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Numerical Solution of Linear Second Order BVPs by Using ExponentialSpline with Finite difference method (FDM)

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Abstract

In this paper, a numerical schemeis proposed for the numerical solutions of the second order boundary value problems using **e**xponential spline with finite differencemethod. The numerical results are obtained for different values of (n). Three test problems have been considered to test the accuracy of the proposed method, and to compare the compute results with exact solutions and other known methods.

Keywords: Exponential spline,finite difference,boundary value problem,exact solution.

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الحل العددي لمسائل القيم الحدودية الخطية من الرتبة الثانية باستعمال السبلاين الاسية مع طريقة الفروقات المحددة

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الخلاصة

في هذا البحث نقترح تطبيق السبلاين الاسية مع طريقة الفروقات المحددة على مسائل القيم الحدودية من الرتبة الثانية. النتائج العددية التي حصلنا عليها كانت لقيم (n) المختلفة, ثلاث مسائل تم اخذها لاختبار دقة الطريقة المقترحة, فضلا عن مقارنتها.

1.Introduction

Solutions of boundary value problems(BVPs) can sufficiently closely be approximated by simple and efficient numerical method . Among these numerical methods are finite difference method, standard 5-point formula, iteration method , relaxation method ect. But here spline method(AbdEl-Salamand Zaki(2010)[2],Caglaret al.(2010)[4]and El-Danaf (2008)[6])with finite difference will be considered.Severd types of physical differential equation can be formulated as boundary value problems. Problems involving the wave equation, such us the determination of normal modes, are often stated as boundary value problems(Sebestyen (2011)[14]). The analysis of these problems involves the eigen functions of a differential operator.

Consider the linear second-order BVP(Gebreslassie et al.(2012)[8] , Keenan (1992)[11]and Sebestyen (2011))

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$
with boundary conditions:

$$y(x_0) = \gamma_1,\ y(x_n) = \gamma_2 \quad (2)$$

where γ_1 and γ_2 are real constants, $p(x), q(x)$ and $r(x)$ are continuous functions defined on the interval $[x_0, x_n]$.

In this paper, we solved this problem numerically using theexponentialspline method and comparing itwith analytic(exact)solution. The paper is organized as follows: In sections

2, we introduce derivation of our method.Analysis of the method is presented in section 3.In section 4, exponentialspline solutions are displayed. The convergence analysis is shown in section 5, In section 6, numerical results are included to show the applications and advantages of exponential spline method.

2. Derivation OfExponentialSpline Schemes

We divide the interval $[a,b]$ into $n+1$ equal subintervals using the grid points $x_i = a + ih,\ i = 0,1,2,\dots n+1$ with $x_0 = a,\ x_{n+1} = b,\ h = \frac{b-a}{n+1}$ where n is arbitrary positive integer.Let $y(x)$ be the exact solution of the problem(1-2) and y_i be an approximation to $y(x_i)$ obtained by the segment $E_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}).Each Exponentialspline segment $E_i(x)$ has the form((Zahra .W (2009)[15]) .

$$E_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x - x_i) + d_i \quad (3)$$

for $i = 0,1,\dots,n$

where a_i, b_i, c_i and d_i are arbitrary non zero constant and k is frequency of the trigonometric functions, which will be used to raise the accuracy of the method and Eq.(3)reduce tocubicpolynomial spline function in $[a,b]$ as $k \to 0$

To derive the expression for the coefficients of Eq.(3) in term of $y_{i+1}, y_i, S_i, S_{i+1}$ we first define:

$$E_i(x_i) = y_i, \qquad E_i(x_{i+1}) = y_{i+1}$$

$$E_i''(x_i) = S_i, E_i''(x_{i+1}) = S_{i+1} \quad (4)$$

After some algebraic manipulation of the Eq. (3) and using the notations defined in Eq. (4), the following expressions follow:

$$a_i = \frac{h^2(-S_i e^{-\theta} + S_{i+1})}{\theta^2(e^{\theta} - e^{-\theta})}$$

$$b_i = \frac{h^2(S_i e^{\theta} - S_{i+1})}{\theta^2(e^{\theta} - e^{-\theta})}$$

$$c_i = \frac{-h(s_{i+1} - k^2 y_{i+1} + k^2 y_i - s_i)}{\theta^2}$$

$$d_i = \frac{(\theta^2 y_i - h^2 S_i)}{\theta^2}$$

where$\theta = hk$ and $i = 1,2, \dots n$. Using the continuity of the first derivative at (x_i, y_i) , that is$E_i'(x_i) = E_{i-1}'(x_i)$ we obtain the following relations ;

$$y_{i-1} - 2y_i + y_{i+1} = \left(\frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right) S_{i-1} + 2\left(\frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right) S_i + \left(\frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}\right) S_{i+1} \quad (5)$$

