# Homogenuous n-parameters Abstract Cauchy Problem

# مسألة كوشي n - متغيرة المجردة المتجانسة

Ayat A. Neamah
Karbala University \ College of Education Pure Sciences

**Abstract**

We use the semigroup theory to study the homogeneus n-parameter ACP

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1,...,t_i,...,t_n) = H_i u(t_1,...,t_i,...,t_n) \\ i = 1,2,...,n \quad , \quad t_i \in (0,T_i] \\ u(0) = x \quad : \quad x \in \bigcap_{i=1}^{n} D(H_i) \end{cases}$$

where X is a Banach space, $H_i: D(H_i) \subseteq X \to X, i = 1,2,...,n$ is a densely – defined closed linear operator. We discuss the existence and uniqueness of solution of n-ACP. In fact, we claim that if $(H_1, H_2, ..., H_n)$ is the generator of a $C_0$- n parameter semigroup $W(t_1, t_2, ..., t_n)$ : $t_i \in (0,T_i]$ then n-ACP with some conditions has a unique solution.

**الملخص**

نستخدم نظرية شبه الزمر لبحث عن مسألة كوشي n - متغيرة المجردة المتجانسة

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1,...,t_i,...,t_n) = H_i u(t_1,...,t_i,...,t_n) \\ i = 1,2,...,n \quad , \quad t_i \in (0,T_i] \\ u(0) = x \quad : \quad x \in \bigcap_{i=1}^{n} D(H_i) \end{cases}$$

حيث يكون X فضاء باناخ و $H_i: D(H_i) \subseteq X \to X, i = 1,2,...,n$ عامل خطي مغلق ، محدد و مكثف. ثم نناقش وجود و تفرد الحل لمسألة كوشي n - متغيرة المجردة المتجانسة . في الواقع ، نحن ندعي أنه إذا كان $(H_1, H_2, ..., H_n)$ المولد من شبه زمر n– متغيرة $t_i \in (0,T_i]$ : $(t_1, t_2, ..., t_n)$ ،يوجد حل فريد من نوعه لمسألة كوشي n - متغيرة المجردة المتجانسة مع بعض الشروط.

## 1.Introduction

**Defenition.1.1[1]:** suppose that $X$ is a Banach space and $Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$ Where $D(A) = \{x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exist}\}$ then $A: D(A) \subseteq X \to X$ is called an infinitesimal generator of one parameter semigroup $\{T(t)\}_{t \geq 0}$ and *D(A)* is called the domain of *A.*

**Defenition.1.2[1]:** Suppose that $X$ is a Banach space and $A: D(A) \subseteq X \to X$ is a linear operator, then $\rho(A) = \{\lambda \in \mathbb{C} \ : \lambda I - A: D(A) \subseteq X \to X \ \ invertable\}$ is resolvent set and if $\lambda \in \rho(A)$ then we define the operator $R(\lambda : A)$ by $R(\lambda : A) = (\lambda I - A)^{-1}$.

**Definition.1.3[1]:** suppose that $X$ is a Banach space and $A: D(A) \subseteq X \to X$ is a linear operator. The initial value problem (1) is called an one parameter Abstract Cauchy Problem (ACP) which depend with the $(A, D(A))$ and $x \in X$

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) \quad : \quad \forall t \geq 0 \\ u(0) = x \end{cases} \qquad \textbf{(1)}$$

**Definition.1.4[1]:** Suppose that $D(H_i) \subseteq X : i = 1, 2, ..., n$ and $x \in X_1 = \bigcap_{i=1}^{n} D(H_i)$, then $\|\cdot\|_1$ is called graph norm where $\|x\|_1 = \|x\| + \sum_{i=1}^{n} \|H_i x\|$.

