Comparison of MINQUE and Simple estimator in general

Gauss -Markov Model

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Abstract:

This paper considers acomparison of MINQUE with simple estimator of σ^2 in the general Gauss-Markov model under the special entropy loss function criterion where the design matrix X need not to have full rank and the dispersion matrix V can be singular A is considered . It is interesting to show that MINQUE is superior to simple estimator OLSE

Introduction:

We consider the general Gauss-Markov model $\{Y, X\beta, \sigma^2\}$, where $E(Y) = X\beta$, $D(Y) = \sigma^2 V$, Eand D being expection and dispersion operators, X and V are known matrices of order $n \times p$ and $n \times n$ respectively, both possibly deficient in rank.

Assume that the model is consistent,i.e. $Y \in R(X \vdots V)$, where R(A) stands for the range(column space) of a matrix A and (A⁝B) denotes the partitioned matrix with A and B placed next to each other.

Suppose we wish to estimate the unknown positive parameter σ^2 by using the competing estimators

$$(1/f)Y'M(MVM)^{+}MY \qquad (1)$$

and

$$(1/f)Y'MY \qquad (2)$$

Where f=rank(X⁝V)-rank(X) is assumed to be positive and $M = I - XX^{+}$, A^{+} stands for the unique Moore-Penrose inverse of a matrix A, and A' denotes the transpose of A.

Formula (1) provides the MINQUE (Minimum Norm Quadrate Unbiased Estimator) for σ^2.

It can also be written as

$$(1/f)Y'(MVM)^{+}Y, \qquad (3)$$

By using Theorem 3.4(a) in Rao (1974), the MINQUE can be represented in further different forms.

An other estimator in Formula (2) is called "simple" estimator or the ordinary least squares estimator, which is obtained simply by replacing V by I in eq (3).

The object of the present note is to make comparison of these two estimators , the criterion we used in this comparison is the risk of the entropy loss function , in which entropy loss function defined by

$$L(\sigma^2, G) = tr(G\sigma^{2^{-1}}) - \log\left|trG\sigma^{2^{-1}}\right| - p \qquad (4)$$

where G is a positive definite matrix.

<u>Comparison of estimators:</u>

The following lemmas are necessary for a proof of our main theorem.

<u>Lemma(1):</u> Let V be $n \times n$ nonnegative definite matrix with rank r and σ^2 be positive parmeters . Random matrix $X \sim N(\mu, \sigma^2 V)$ if and only if $X = \mu + AU$, where A is $n \times r$ matrix with rank r and $AA' = V$, $U \sim N(0, \sigma^2 I_r)$

<u>Proof</u>: Obviously, $X \sim N(\mu, \sigma^2 V)$ if and only if

$Y = (X - M)\sigma^{-\frac{1}{2}} \sim, N(0, I \otimes V)$, that means, $Y_1, Y_2, ..., Y_p\, iid \sim N(0, V)$, where $Y_1, Y_2, ..., Y_p$, are the column vectors of y, we know that for every i=1,..,p, $y_i \sim N(0, V)$ if and only if $Y_i = AW_i$, where A is $n \times r$ matrix with rank r and $AA' = V$, $W \sim N(0, I_r)$.

A proof of this proposition can be found in Rao (1973), P.521.

Let W=(W_1,…,W_P).Then $W \sim \mathrm{N}(0, \mathrm{I})$ Hence $Y \sim \mathrm{N}(0, \mathrm{I} \otimes \mathrm{V})$ if and only if Y= AW, that implies $\mathrm{X} \sim \mathrm{N}\ (\mu, \sigma^2 \mathrm{V})$ if and only if $\mathrm{X} = \mu + AU$ where $U = W\sigma^{-\frac{1}{2}} \sim N\ (0, \sigma^2 I_r)$.The proof of lemma (1) is completed .

<u>Lemma(2)</u>:let X be an $n \times p$ matrix and V $n \times n$ nonnegative definite matrix. Then $rank(VM) = rank(X \vdots V) - rank(X)$ where $M = I - XX^+$.