For simplicity, Equation(5) can be re-written in the following form:

$$y_{i-1} - 2y_i + y_{i+1} = h^2(\alpha S_{i-1} + 2\beta S_i + \alpha S_{i+1}) \quad (6)$$

where

$$\alpha = \frac{(\theta e^{-\theta} - 2\theta e^{\theta} + e^{2\theta} + e^{-2\theta} - 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}$$

$$\beta = \frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}$$

the local truncation errors,t_i, $i = 1, 2, \dots, n-1$, in equation (6) can beobtained as follows:

First we re-write the equation.(6) in the form ,

$$y_{i-1} - 2y_i + y_{i+1} = h^2\left(\alpha y_{i-1}^{(2)} + 2\beta y_i^{(2)} + \alpha y_{i+1}^{(2)}\right) \quad (7)$$

the terms $y_{i-1}^{(2)}$, $y_i^{(2)}$ and $y_{i+1}^{(2)}$ in equation (7) are expanded around the point x_i using Taylor series and the expressions for$t_i, i = 1,2, \dots, n$.can be obtained,

$$t_i = h^2(1 - 2\alpha - 2\beta) y_i'' + \frac{h^4}{12}(1 - 12\alpha) y_i^{(4)} + \frac{h^6}{360}(1 - 30\alpha) y_i^{(6)} + O(h^8) \quad (8)$$

Now the Equation (8) gives rise to the class of methods of different orders as follow:

Second order method:

For any choice of arbitrary α and β with $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$ and $\alpha+\beta=\frac{1}{2}$,then local truncation error is

$$t_i=\frac{-h^4}{12}y_i^{(4)}+O\left(h^6\right), \qquad \text{for } i=1,2,\ldots,n, \qquad (9)$$

Fourth order method:

If $\alpha=\frac{1}{12}$ and $\beta=\frac{5}{12}$ $t_i=O(h^6)$then the resulting method is fourth order method. Then the local truncation error is :

$$t_i=\frac{-h^6}{240}y_i^{(6)}+O\left(h^8\right), \qquad \text{for } i=1,2,\ldots,n, \quad (10)$$

3. Analysis of the method

To illustrate the application of the spline method developed in the previous section, we consider the linear of second–order BVP that is given in Eq. (1). At the grid point (x_i,y_i) ,the proposed problem in Equation (1) may be discretized by

$$S_i+p(x)y'+q(x)y=r(x) \quad (11)$$

Solving Equation(11)for S_i, we get

$S_i=-p(x)y'-q(x)y+r(x)$ (12)and approximate first derivative by using finite-difference .

The following approximation for the first-order derivative of y in Eq. (12) can be used (Caglar et al (2010)[4] and Rashidinia et al.(2008)[13]).

$$y'_{i-1}=\frac{-y_{i+1}+4y_i-3y_{i-1}}{2h}$$

$$y'_i=\frac{y_{i+1}-y_{i-1}}{2h}$$

$$y'_{i+1}=\frac{3y_{i+1}-4y_i+y_{i-1}}{2h}$$

So Eq. (12) becomes

$$\left.\begin{aligned} S_{i-1} &= -\frac{p(x_{i-1})(-y_{i+1}+4y_i-3y_{i-1})}{2h} - q(x_{i-1})y_{i-1} + r(x_{i-1}) \\ S_i &= -\frac{p(x_i)(y_{i+1}-y_{i-1})}{2h} - q(x_i)y_i + r(x_i) \\ S_{i+1} &= -\frac{p(x_{i+1})(3y_{i+1}-4y_i+y_{i-1})}{2h} - q(x_{i+1})y_{i+1} + r(x_{i+1}) \end{aligned}\right\} \quad (13)$$

Substituting Eq. (13)in Eq. (6),we get the following equation:

$$\begin{aligned} &\left[-1+h\alpha\left(\frac{3p(x_{i-1})}{2}-\frac{p(x_{i+1})}{2}\right)+h\beta p(x_i)-h^2\alpha q(x_{i-1})\right]y_{i-1} \\ &+\left[2+2h\alpha\left(-p(x_{i-1})+p(x_{i+1})\right)-2h^2\beta q(x_i)\right]y_i \\ &+\left[-1+h\alpha\left(\frac{p(x_{i-1})}{2}-\frac{3p(x_{i+1})}{2}\right)-h\beta p(x_i)-h^2\alpha q(x_{i+1})\right]y_{i+1} \\ &\qquad = -h^2\alpha r(x_{i-1})-2h^2\beta r(x_i)-h^2\alpha r(x_{i+1}) \end{aligned} \quad (14)$$

4.ExponentialSpline Solutions.