**Theorem.1.5:** suppose that $A: D(A) \subseteq X \to X$ is a closed operator, then the following statements are equivalent:

**a.** For each $x \in D(A)$ there exists a unique solution for (1).

**b**. The part $A_1 = A\big|_{X_1}$ is the infinitesimal generator of a $C_0$-one-parameter semigroup of operators on the Banach space $X_1 = (D(A), \|\cdot\|_A)$ where $\|\cdot\|_A$ is the graph norm on $D(\mathrm{A})$.

**Proof:** see [2]

**Defenition.1.6[2]:** Let $X$ is a Banach space, *B(X)* is the Banach space of all bounded linear operators on $X$ and $\mathbb{R}_+^n = \{ (t_1, t_2, \dots, t_n) \ : \ t_i \geq 0 \ , i = 1, 2, \dots, n \}$. By a n-parameter semigroup of operators, we mean a homomorphism $W: (\mathbb{R}_+^n, +) \to B(X)$ for which *W(0)=I* and denote it by $(X, \mathbb{R}_+^n, W)$.

**Definition.1.7[1]:** Let $\{e_i\}_{i=1}^n$ be the standard basis of $R^n$ then the function $u_i : R^+ \to B(X)$ where $u_i(s) = W(se_i) \ : \ i = 1, 2, ..., n \ , \ \forall s \in R^+$ is called the component of W.

**Proposition.1.8[7]:**a.Let $(X, \mathbb{R}_+^n, W)$ be a n-parameter semigroup then the component $\{u_i(s)\}_{s \geq 0} : \forall i = 1, 2, ..., n$ of W is an one parameter semigroup of operators.

b. If for each $\{u_i(s)\}_{s \geq 0} : \forall i = 1, 2, ..., n$ be an one parameter semigroup of operators such that for each $1 \leq i, j \leq n$ : $u_i(s)\, u_j(s') = u_j(s')\, u_i(s)$, then $W(t_1, t_2, ..., t_n) = u_1(t_1)\, u_2(t_2) ... u_n(t_n)$ is a n-parameters semigroup.

**Definition.1.9[1]:** The n-parameter semigroup $(X, \mathbb{R}_+^n, W)$ is called strongly continuous($C_0$ –n parameter semigroup)if the component $\{u_i(s)\}_{s \geq 0} : \forall i = 1, 2, ..., n$ are strongly continuous.

**Definition.1.10[1]:** The n-parameter semigroup $(X, \mathbb{R}_+^n, W)$ is called uniformly continuous if the component $\{u_i(s)\}_{s \geq 0} : \forall i = 1, 2, ..., n$ are uniformly continuous.

**Theorem.1.11:** The n-parameter semigroup $(X, \mathbb{R}_+^n, W)$ is strongly continuous if and only if $\lim_{t \to 0} W(t)x = x \ : \ t \in R_+^n$ and it is uniformly continuous if and only if $\lim_{t \to 0} W(t) = I \ : \ t \in R_+^n$.

**Proof :** see [7].

**Definition.1.12[1]:** Let $(X, \mathbb{R}_+^n, W)$ be a n-parameter semigroup and $H_i$ 's : i=1,2,…,n be the infinitesimal generators of $\{u_i(s)\}_{s \geq 0}$ of *W*, then $(H_1, H_2, ..., H_n)$ is called an infinitesimal generator of $(X, \mathbb{R}_+^n, W)$.

If *W* be a $C_0$-n parameter semigroup of linear operators, then by Hill–Yosida theorem , $H_i$ are closed linear operators and $\overline{D(H_i)} = X$ .

**Theorem.1.13:** Let $(X, R_+^n, W)$ be a $C_0$–n parameter semigroup and $(H_1, H_2, ..., H_n)$ be an infinitesimal generator , then

*a.* If $x \in D(H_i)$, then for each $W(t)x \in D(H_i)$ , $t \in R_+^n$: $H_i W(t)x = W(t) H_i x \quad : \quad i = 1, 2, ..., n$

*b.* $\bigcap_{i=1}^{n} \bigcap_{j=1}^{\infty} D(H_i^{\,j})$ is dense in X.

