $I(X)$ and for $t \in [0,1]$, let f_t and g_t be maps from $IF(X)$ to $I(X) \cup \emptyset$ defined by $f_t(A) = U(\alpha_A, t)$ and $g_t(A) = L(\beta_A, t)$, for all $A =< \alpha_A, \beta_A > \in IF(X)$, then f_t and g_t are clearly well define .

Theorem (3.20) : For any $t \in (0,1)$ the maps f_t and g_t are surjective from $IF(X)$ to $I(X) \cup \emptyset$.

proof : Let $t \in (0,1)$. for a fuzzy set 0 define by $0(x) = 0$ and 1 defined by $1(x) = 1$ for all $x \in X$ It clear that $0_\sim = (0,1)$ in $IF(X)$, and $f_t(0_\sim) = U(0,t) = \emptyset = L(1,t) = g_t(0_\sim)$.
Let $D(\neq \emptyset) \in I(X)$. For $D_\sim = \left(X_D, \tilde{X}_D\right) \in IF(X)$,We have $f_t(D_\sim) = U(X_D,t) = D$ and $g_t(D_\sim) = L\left(\tilde{X}_D,t\right) = D$. The proof is completed.

When there exists a one - to one corresponding whose domain is a set A and whose range is a set B . This set $A$is said to be equipotent to B , and this relation is an equivalence relation .

Theorem (3.21) : For any $t \in (0,1)$ the quotient sets $IF(X)/U^t$ and $IF(X)/L^t$ are equipotent to $I(X) \cup \emptyset$.

proof : For $t \in (0,1)$, Let $f_t^*: IF(X)/U^t \to I(X) \cup \emptyset$ a map defined by $f_t^*([A]_{U^t}) = f_t(A)$ for all $A =< \alpha_A, \beta_A > \in IF(X)$. If $U(\alpha_A,t) = U(\alpha_B,t)$ and $L(\beta_A,t) = L(\beta_B,t)$ for $A =< \alpha_A, \beta_A >$ and $B =< \alpha_B, \beta_B >$ in $IF(X)$, then $(A,B) \in U^t$ and $(A,B) \in L^t$. Hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. So f_t^* is injective .
Now , Let $D(\neq \emptyset) \in I(X)$. For $D_\sim = \left(X_D, \tilde{X}_D\right) \in IF(X)$. Then we get $f_t^*([D_\sim]_{U^t}) = f_t(D_\sim) = U(X_D,t) = D$. Finally, for $0_\sim = (0,1)$ is in sets $IF(X)$ we have
$f_t^*([0_\sim]_{U^t}) = f_t(0_\sim) = U(0,t) = \emptyset$. Thus f_t^* is surjective .
In the same way we can prove that g_t^* is surjective map and this. and the proof is competed

We can define R^t as a relation on $IF(X)$ for any $t \in [0,1]$ as follows :
$(A,B) \in R^t \Leftrightarrow U(\alpha_A,t) \cap L(\beta_A,t) = U(\alpha_B,t) \cap L(\beta_B,t)$ for $A =< \alpha_A, \beta_A >$ and $B =< \alpha_B, \beta_B >$ in $IF(X)$. And R^t is an equivalence relation on $IF(X)$.

Theorem (3.22) : Let $t \in (0,1)$. Then $\emptyset_t: IF(X) \to I(X) \cup \emptyset$ is a surjective map defined by $\emptyset_t(A) = f_t(A) \cap g_t(A)$ for any $A =< \alpha_A, \beta_A > \in IF(X)$.

proof : Let $0_\sim = (0,1) \in IF(X)$, so $\emptyset_t(0_\sim) = f_t(0_\sim) \cap g_t(0_\sim) = U(0,t) \cap L(1,t) = \emptyset$ for any $t \in (0,1)$. Now let $C \in IF(X)$, so we get $C_\sim = \left(X_C, \tilde{X}_C\right) \in IF(X))$, such that
$\emptyset_t(C_\sim) = f_t(C_\sim) \cap g_t(C_\sim) = U(X_C,t) \cap L\left(\tilde{X}_C,t\right) = C.$

Theorem (3.23) : The quotient sets $IF(X)/R^t$ are equipotent to $I(X) \cup \emptyset$, for any $t \in (0,1)$

proof : Let the map $\emptyset_t^*: IF(X)/R^t \to I(X) \cup \emptyset$ defined by $\emptyset_t^*([A]_{R^t}) = \emptyset_t(A)$
for all $[A]_{R^t} \in IF(X)/R^t$ and for any $t \in (0,1)$. If $\emptyset_t^*([A]_{R^t}) = \emptyset_t^*([B]_{R^t})$ for any $[A]_{R^t} = [B]_{R^t} \in IF(X)/R^t$, so we get $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, and this give that $U(\alpha_A,t) \cap L(\beta_A,t) = U(\alpha_B,t) \cap L(\beta_B,t)$
Thus $(A,B) \in R^t$, and we get $[A]_{R^t} = [B]_{R^t}$ and $[A]_{L^t} = [B]_{L^t}$. So $\emptyset_t^*$ is injective .
Now if $0_\sim = (0,1) \in IF(X)$, we have
$\emptyset_t^*(0_\sim) = \emptyset_t(0) = f_t(0_\sim) \cap g_t(0_\sim) = U(0,t) \cap L(1,t) = \emptyset$
and if $C \in IF(X)$, there exist $C_\sim = \left(X_C, \tilde{X}_C\right) \in IF(X))$, such that

$$\emptyset_t^*(C_\sim) = [C_\sim]_{R^t} = \emptyset_t(C_\sim) = f_t(C_\sim) \cap g_t(C_\sim) = U(X_C,t) \cap L\left(\tilde{X}_c,t\right) = C$$

Hence $\emptyset_t^*$ is surjective , and this complete the proof .

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