The tri-diagonal linear system (14) can be written in the following matrix form,

$$AY + h^2 DR = G \quad (15)$$

where $A = N + hBQ - h^2 Bq$ and A is tri-diagonal and diagonally matrix of order $(n-1)$

Here $N = (L_{ij})$ is a tri-diagonal matrix defined by

$$L_{ij} = \begin{cases} 2 & i = j = 1, 2, \ldots, n-1, \\ -1 & |i-j| = 1, \\ 0 & otherwise. \end{cases}$$

And $BQ = Z_{ij}$ and $Bq = U_{ij}$ are tri-diagonal matrices defined by

$$Z_{ij}=\begin{cases}2\alpha\left(-p\left(x_0\right)+p\left(x_2\right)\right), & i=j=1,\\ \alpha\left(\dfrac{3p\left(x_{i-1}\right)}{2}-\dfrac{p\left(x_{i+1}\right)}{2}\right)-h\beta p\left(x_i\right), & i>j,\\ 2\alpha\left(-p\left(x_{i-1}\right)+p\left(x_{i+1}\right)\right), & i=j,\\ \alpha\left(\dfrac{p\left(x_{i-1}\right)}{2}-\dfrac{3p\left(x_{i+1}\right)}{2}\right)-h\beta p\left(x_i\right), & i<j,\\ 2\alpha\left(-p\left(x_{n-2}\right)+p\left(x_n\right)\right), & i=j=n-1,\end{cases}$$

and

$$U_{ij}=\begin{cases}2\beta q\left(x_i\right), & i=j=1,2,\ldots,n-1.\\ \alpha q\left(x_{i-1}\right), & i>j\\ \alpha q\left(x_{i+1}\right), & i<j\end{cases}$$

$$R=\left(r\left(x_1\right),r\left(x_2\right),\ldots,r\left(x_{n-2}\right),r\left(x_{n-1}\right)\right)^T,\quad Y=\left(y_1,y_2,\ldots,y_{n-2},y_{n-1}\right)^T$$

The tri-diagonal matrix D is defined by

$$\begin{bmatrix}2\beta & \alpha & & & & \\ \alpha & 2\beta & \alpha & & & \\ 0 & \alpha & 2\beta & \alpha & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha & 2\beta & \alpha & \\ & & & \alpha & 2\beta & \end{bmatrix}$$

$$G=\left(g_1,0,0,0,\ldots,0,g_{n-1}\right)^T$$

Where

$$g_1=-h^2\alpha r\left(x_0\right)-\left(-1+h\alpha\left(\frac{3p\left(x_0\right)}{2}-\frac{p\left(x_2\right)}{2}\right)+h\beta p\left(x_1\right)-h^2\alpha q\left(x_0\right)\right)\gamma_1$$

$g_i=0$, for $i=2,3,\ldots,n-2,$

$$g_{n-1}=-h^2\alpha r\left(x_n\right)-\left(-1+h\alpha\left(\frac{p\left(x_{n-2}\right)}{2}-\frac{3p\left(x_n\right)}{2}\right)-h\beta p\left(x_{n-1}\right)-h^2\alpha q\left(x_{n-1}\right)\right)\gamma_n$$

We assume that ,

$$\overline{Y}=\left(y\left(x_1\right),y\left(x_2\right),...,y\left(x_{n-1}\right)\right)^T$$

Be the exact solution of the given boundary value problem (1) at nodal point x_i ,for $i=0,1,...,n-1.$ and then we have

$$A\overline{Y}+h^2DR=T\left(h\right)+G \quad (16)$$

If we subtract equation (15) from equation (16) we get the following

$$A\left(\overline{Y}-Y\right)=AE=T\left(h\right) \quad (17)$$

5. Convergence Analysis

Our main purpose now is to derive a bound $\|E\|_\infty$ We now turn back to the error equation in (17) and rewrite it in the form ,

$$E=A^{-1}T=\left[N+hBQ-h^2Bq\right]^{-1}T=\left[I+N^{-1}\left(hBQ-h^2Bq\right)\right]^{-1}N^{-1}T$$

$$\|E\|_\infty\leq\left\|\left[I+N^{-1}\left(hBQ-h^2Bq\right)\right]^{-1}\right\|_\infty\left\|N^{-1}\right\|_\infty\|T\|_\infty \quad (18)$$

In order to derive the bound on $\|E\|_\infty$, the following two lemmas are needed.

Lemma 1(Quarteroni el at (2000)[12]):If G is a square matrix of order n and $\|G\|<1$, then the $(I+G)^{-1}$ exists and $\|(I+G)^{-1}\|\leq(1-\|G\|)^{-1}$.