*c.* For each $1 \le i, j \le n$ , $D(H_i) \bigcap D(H_i H_j) \subseteq D(H_j H_i)$ and for each $x \in D(H_i) \bigcap D(H_i H_j)$ : $H_i H_j(x) = H_j H_i(x)$.

**Proof :** see [1].

## 2.Basic results

**Definition.2.1[2]:** Let $X$ be a Banach space and $H_i : D(H_i) \subseteq X \to X$ be a closed linear operator and for each $i = 1, 2, ..., n$, $T_i > 0$, then $u : [0, T_1] \times ... \times [0, T_n] \to X$ is a solution of the initial value problem.

$$\begin{cases} \dfrac{\partial}{\partial t_i} u(t_1, ..., t_i, ..., t_n) = H_i u(t_1, ..., t_i, ..., t_n) \\ i = 1, 2, ..., n \quad , \quad t_i \in (0, T_i] \\ u(0) = x \quad : \quad x \in \bigcap_{i=1}^{n} D(H_i), \end{cases} \qquad \textbf{(2)}$$

where $(t_1, ..., t_n) \in R_+^n$, $u(t_1, ..., t_n) \in \bigcap_{i=1}^{n} D(H_i)$ and $u$ is satisfies in n-parameters abstract Cauchy problem.

**Definition.2.2[2]:** Let $T = (T_1, ..., T_n) \in R_+^n$ where $T_i > 0$ : $i = 1, 2, ..., n$ , then $I_T = [0, T_1] \times ... \times [0, T_n]$ is called n-cell.

**Theorem.2.3:** suppose that $I_T$ be a n-cell and $(H_1, ..., H_n)$ be the infinitesimal generator of $C_0$-n-parameter semigroup $(X, R_+^n, W)$ then (2) has the unique solution for each $x \in \bigcap_{i=1}^{n} D(H_i)$.

**Proof:** suppose that $I_T$ be an arbitrary, $\{e_i\}_{i=1}^{n}$ be a standard basis and $H_i : i = 1, 2, ...n$ be the infinitesimal generator of one-parameter semigroup $\{W(se_i)\}_{s \ge 0}$ and Let $x \in \bigcap_{i=1}^{n} D(H_i)$, then we define the function $u : I_T \to X$ where $u(t) = W(t)x$. First, $u(t)$ is a solution of n-ACP because $\dfrac{\partial}{\partial t_i} u(t_1, ..., t_i, ..., t_n) = \dfrac{\partial}{\partial t_i} W(t_1, ..., t_i, ..., t_n)x = H_i W(t_1, ..., t_i, ..., t_n)x = H_i u(t_1, ..., t_i, ..., t_n)$ and $u(0) = W(0)x = I(x) = x$. For proving the uniqueness of solution it is enough to show that

(2) has no proper (that is, nonzero) solution for the initial value x = 0. Theorem.1.4 shows that for each $i = 1, 2, \dots, n$, the initial value problem

$$\begin{cases} \dfrac{du^i(s)}{ds} = H_i u^i(s) \\ u^i(0) = x \end{cases} \qquad \textbf{(3)}$$

has a unique solution for each $x \in D(H_i)$. By definition of solution, we know that for $t \in I_T : u(t) \in \bigcap_{i=1}^{n} D(H_i)$ which is a solution of (2). So for initial value $x = u(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_n) \in D(H_i)$,

$u^i(s) = u(t_1,\dots,t_{i-1},s,t_{i+1},\dots,t_n)$ and $v^i(s) = W(se_i)u(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_n)$ are two solutions of (3). Uniqueness of solution of (3) implies that for each $i = 1,2,\dots,n,\ j = 1,2,\dots,n$ such that $i \neq j$ and $0 \le t_j \le T_j$ , $0 \le s \le T_i$ :

$$W(se_i)\, u(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_n) = v^i(s) = u^i(s) = u(t_1,\dots,t_{i-1},s,t_{i+1},\dots,t_n)$$

