Journal University of Kerbala , Vol. 16 No.1 Scientific . 2018

Convex and monotone approximation on ordered vector space

التقريب المحدب والرتيب على الفضاءات المرتبة خطياً

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Abstract

The main aim of this paper is to introduce a result for the shape preserving for function in L_p space on the ordered vector space in terms of the K-th modulus of smoothness.

المستخلص

تناول عملنا التقريب بقيود لتطبيقات في الفضاءات $L_p(I)$ عندما $0 < p < 1$ التي يكون مجالها المقابل مجموعة جزئية من فضاء خطي مرتب.
للتطبيقات التي تنتمي لتلك الفضاءات $L_p(I)$ قمنا بتعريف معيار كاذب.
برهنا نظرية مباشرة للتقريب المحدب للتطبيقات التي تنتمي للفضاءات اعلاه. وكنتيجة لهذه المبرهنة حصلنا على مبرهنة مباشرة للتقريب الرتيب في نفس الفضاءات المعرفة اعلاه.

1. Introduction and Basic Definitions

In leviaten [4] introduced point wise estimations for convex polynomial approximation.

In Gal [2] defined linear operators to prove direct theorem for approximation on normed linear space.

In Gal [3] used classical operators to prove shape preserving estimations in terms of the global smoothness.

In Kopotun, Leviatan, and Shevchuk [5] introduced an articale for the convex polynomial approximation in the uniform norm for real continuous function.

George and Sorin [1] proved a direct inequality for the convex shape preserving approximation of continuous function on ordered space. The direct inequality is a result on the constrained uniform approximation in terms of the first order modulus of smoothness.

In our work, we improve the result of George and sorin [1] for functions in L_p spaces with $p < 1$, and prove direct inequality in terms of k-th modulus of smoothness.

The following definition are needed.

Definition 1.1:[1]

Let $(Y, \|\cdot\|_Y)$ be a normed space. The algebraic polynomial of degree not exceeding $n \in N$ and coefficients C_K in Y has the form

$$P_n(x) = \sum_{k=1}^{n} c_k x^k , \quad x \in [a, b].$$

Definition 1.2:

If f is a map on $[-1,1]$ and of value in Y. Then kth Ditzian–Totik modulus of smoothness of f defind by

$$\omega_{\emptyset}^{k}(f;\delta)_p = \sup_{0\le h\le\delta} \left\|\bar{\Delta}_{h\emptyset(x)}^{k} f(x)\right\|_p ,$$

where $\emptyset^2(x) = 1 - x^2$ and

$$\|f\|_p = \sup_{n\in N} \left(\sum_{i=1}^{n} \frac{c}{n} |f(x_i)|^p\right)^{1/p}, x_i \in [-1,1]$$ where x_i are not equally spaced knots.

$|x_i - x_{i-1}| \leq \frac{c}{n}, c \in IR^+$
$L_p[-1,1] = \{f: [-1,1] \to Y; \|f\|_p < \infty\}$

Definition 1.3:

For a map on $[-1,1]$ and value on Y, the K-th modulus of smoothness is defined by

$\omega_k(f;\delta)_p = \sup_{0\leq h\leq\delta} \{\sup_{n\in N}\{\left\|\Delta_h^k f(x)\right\|_p; x,\ x+kh \in [-1,1]\}\}$

Here $\Delta_h^k f(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x+jh)$.

2. Constrained Approximation

On the normed space $(Y, \|\cdot\|_Y)$ let us define the order relation $\leq_Y$ by the relation that satisfy:

1. If $x \leq_Y y, a \geq 0$, then $ax \leq_Y ay$;
2. If $x \leq_Y y$ and $z \leq_Y w$, then $x + z \leq_Y y + w$.

Definition 2.1:

The map f on $[a,b]$,with value in Y, is called

(i) increasing on $[a,b]$ *if* $x \leq y$,then $f(x) \leq_Y f(y)$;

(ii) convex on $[a,b]$ *if* $f(\alpha x + (1-\alpha)y) \leq_X \alpha f(x) + (1-\alpha)f(y)$,
$\forall x, y \in [a,b], \alpha \in [0,1]$.

Theorem 2,2:

If f is a convex map in $L_p[-1,1]$,$p < 1$, $n \in N$ there exists a convex algebraic polynomial P_n of degree not exceeding n satisfying $\|f - P_n\|_p \leq C_{(p)}\omega_\emptyset^k(f;1/n)_p$
where $C_{(p)}$ is a constant depending on p and it may vary on each step.

Proof: *let* $P_n(f)(x) = \sum_{j=1}^{n} s_j B_j(x)$, Where $s_j = \frac{\Delta_h^k f(x_j)}{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}}$,

$j = 0, \dots, n$ *and* $B_j(x)$ are convex functions of value in *IR* and $x \in [-1,1]$

Now assume $x,\ y\ in[-1,1]$, with $x \leq y$ and that f is convex on $[-1,1]$. It follows that $0_X \leq s_j$, which immediately then

$$P_n(f)[\alpha x + (1-\alpha)y] = \sum_{j=1}^{n} s_j B_j(\alpha x + (1-\alpha)y) ,$$
$$\leq_Y \sum_{j=1}^{n} s_j\left[\alpha B_j(x) + (1-\alpha)B_j(y)\right]$$
$$= \alpha\sum_{j=1}^{n} s_j B_j(x) + (1-\alpha)\sum_{j=1}^{n} s_j B_j(x)$$
$$= \alpha P_n(f)(x) + (1-\alpha)P_n(f)(y)$$

This implies that $P_n(f)$ is convex on $[-1,1]$.

Let us now true the light to the estimate

$$\|f(x) - P_n(x)\|_p = \left\|\sum_{j=1}^{n} f(x) - \sum_{j=0}^{n} s_j B_j(f)(x)\right\|_p$$
$$= \left\|\sum_{j=1}^{n} f(x) - \sum_{j=0}^{n} \frac{\Delta_h^k f(x_j)}{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}} B_j(f)(x)\right\|_p$$
$$= \left\|\frac{\sum_{j=0}^{k} -1^{k-j}\binom{k}{j} f(x) - \sum_{j=1}^{n}\Delta_h^k f(x_j) R_j(f)(x)}{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}}\right\|_p$$
$$= \left\|\frac{\sum_{j=1}^{n}\Delta_h^k f(x_j)[1 - R_j(f)(x)]}{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}}\right\|_p$$
$$\leq_X C_{(p)}\left\|\sum_{j=1}^{n}\Delta_h^k f(x_j)\right\|_p$$

$\leq_X \; C_{(p)}\omega_k(f;\delta)_p$

This ends the proof of the estimate. ■

Corollary 2.3:

If f is an increasing map on *[-1,1]* and of value in Y , and let $n \in N$, then there exists an increasing polynomial of degree not exceeding *n* such that

$If\, f(x) \leq_Y f(y)\ then\ P_n(f)(x) \leq_Y P_n(f)(y)$

Proof: Let f be an increasing function in $L_p[-1,1]$ then so as $P_n(f)(x) = \sum_{j=1}^{n} s_j B_j(x)$, using the same lines of the proof of theorem (2.2) we can get the proof of the result above.

$P_n(f)(x) = \sum_{j=1}^{n} s_j B_j(x) \leq_X \sum_{j=1}^{n} s_j B_j(y) = P_n(f)(y)$ ■

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