Lemma 2:The matrix $\left(N+hBQ-h^2BR\right)$ is nonsingular if $\|p\|_\infty<\dfrac{8h\varepsilon}{\left(a-b\right)^2\left(8\alpha+2\beta\right)}$ and $\|q\|_\infty<\dfrac{8\left(1-\varepsilon\right)}{\left(a-b\right)^2}$ where $0<\varepsilon<1$.

Proof:

Since, $A=\left(N+hBQ-h^2BR\right)=\left[I+N^{-1}\left(hBQ-h^2Bq\right)\right]N$ and the matrix N is nonsingular, so to prove A nonsingular it is sufficient to show $\left[I+N^{-1}\left(hBQ-h^2Bq\right)\right]$ nonsingular.

Since

$$\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le \left\|N^{-1}\right\|_\infty\left(\left\|hBQ-h^2Bq\right\|_\infty\right)\le\left\|N^{-1}\right\|_\infty\left(\left\|hBQ\right\|_\infty+\left\|h^2Bq\right\|_\infty\right) \quad (19)$$

Moreover,

$$\left\|N^{-1}\right\|_\infty \le \frac{(b-a)^2}{8h^2}$$ (Rashidinia et al.(2008)[13]).

, $\left\|hBQ\right\|_\infty \le h\left(8\alpha+2\beta\right)\left\|p\right\|_\infty$ and $\left\|hBq\right\|_\infty \le h^2\left\|q\right\|_\infty$

Where

$\left\|p\right\|_\infty = \max_{a\le x_i\le b}\left|p\left(x_i\right)\right|$ and $\left\|q\right\|_\infty = \max_{a\le x_i\le b}\left|q\left(x_i\right)\right|$

There for ,substituting $\left\|N\right\|_\infty$, $\left\|hBQ\right\|_\infty$, $\left\|h^2Bq\right\|_\infty$ in equation (19) we get ,

$$\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le \frac{(b-a)^2}{8h}\left(8\alpha+2\beta\right)\left\|p\right\|_\infty+\frac{(b-a)^2}{8}\left\|q\right\|_\infty . \quad (20)$$

Since ,

$$\begin{cases}\left\|p\right\|_\infty < \dfrac{8h\varepsilon}{(a-b)^2\left(8\alpha+2\beta\right)}\\ \left\|q\right\|_\infty < \dfrac{8(1-\varepsilon)}{(a-b)^2}\end{cases} \quad (21)$$

There for equations (20) and (21) leads to $\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le 1$ .From Lemma (1) , it show that the matrix A is nonsingular. Since $\left\|N^{-1}\left(hBQ-h^2Bq\right)\right\|_\infty \le 1$ so using Lemma(1) and equation (18) follow that

$$\left\|E\right\|_\infty \le \frac{\left\|N^{-1}\right\|_\infty\left\|T\right\|_\infty}{1-\left\|N^{-1}\right\|_\infty\left\|\left(hBQ-h^2Bq\right)\right\|_\infty} .$$

From equation(9) we have

$$\left\|T_i\right\|_\infty = \frac{1}{12}h^4M_4 , \quad M_4=\max_{a\le x\le b}\left|y^{(4)}(x)\right|$$

then,

$$\|E\|_\infty \le \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1-\|N^{-1}\|_\infty \|(hBQ-h^2Bq)\|_\infty} \cong O(h^2) \quad . \quad (22)$$

Also from equation (10) we have

$$\|T_i\|_\infty = \frac{1}{240} h^6 M_6 \;,\quad M_6 = \max_{a\le x\le b} \left|y^{(6)}(x)\right|$$

Then,

$$\|E\|_\infty \le \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1-\|N^{-1}\|_\infty \|(hBQ-h^2Bq)\|_\infty} \cong O(h^4) \quad . \quad (23)$$

Theorem 1

Let $y(x)$ is the exact solution of the continuous BVP (1) with the boundary condition (2) and let $y(x_i)$,for $i=1,2,\dots,n-1$, satisfies the discrete BVP (15). Further, if $e_i = y(x_i) - y_i$ then

1- $\|E\|_\infty \cong O(h^2)$ for second order convergent method.

2- $\|E\|_\infty \cong O(h^4)$ for fourth order convergent method.

which are given by (22) and (23), neglecting all errors due to round off.

6. Numerical Examples

In this section we illustrate the numerical technique discussed in the previous sections by the following BVP ofsystem(1-2), in order to illustrate the comparative performance of our method(14) over other existing methods. All calculations are implemented by Maple 13.