Hence, for $t = \sum_{i=1}^{n} t_i e_i \in I_T$ , we have

$$\begin{aligned} u(t) &= u(t_1,\dots,t_n) \\ &= W(t_1e_1)\, u(0,t_2,\dots,t_n) && i = 1 ,\ s = t_1 \\ &= W(t_1e_1)\big(W(t_2e_2)\, u(0,0,t_3,\dots,t_n)\big) && i = 2 ,\ s = t_2 \\ &= \dots \\ &= W(t_1e_1)\, W(t_2e_2) \dots\ W(t_ne_n)\, u(0,0,\dots,0) \\ &= W(\sum_{i=1}^{n} t_i e_i)\, u(0) \\ &= W(t)(0) \\ &= 0 \end{aligned}$$

So, for each $t \in I_T : u(t) = 0$ . Hence, n-ACP can not have a proper solution for x=0 or for each $x \in \bigcap_{i=1}^{n} D(H_i)$ has a unique solution. ∎

In the next theorem, we try to close to the reverse of the previous theorem. In fact, we are going to show that for positive n-cells $I_T, I_{T'}$ where $I_{T'} \subseteq I_T$ such that if (2) has a unique solution for each $x \in X_1$ then there exists a $C_0$-n-parameter semigroup $(X_1, R_+^n, W)$ with the infinitesimal generator $(K_1,\dots,K_n)$ for which $W(t)x = u(t,x)$ the unique solution of (2), also for $x \in D(K_i) : H_i(x) = K_i(x)$.

**Theorem.2.4:** Suppose $H_i : i = 1,2,\dots,n$ are closed linear operators and for positive n-cells $I_T, I_{T'}$ where $I_{T'} \subseteq I_T$ if n-ACP(2) has a unique solution for each $x \in X_1$ then there exist a $C_0$-n-parameter semigroup $(X_1, R_+^n, W)$ of linear bounded operators with the infinitesimal generator

$(K_1,...,K_n)$ such that for $t \in I_T, x \in X_1 : W(t)x = u(t,x)$ where $u(t,x)$ is the unique solution of (2) for the initial value x. and for $x \in D(K_i) : H_i(x) = K_i(x)$.

**Proof:** Let $u(t,x)$ be the unique solution of (2) for $x \in X_1$. For $t \in I_T$, we define the operator $W_1(t) : X_1 \to X$ by $W_1(t)x = u(t,x)$. It is clear that $W_1(t)$ is well-defined and a linear operator, we claim that $W_1(t)$ is bounded. Define the mapping $\phi : X_1 \to C^1(I_T, X_1)$ by $\phi(x)(t) = W_1(t)x$, where $C^1(I_T, X_1)$ is the Banach space of all continuous $X_1$-valued functions on $I_T$ with continuous partial derivative, equiped with the supremum norm, $\phi$ is linear, we prove it is closed. Suppose in $X_1$ , $x_m \to x$ and in $C^1(I_T, X_1)$, $\phi(x_m) \to f$ . So by integrating (2) implies that for each $i = 1,2,..., n$, $n \in N$ , $t = (t_1, t_2,..., t_n) \in I_T$ we have

$$W_1(t_1,...,t_n)x_m = W_1(t_1,...,t_{i-1},0,t_{i+1},...,t_n)x_m + \int_0^{t_i} H_i W_1(t_1,...,t_{i-1},s,t_{i+1},...,t_n)x_m ds \qquad \textbf{(4)}$$

Let $m \to \infty$ , so $\sup_{t \in I_T} \| \varphi(x_m)t - f(t) \|_1 \to 0$, so by (4) and the closedness of $H_i$, we have for each $i = 1, 2, ..., n$

$$f(t_1,...,t_n) = f(t_1,...,t_{i-1},0,t_{i+1},...,t_n) + \int_0^{t_i} H_i f(t_1,...,t_{i-1},s,t_{i+1},...,t_n) ds \qquad \textbf{(5)}$$