A proof can be found in Wang and Chow (1994)

<u>Lemma(3)</u>:

$$\hat{\sigma}_m^2 = \frac{\sum_{i=1}^{k} U_i U_i'}{k} \qquad (5)$$

$$\hat{\sigma}_s^2 = \frac{\sum_{i=1}^{k} \lambda_i U_i U_i'}{k}, \qquad (6)$$

Where U_1,…,U_k are iid ~N(0, σ^2) and $\lambda_1 \geq \ldots \geq \lambda_k > 0$ are the positive eigenvalues of MV.

<u>Proof:</u> since MV=0, thus $\hat{\sigma}_m^2$ and $\hat{\sigma}_s^2$ can be written as

$$\hat{\sigma}_m^2 = \varepsilon' M (MVM)^+ M\varepsilon / k$$

$$\hat{\sigma}_s^2 = \varepsilon' M\varepsilon / k$$

In view of lemma 1 and $\varepsilon \sim \mathrm{N}(0, \sigma^2 \mathrm{V})$, r= rank (v) we note that there is an $n \times r$ matrix A with rank r such that

$$\varepsilon \sim \mathrm{A}\delta, \qquad \delta \sim \mathrm{N}(0, \sigma^2 I), \qquad AA' = V,$$

Thus

$$\hat{\sigma}_m^2 = \delta' Q_1 \delta / f$$

(7)

$$\hat{\sigma}^2{}_s = \delta' Q_2 \delta / K$$

(8)

Where

$$Q_1 = A'M(MAA'M)^+ MA, \quad Q_2 = A'MA$$

It is not difficult to verify that $Q_1Q_2=Q_2Q_1$, which implies (see for example , Rao (1973),p.41) that there is an $n \times r$ orthogonal matrix T such that both $T'Q_1T$ and $T'Q_2T$ are diagonal. By using lemma(2), it is can be shown that

$$\text{rank}(Q_1)=\text{rank}(A'MA)= \text{rank}(A'M)=\text{rank}(VM)$$

$$= \text{rank } (V:X)-\text{rank } (X)=k$$

(9)

We note that Q_1 is projection matrix, thus

$$T'Q_1T = \begin{pmatrix} I_K & o \\ o & o \end{pmatrix}$$

(10)

$$T'Q_2T = \begin{pmatrix} \Lambda_K & o \\ o & o \end{pmatrix}$$

(11)

Where $\Lambda_k = diag(\lambda_1,...,\lambda_k)$

Denote

$$U=T'\delta$$

(12)

Then

$$U \sim \text{N } (0,\sigma^2 I \ )$$

(13)

Let $U'=(U_1,\dots,U_r)$ .Then $U_1,\dots,U_r$ are iid ~N(0, σ^2).

Substitute (10),(12) in (8) yields (5)

Also substituting (11),(13) in (8) yields (6)

The proof of lemma 3 is completed

Theorem: under entropy loss function

$$(a) \qquad R(\hat{\sigma}^2{}_m) = R(\hat{\sigma}_s^2), \quad if \quad \lambda_1 = \ldots = \lambda_k = 1,$$

$$(b) \qquad R(\hat{\sigma}^2{}_m) < R(\hat{\sigma}_s^2), \qquad otherwise.$$

Where $R(\hat{\sigma}^2{}_m)$, $R(\hat{\sigma}^2{}_s)$ are the risk of $\hat{\sigma}^2{}_m$, $\hat{\sigma}^2{}_s$ respectively.

Proof:

From eq(4) we have

$$L(\sigma^2, \hat{\sigma}^2{}_s) = (\hat{\sigma}^2{}_s\,\sigma^{-1}) - \log\left|\hat{\sigma}^2{}_s\,\sigma^{-1}\right| - p$$

$$= (\sigma^{-\frac{1}{2}}\,\hat{\sigma}^2{}_s\,\sigma^{-\frac{1}{2}}) - \log\left|\sigma^{-\frac{1}{2}}\,\hat{\sigma}^2{}_s\,\sigma^{-\frac{1}{2}}\right| - p$$

also

$$L(\sigma, \hat{\sigma}^2{}_m) = (\sigma^{-\frac{1}{2}}\,\hat{\sigma}^2{}_m\,\sigma^{-\frac{1}{2}}) - \log\left|\sigma^{-\frac{1}{2}}\,\hat{\sigma}^2{}_m\,\sigma^{-\frac{1}{2}}\right| - p$$