Example1 : Consider the linear BVP of the form(Kalyani andRama Chandra Rao (2013)[10]):

$$y'' - 2y' - 2y = -2 \quad 0 \le x \le 1$$

$$y(0) = 0, y(1) = 0 ,$$

with exact solution

$$y=\frac{(e^{(1-\sqrt{3})}-1)e^{(1+\sqrt{3})x}}{e^{(1+\sqrt{3})}-e^{(1-\sqrt{3})}}+\frac{(1-e^{(1+\sqrt{3})x})}{e^{(1+\sqrt{3})}-e^{(1-\sqrt{3})}}+1 .$$

The numerical solutions of the Example (1) are present in the Table 1 that contains results for the proposed methodfor different values of$h$, Table 2 shows compared of our method with B-spline method (Cagar el at.(2006)[3]), finite differencemethod (Fang et at.(2002)[12]) ,Non-polynomial spline(Kalyani andRama Chandra Rao(2013)[10]) and exact solution. Figure 1 shows the exact and numerical solution for $h=0.1$.

Example2 : Consider the linear BVP of the form(Islam (2005)[9]):

$$y''+\frac{2}{x}y'-\frac{2}{x^2}y=\frac{\sin(lnx)}{x^2} \quad 1\le x\le 2$$

$$y(1)=1, y(2)=2 .$$

with exact solution is given by

$$y=c_1x+\frac{c_2}{x^2}-\frac{3}{10}\sin(lnx)-\frac{1}{10}\cos(lnx)$$

where $c_1=1.1392070132$ and $c_2=-0.0392070132$.

The numerical solutions of the Example(2) are presented in Table 3 for different values of h and subintervals n. Table 4 shows compared of the proposed method with Non-polynomial (Islam(2005)) and exact solution .The Figure2 shows the comparison of the exact and numerical solutions (for choosing $h=\frac{1}{32}$)

Example 3 :Consider the linear BVP of the form (Rashidiniaet al.(2008)[13]):

$$-\frac{d}{dx}\left(e^{1-x}\frac{dy}{dx}\right)=1+e^{1-x} \quad 0\le x\le 1,$$

$$y(0)=0, y(1)=0 .$$

with exact solution,

$y(x) = x(1 - e^{x-1})$.

The numerical solutions of the Example(3) are presented in Table5for different values of subintervals n.Table 6shows the exact solution and numerical solution for subintervals n ,Table5shows compared of the proposed methodwithCubic spline method(Rashidinia et al.(2008)[13])and exact solution for example(3).

Table 1. The numerical solution and exact solution of example (1) at different values of subintervals

x	Numerical solution				Exact Solution
	$h=\frac{1}{10}$	$h=\frac{1}{20}$	$h=\frac{1}{40}$	$h=\frac{1}{80}$	
0.1	0.05714756655	0.05717993120	0.05717460293	0.05717101498	0.0571708338
0.2	0.1061514945	0.1061003070	0.1060001848	0.1060925428	0.1060901588
0.3	0.1461674618	0.1460685360	0.14603215870	0.14601677343	0.1460159614
0.4	0.1760853324	0.1759177761	0.1758537871	0.1758449733	0.1758424928
0.5	0.1944229959	0.1941016040	0.1939217769	0.1939418051	0.1939952135
0.6	0.1986860689	0.1983149125	0.1982227516	0.1982957012	0.1982920867
0.7	0.1862827254	0.1858004522	0.1857730781	0.1857677320	0.1857606816
0.8	0.1527795521	0.1525071170	0.1524171796	0.1523047004	0.1523972596
0.9	0.09217985516	0.09290168746	0.09286012672	0.09284474541	0.0928496490

Table 2. Comparison ofthe proposed methodwith other methods for example(1),when $h=\frac{1}{10}$

x	Finite difference method (Fang et at.(2002))	B-spline method (Cagar el at.(2006))	Non-polynomial spline(Kalyaniand Rama Chandra Rao(2013))	The proposed method	Exact solution
0.1	0.0399	0.05657	0.05730	0.05724756655	0.0571708338
0.2	0.0897	0.104297	0.106357	0.1062514945	0.1060901588
0.3	0.1302	0.1464167	0.146421	0.1462674618	0.1460159614
0.4	0.1604	0.1763667	0.176383	0.1761853324	0.1758424928
0.5	0.1787	0.193999	0.194657	0.1944229959	0.1939952135
0.6	0.1827	0.1982966	0.199044	0.1987860689	0.1982920867
0.7	0.1695	0.18655	0.186544	0.1862827254	0.1857606816
0.8	0.1350	0.153771	0.15311	0.1528795521	0.1523972596
0.9	0.0735	0.909366	0.093335	0.09317985516	0.0928496490