But $f \in C^1(I_T, X_1)$. So by (5), we have for each $i = 1, 2,..., n$ and $t \in (0, T_i]$ :

$$\begin{cases} \dfrac{\partial}{\partial t_i} f(t_1,..., t_i,..., t_n) = H_i f(t_1,..., t_n) \\ f(0) = \lim\limits_{m\to\infty} \phi(x_m)(0) = \lim\limits_{m\to\infty} W_1(0)x_m = \lim\limits_{m\to\infty} x_m = x \end{cases}$$

Hence, $f$ is a solution of (2) for the initial value $x$. But, the solution is unique so, $f(t) = W_1(t)x = \phi(x)t \quad : \quad t \in I_T$ . It means that $\phi$ is closed operator. Thus,

$$\sup_{\|x\|_1 \le 1} \| W_1(.)x \|_\infty = \sup_{\|x\|_1 \le 1} \left( \sup_{t \in I_T} \| W_1(t)x \|_1 \right) = M < \infty$$

Thus, for each $t \in I_T$ , $W_1(t)$ is a bounded operator on $X_1$. Now, let $T'' = (T_1'',..., T_n'')$ where $T_i'' = \min\{ T_i' , T_i - T_i' \}$. We are going to show that for each $t, t' \in I_{T''} : W_1(t + t') = W_1(t)W_1(t')$ , First we notice that for $t \in I_{T'}$ and $t' \in I_{T''}$, $t_i \le T_i'$ , $t_i' \le T_i - T_i'$ . so $t_i + t_i' \le T_i$ or $t + t' \in I_T$. Let $t' \in I_{T''}$ be fixed, for $x \in X_1$ define $v(.) : I_{T'} \to X_1$ by $v(t) = W_1(t + t')x$. Trivially v(t) is a solution of

$$\begin{cases} \dfrac{\partial}{\partial t_i} v(t_1,...,t_i,...,t_n) = H_i v(t_1,...,t_i,...,t_n) \\ i = 1,2,...,n \quad , \quad t_i \in (0,T_i] \\ v(0) = W_1(t')x \quad : \quad x \in \bigcap_{i=1}^{n} D(H_i) \end{cases}$$

and $u(.) : I_{T'} \to X_1$ by $u(t) = W_1(t)W_1(t')x$ is also a solution of above problem but the solution of the above problem is unique in $I_{T'}$ ,so $W_1(t+t')x = v(t) = u(t) = W_1(t)W_1(t')x$ **(6)**

Now, we can extend $W_1(t)$ to a n-parameter $C_0$-semigroup of operators. Let $s = (s_1, s_2,..., s_n) \in R_+^n$ and choose $m_i \in N$ and $r_i \in (0, \frac{T_i''}{2}]$ such that $s_i = m_i \frac{T_i''}{2} + r_i : i = 1,2,...,n$ .Suppose $r = (r_1, r_2,..., r_n) \in I_{T''/2}$ so, define $W(s)x = W_1(r)\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i}\right](x)$. By the previous parts of proof the operators in the right hand side of the last equality commute and are bounded linear operators on $X_1$. Now, if $s, s' \in R_+^n, s_i' = m_i' \frac{T_i''}{2} + r_i'$ then

$s + s' = (m_i + m_i' + m_i'')\frac{T_i''}{2} + r_i''$ and $r_i + r_i' = m_i'' \frac{T_i''}{2} + r_i''$ where $m_i''$ is one or zero. So,

$$W(s+s')x = W_1(r_1'',...,r_n'')\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i+m_i'+m_i''}\right](x)$$

$$= W_1(r_1'',...,r_n'')\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i''}\right]\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i+m_i'}\right](x)$$