It follows from (6) that

$$L(\sigma, \hat{\sigma}^2{}_s) = \left(\frac{\sum_{i=1}^{n} \lambda_i \sigma^{-\frac{1}{2}} U_i U_i' \sigma^{-\frac{1}{2}}}{k}\right) - \log\left|\frac{\sum_{i=1}^{n} \lambda_i \sigma^{-\frac{1}{2}} U_i U_i' \sigma^{-\frac{1}{2}}}{k}\right| - p$$

$$= \left(\frac{\sum \lambda_i V_i V_i'}{k}\right) \log\left|\frac{\sum \lambda_i V_i V_i'}{k}\right| - p,$$

Where

$$V_i = \sigma^{-\frac{1}{2}} U_i \qquad\qquad V_1, \ldots, V_K \; i.i.d \; N_P(0,1)$$

From (5) we have

$$L(\sigma, \hat{\sigma}^2{}_m) = \left[\left(\frac{\sum V_i V_i'}{k}\right) - \log\left|\frac{\sum V_i V_i'}{k}\right| - p\right]$$

Now :

$$R(\hat{\sigma}^2{}_s) = E\left[\, L(\sigma, \hat{\sigma}^2{}_s)\right]$$

$$= E\left[\, tr\left(\frac{\sum_{i=1}^{k} \lambda_i V_i V_i'}{k}\right) - \log\left|\frac{\sum_{i=1}^{k} \lambda_i V_i V_i'}{k}\right| - p \,\right],$$

$$R(\hat{\sigma}^2{}_m) = E\left[\, L(\sigma, \hat{\sigma}^2{}_m)\right]$$

$$= E\left[\, tr\left(\frac{\sum_{i=1}^{k} V_i V_i'}{k}\right) - \log\left|\frac{\sum_{i=1}^{k} V_i V_i'}{k}\right| - p \,\right],$$

Obviously,

$$R(\hat{\sigma}^2{}_s) = R(\hat{\sigma}^2{}_m) \qquad , \lambda_1 = \ldots = \lambda_k = 1.$$

Let $L(\lambda_1, \ldots, \lambda_k) = E\left\{ tr\left(\frac{\sum_{i=1}^{k} \lambda_i V_i V_i'}{k}\right) - \log\left|\frac{\sum_{i=1}^{k} \lambda_i V_i V_i'}{k}\right| - p \right\}$

Because $L(\lambda_1, \ldots, \lambda_k)$ is a symmetric and convex function, $L(\lambda_1, \ldots, \lambda_k)$ has minimum value, and when $\lambda_1 = \ldots = \lambda_k = \lambda$, $L(\lambda_1, \ldots, \lambda_k)$ takes minimum value.

Now , Let

$$h(\lambda) = E\left\{ tr\left(\frac{\lambda \sum_{i=1}^{k} V_i V_i'}{k} \right) - \log \left| \frac{\lambda \sum_{i=1}^{k} V_i V_i'}{k} \right| - p \right\}$$

$$= E\left\{ tr\left(\frac{\lambda \sum_{i=1}^{k} V_i V_i'}{k} \right) - \log \left(\frac{\lambda^P}{} \left| \frac{\sum_{i=1}^{k} V_i V_i'}{k} \right| \right) - p \right\}$$

$$= E\left\{ tr\left(\frac{\lambda \sum_{i=1}^{k} V_i V_i'}{k} \right) - \log\left(\lambda^P\right) - \log \left| \frac{\sum_{i=1}^{k} V_i V_i'}{k} \right| - p \right\}$$

Then

$$h'(\lambda) = E\left\{ tr\left(\frac{\sum_{i=1}^{k} V_i V_i'}{k} \right) - \frac{p\lambda^{P-1}}{\lambda^P} \right\}$$