Table 3. The numerical solution and exact solution of example (2) at different values of subintervals

x	**Numerical solution**				Exact Solution
	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$	
$\frac{1}{1}$	1.057684044	1.057684614	1.057684773	1.057684829	1.057684826
$\frac{1}{1}$	1.116067219	1.116068588	1.116068955	1.116069075	1.116069082

$\frac{1}{1}$	1.175173380	1.175175547	1.175176120	1.175176299	1.175176319
$\frac{2}{1}$	1.235002859	1.235005712	1.235006459	1.235006685	1.235006720
$\frac{2}{1}$	1.295541307	1.295544686	1.295545563	1.295545825	1.295545868
$\frac{2}{1}$	1.356765435	1.356769160	1.356770124	1.356770405	1.356770456
$\frac{2}{1}$	1.418646779	1.418650674	1.418651677	1.418651959	1.418652026
$\frac{2}{1}$	1.481154166	1.481158064	1.481159066	1.481159333	1.481159417
$\frac{2}{1}$	1.544255337	1.544259093	1.544260055	1.544260294	1.544260395
$\frac{2}{1}$	1.607918004	1.607921491	1.607922378	1.607922598	1.607922696
$\frac{2}{1}$	1.672110531	1.672113637	1.672114420	1.672114619	1.672114704
$\frac{2}{1}$	1.736802353	1.736804980	1.736805636	1.736805819	1.736805879
$\frac{2}{1}$	1.801964231	1.801966295	1.801966811	1.801966974	1.801967000
$\frac{3}{1}$	1.867568386	1.867569818	1.867570175	1.867570296	1.867570305
$\frac{3}{1}$	1.933588556	1.933589297	1.933589481	1.933589562	1.933589548

Table 4. Comparison of the proposed method with other method ((Islam

(2005)) for example (2) , when $h = \frac{1}{16}$

x	**Numerical solution**				
	Non-polynomial (Islam (2005))	The proposed method	Absolute error The proposed method	Absolute error Non-polynomial ((Islam (2005))	Exact solution
$\frac{1}{1}$	1.057689584	1.057684044	0.000004758	0.000006438	1.057684826
$\frac{1}{1}$	1.116075308	1.116067219	0.000006226	0.000009312	1.116069082
$\frac{1}{1}$	1.175182204	1.175173380	0.000005885	0.000010111	1.175176319
$\frac{2}{1}$	1.235011351	1.235002859	0.000004631	0.000009738	1.235006720
$\frac{2}{1}$	1.295548878	1.295541307	0.000003010	0.000008748	1.295545868
$\frac{2}{1}$	1.356771804	1.356765435	0.000001348	0.000007482	1.356770456
$\frac{2}{1}$	1.418651862	1.418646779	0.0000001641	0.000006140	1.418652026
$\frac{2}{1}$	1.481158003	1.481154166	0.000001414	0.000004852	1.481159417

$\frac{2}{1}$	1.544258044	1.544255337	0.000002351	0.000003681	1.544260395
$\frac{2}{1}$	1.607919746	1.607918004	0.000002950	0.000002662	1.607922696
$\frac{2}{1}$	1.672109294	1.672117914	0.000003210	0.00000541	1.672114704
$\frac{2}{1}$	1.672108584	1.736802734	0.000003145	0.00000411	1.736805879
$\frac{2}{1}$	1.801960270	1.801969770	0.000002770	0.00000673	1.801967000
$\frac{3}{1}$	1.801965760	1.867572410	0.000002105	0.00000124	1.867570305
$\frac{3}{1}$	1.933597558	1.933588374	0.000001174	0.00000801	1.933589548

Table 5. The numerical solution and exact solution of example (3) at different values ofsubintervals**.**