And in other side, we have

$$W(s)W(s')x = W_1(r)\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i}\right]W_1(r')\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i'}\right](x)$$

$$= W_1(r)W_1(r')\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i+m_i'}\right](x)$$

So because $r, r' \in I_{\frac{T''}{2}}$ and $r + r' \in I_{T''} \subseteq I_{T'}$. So, $W_1(r+r') = W_1(r)W_1(r')$ and again from (6), we have

$$W_1(r+r') = W_1\left(\sum_{i=1}^{n}(m_i'' \frac{T_i''}{2} + r_i'')e_i\right)$$

$$= W_1(r_1'', r_2'', ..., r_n'')\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i''}\right]$$

So because $m_i'' = 1$ or $m_i'' = 0$ hence $W(s+s') = W(s)W(s')x$ and $u(t,x)$ is continuous so $W$ is a $C_0$-n parameter semigroup and for $s \in I_T$, $W(s)x = W_1(s)(x)$ because

$$\frac{\partial}{\partial s_i}W(s)x = \frac{\partial}{\partial r_i}W_1(r)\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i}\right](x) = H_i\, W_1(r)\left[\prod_{i=1}^{n}\left(W_1(\frac{T_i''}{2}e_i)\right)^{m_i}\right](x) = H_i\, W(s)x$$

Now if $(K_1,...,K_n)$ is the generator of $W$ and $x \in D(K_i) \subseteq X_1 = \bigcap_{i=1}^{n} D(H_i)$, then

$\lim_{s\to 0}\frac{W(se_i)x - x}{s} = K_i(x)$, but $x \in D(H_i)$ so

$$\lim_{s\to 0}\frac{W(se_i)x - x}{s} = \frac{\partial}{\partial t_i}W(0,0,...,0)x = H_i W(0)x = H_i x$$

So $H_i(x) = K_i(x)$ ■

In the previous Theorem, we could replace the assumption of the existence of a unique solution for (2) in $I_T$ and $I_{T'}$, by the assumption that (2) has a unique solution in $I_T$ and whole of $R_+^n$, which appears this assumption is stronger than our hypothesis. As another application of $C_0$-n-parameter semigroups, we shall show that for a closed linear operator $A: D(A) \subseteq X \to X$, the next n-parameter initial value problem does not have a unique solution in both $I_T$ and $I_{T'}$ where $I_{T'} \subseteq I_T$ for each $x \in D(A)$

$$\begin{cases} \sum_{i=1}^{n}\frac{\partial}{\partial t_i}u(t_1,...,t_n) = Au(t_1,...,t_n) \\ t = (t_1,...,t_n) \in I_T \\ u(0) = x \quad : \quad x \in D(A) \end{cases} \qquad \textbf{(7)}$$

But this initial value problem can have a solution, for example if $(H_1,...,H_n)$ is a generator of a $C_0$-n-parameter semigroup $(X, R_+^n, W)$ and $A = H_1 + ... + H_n$, then obviously for $x \in \bigcap_{i=1}^{n} D(H_i)$ : $u(t) = W(t)x$ is a solution of (7) in any positive n-cell $I_T$. We recall that the closure of $\bigcap_{i=1}^{n} D(H_i)$ in $\|\cdot\|_A$ is $D(A)$.

Before proving our claim we need the following lemmas.

**Lemma.2.6:** suppose that $J$ be a real interval and $P, Q: J \to B(X)$ be two strongly continuous operators on $J$, and suppose for each $x \in D$, $P(.)x: J \to X$, $Q(.)x: J \to X$ be differentiable functions where $D$ is a subspace of $X$ and $D$ is Invariant under Q(t), then for each $x \in D$, the function $(PQ)(.)x : J \to X$ by $PQ(t)x = P(t)\,Q(t)x$ differentiable and

$$\frac{d}{dt}(P(.)\ Q(.)x)(t_0) = \frac{d}{dt}(P(.)\ Q(t_0)x)(t_0) + P(t_0)\left(\frac{d}{dt}(Q(.)x)\right)(t_0)$$