So

$$h'(\lambda) = 0 \quad \Rightarrow \quad E\left\{ tr\left(\frac{\sum_{i=1}^{k} V_i V_i'}{k} \right) \right\} - \frac{p}{\lambda} = 0.$$

then $V_j \sim N_p(0, I)$ then $V_j = (V_{j1}, \ldots, V_{jp})'$,

$V_{j1}, \ldots, V_{jp}\, i.i.d \sim \mathrm{N}(0,1)$ so that

$$V_jV_j' = \begin{bmatrix} V_{j1} \\ . \\ . \\ . \\ V_{jp} \end{bmatrix} \begin{pmatrix} V_{j1} & . & . & . & V_{jp} \end{pmatrix}$$

So

$$Etr(V_jV_j') = \sum EV^2{}_{jk} = p$$

Thus

$$h'(\lambda) = E\left\{ tr\sum_{i=1}^{k} V_iV_i' - \frac{P\lambda^{P-1}}{\lambda^P} \right\} = 0$$

$$= \left[P - \frac{P}{\lambda} \right] = 0$$

$$\rightarrow P = \frac{P}{\lambda}$$

$$\rightarrow P = \lambda P$$

$$\therefore \lambda = 1$$

The proof of theorem is completed.■

References:

[1] AL-Mouel,A.S.and AL-Mousaway,N.H. A note on equality of MINQUE and simple estimator in multivariate linear model, Journal of Thi-Qar university 2008,.4,No.3,98-104.

[2] AL-Mouel,A.S.and AL-Mousaway,N.H.Under entropy loss function comparison of MINQUE and simple estimator of covariance matrix.Joural of AL-Qadisihah for pure science 2008,13, No 4, 127-134.

[3] AL-MOUEL A.S.Comparison of MINQUE and Simple Estimator of Covariance Matrix.Journal of East China Normal university (Natural Science),2004,40-45.

[4] Baksalary, J.K,1984.Nonnegative definite and positive definite solutions to the matrix equation $AXA^*=B$. Linear and Multilinear

Algebra16,133-139.

[5] Baksalary, J.K,Hauk, J.,Kala,R.,1980. Nonnegative definite and positive definite solutions to some matrix equations occurring in distribution theory of quadratic forms.Sankhya¯,ser. A42,283-291.

[6] Baksalary, J.K.,Puntanen,S.,Styan, G.P.H.,1990.A property of the dispersion matrix of the best linear unbiased estimator in the general Gauss-Markov model. Sankhya¯,ser.A 52,279-296.

[7] Bhimasankaram,p.,Majumder,D.,1980.Hermitian and nonnegative definite solutions of some matrix equations connected with distribution of quadratic forms.Sankhya ¯,ser.A 42,272-282.

[8] Chikuse, Y.,1981.Representations of the covariance matrix for the robustness in the Gauss-Markov model. Comm.Statist. Theory Methods10,1997-204.

[9] Groβ , J.(1997).A note on equality of MINQUE and simple estimator in the general Gauss-Markov model. Statistics probability letters 35,335-339.

[10] Khatri,C.G.,Mitra, S.K., 1976.Hermitian and Nonnegative definite solutions of linear matrix equation . SIAM J.Appl. Math.31,579-585.

[11] Marsaglia, G.,Styan ,G.P.H., 1974.Equalities and inequalities for ranks of matrices Linear and Multilinear Algebra 2,269-292

[12] Mitra ,S.K.,1968.On a generalized inverse of a matrix and applications.Sankhy ser.A 30,107-114.

[13] Montgomery DC. Design and Analysis of Experiments [M].New York ,Wiley, 1977.

[14] Puntanen,S.,Styan,G.P.H., 1989.The equality of the ordinary least squares estimator and the best linear unbiased estimator .Amer .Statis .43,153-164.

[15] Rao,C.R,1973.Linear Statistical Inference and Its Aapplications.2nd ed .Wiley New York.

[16] Rao,C.R,1974.Projectors, generalized Inference and the BLUEs'
J.Ray.Statist Soc.Ser B36,442-448.

[17] Rao,C.R.,Mitra, s.k.,1971.Generalized Inverse of Matrices and
it.Applications.Wiley ,NewYork ,264,313~323.

[18] Wang S C,Chow S C.A Advanced Linear Models [M] .New York
:Marcel dekker Inc,1994.