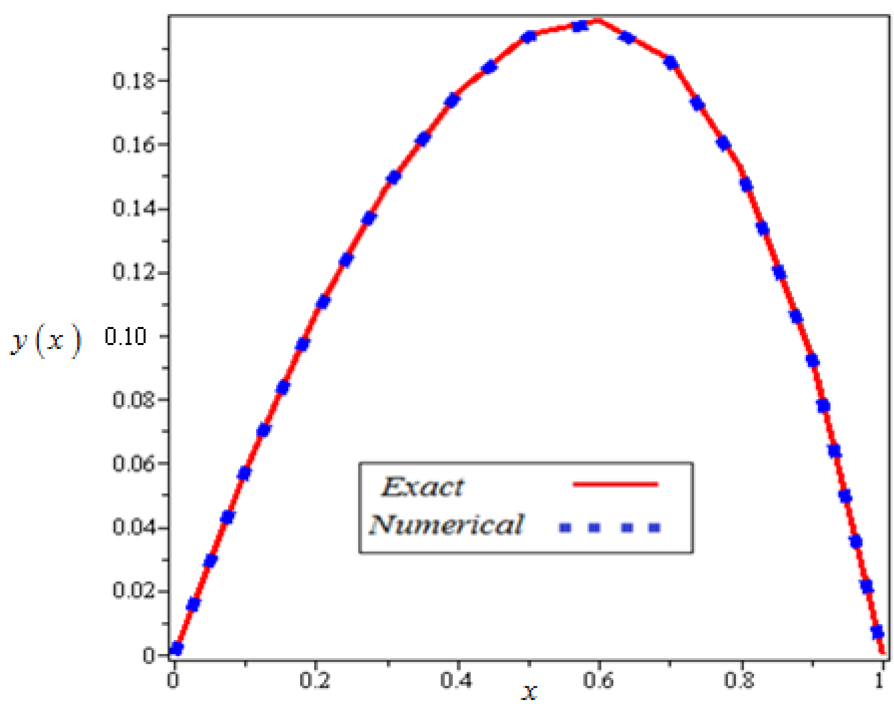
x	**Numerical solution**				Exact Solution
	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$	
[illegible]	0.03803901942	0.03802823942	0.03802554599	0.03802487270	0.03802464833
[illegible]	0.07292034196	0.07289926781	0.07289400240	0.07289268622	0.07289224754

3/1	0.1043383717	0.1043076226	0.1042999399	0.1042980194	0.1042973794
4/1	0.1319612268	0.1319215717	0.1319116640	0.1319091874	0.1319083618
5/1	0.1554286194	0.1553809962	0.1553690977	0.1553661229	0.1553651319
6/1	0.1743495725	0.1742951074	0.1742814996	0.1742780977	0.1742769643
7/1	0.1882999590	0.1882399903	0.1882250075	0.1882212625	0.1882200142
8/1	0.1968198529	0.1967559551	0.1967399910	0.1967360004	0.1967346702
9/1	0.1994106748	0.1993446855	0.1993281985	0.1993240776	0.1993227039
1/1	0.1955321191	0.1954661690	0.1954496922	0.1954455733	0.1954442008
1/1	0.1845988460	0.1845353918	0.1845195387	0.1845155759	0.1845142551
1/1	0.1659769186	0.1659187788	0.1659042537	0.1659006232	0.1658994127
1/1	0.1389799701	0.1389303649	0.1389179716	0.1389148741	0.1389138415
1/1	0.1028650771	0.1028276703	0.1028183249	0.1028159888	0.1028152102
1/1	0.05682831897	0.05680726615	0.05680200654	0.05680069185	0.0568002536

Table 6. Compared of the proposed method with other method (Rashidinia et al.(2008)) and exact solution when $h = \frac{1}{16}$.

x	*Exact solution*	*The proposed method*	*Cubic spline method (Rashidinia et al.(2008))*	*Absolute error The proposed method*	*Absolute error Cubic spline method (Rashidinia et al.(2008))*
[illegible]	*0.03802464833*	*0.03803901942*	*0.03804327288*	*0.00001437109*	*0.00001862455*
[illegible]	*0.07289224754*	*0.07292034196*	*0.07292860910*	*0.00002809442*	*0.00003636156*
[illegible]	*0.1042973794*	*0.1043383717*	*0.1043503642*	*0.0000409923*	*0.0000529848*
[illegible]	*0.1319083618*	*0.1319612268*	*0.1319766022*	*0.0000528650*	*0.0000682404*
[illegible]	*0.1553651319*	*0.1554286194*	*0.1554469757*	*0.0000634875*	*0.0000818438*
[illegible]	*0.1742769643*	*0.1743495725*	*0.1743704418*	*0.0000726082*	*0.0000934775*
[illegible]	*0.1882200142*	*0.1882999590*	*0.1883228008*	*0.0000799448*	*0.0001027866*
[illegible]	*0.1967346702*	*0.1968198529*	*0.1968440464*	*0.0000851827*	*0.0001093762*
[illegible]	*0.1993227039*	*0.1994106748*	*0.1994355106*	*0.0000879709*	*0.0001128067*

$\frac{1}{1}$	0.1954442008	0.1955321191	0.1955567909	0.0000879183	0.0001125901
$\frac{1}{1}$	0.1845142551	0.1845988460	0.1846224409	0.0000845909	0.0001081858
$\frac{1}{1}$	0.1658994127	0.1659769186	0.1659984064	0.0000775059	0.0000989937
$\frac{1}{1}$	0.1389138415	0.1389799701	0.1389981924	0.0000661286	0.0000843509
$\frac{1}{1}$	0.1028152102	0.1028650771	0.1028787347	0.0000498669	0.0000635245
$\frac{1}{1}$	0.0568002536	0.05682831897	0.05683595867	0.00002806537	0.00003570507



Figure(1): Comparison of exact and numerical solutions of example(1) for $h = 0.1$.

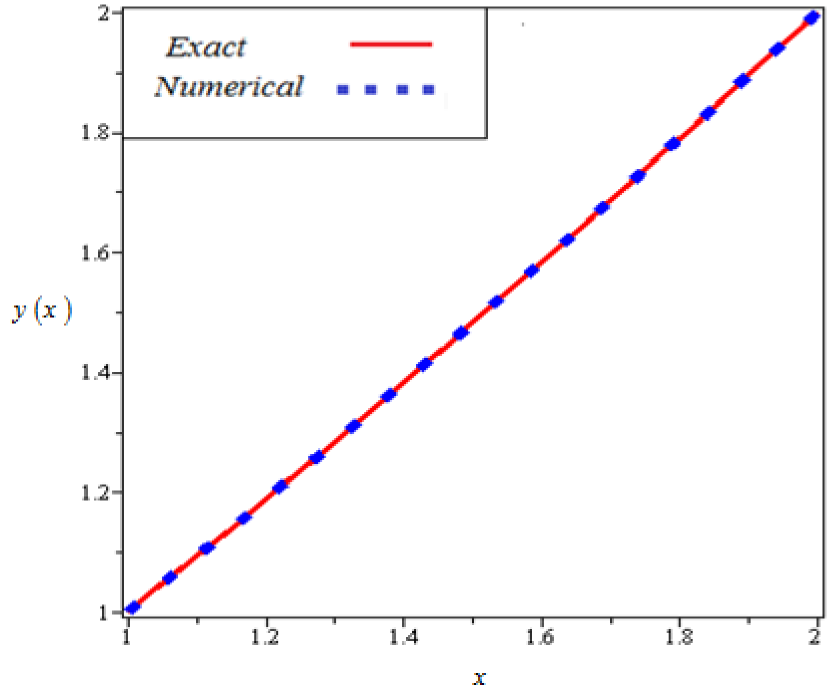


Figure (2): Comparison of exact and numerical solutions of example (2) for $h = \frac{1}{32}$.

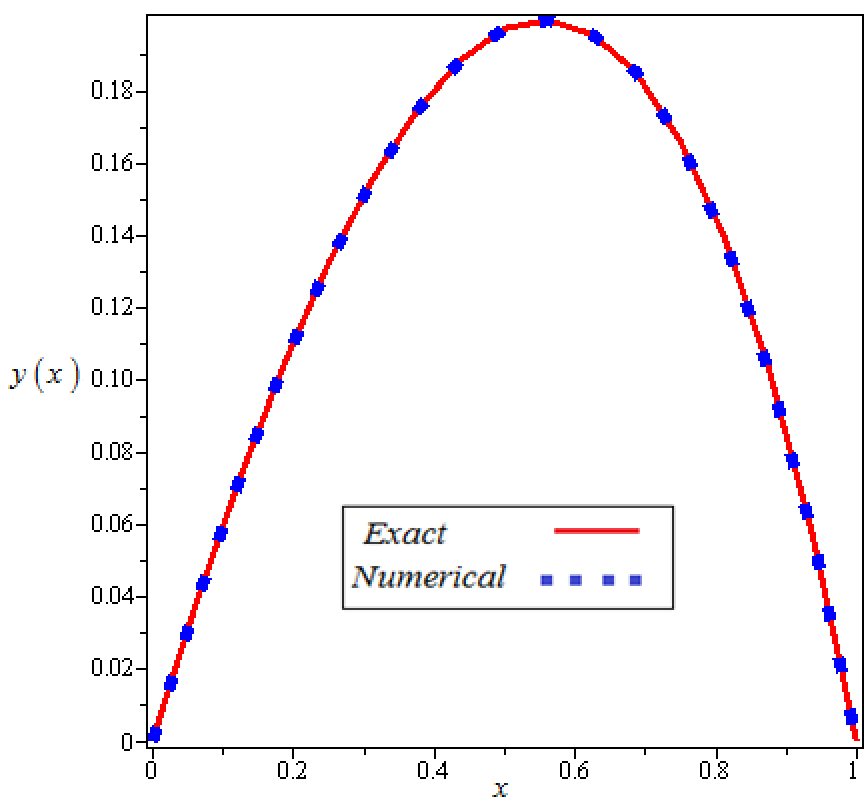


Figure3. Comparison of exact and numerical solutions

of example(3) for $h = \frac{1}{16}$.

7. Conclusion

In this paper, Exponentialspline functions are used to develop a class of numerical methods for solving boundary value problems. The numerical results obtained by using the method described in this study give acceptable results. We have concluded that numerical results converge to the exact solution when h approach to zero.

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