**Proof :** see [2]. ■

**Lemma.2.7 :** Suppose that $\{T(t)\}_{t\geq 0}$ is a $C_0$-one parameter semigroup of operators with the infnitesimal generator $A$, and $B \in B(X)$, then $A + B$ is the infinitesimal generator of a $C_0$-semigroup $\{S(t)\}_{t\geq 0}$ on $X$ that for each $t \geq 0$ and $x \in X$ we have

$$S(t)x = T(t)x + \int_0^t T(t-s)\,B\,S(s)x\,ds$$

**Proof :** suppose that $C = A + B$ , $x \in D(A)$ , so the function $f(.)x \ : s \mapsto T(t-s)\,S(s)x \in X$ so

$$\frac{d}{ds} f(s)x = T(t-s)\,C\,S(s)x - T(t-s)\,A\,S(s)x = T(t-s)\,B\,S(s)x$$ . So,

$$S(t)x - T(t)x = f(t)x - f(0)x = \int_0^t \frac{d}{ds} f(s)x\,ds = \int_0^t T(t-s)BS(s)x\,ds$$ But

$\overline{D(A)} = X$ , so for each $x \in X$ , $S(t)x = T(t)x + \int_0^t T(t-s)\,B\,S(s)x\,ds$ ■

We recall that if $H$ be a linear operator on Banach space $X$ then $\rho(H)$ is the resolvent set of $H$ and if $\lambda \in \rho(H)$ then $R(\lambda : H)$ means $(\lambda I - H)^{-1}$

We observed that for n-parameter semigroup $(X, R_+^n, W)$ the component $\{u^i(s)\}_{s\geq 0}$ are one parameter semigroups on a Banach space $X$. An important question that arises here is under which conditions the inverse of above statement is true, means if $\{u^i(s)\}_{s\geq 0} : i = 1, 2, \ldots, n$ are one parameter semigroups, then when we can say $W(t_1,\ldots,t_n) = u^1(t_1)\,u^2(t_2)\ldots u^n(t_n)$ is a n-parameter semigroup? We know that for each $1 \leq i, j \leq n$ and $s, t \in R_+^n$ , we have $u^i(s)u^j(t) = u^j(t)u^i(s)$ then $(X, R_+^n, W)$ is a n-parameter semigroup.

**Lemma.2.8:** Suppose that $\{u^i(s)\}_{s\geq 0}$ be a $C_0$-one-parameter semigroup of operators on Banach space X with the infinitesimal generator $H_i : i = 1, 2, \ldots, n$, then $W(t_1,\ldots,t_n) = u^1(t_1)\,u^2(t_2)\ldots u^n(t_n)$ is a $C_0$-n-parameter semigroup of operators if and only if there is an $w > 0$ such that for each $i = 1,2,\ldots,n$, $[w,\infty) \subseteq \rho(H_i)$ and for each integers $1 \leq i, j \leq n$ , $\lambda', \lambda \geq w$, we have

$$R(\lambda' : H_j)R(\lambda : H_i) = R(\lambda : H_i)R(\lambda' : H_j).$$

**Proof:** $\Rightarrow$) suppose W is a $C_0$-n-parameter semigroup of operators. Since $H_i$ is the infinitesimal generator of $\{u^i(s)\}_{s\geq 0}$ , by the Hille-Yosida Theorem, there is an $w_i > 0$ such that for each

$\lambda \geq w_i$ , $R(\lambda : H_i)$ exist and are bounded operators. Let $w = \max\{w_i \ : i = 1,2,...,n\}$. If $\lambda \geq w$, then $R(\lambda : H_i)x = \int_0^\infty e^{-\lambda s} u^i(s) x\, ds$. Also, we know that for each $1 \leq i, j \leq n$

$u^i(s)u^j(t) = W(se_i)W(te_j) = W(se_i + te_j) = W(te_j + se_i) = W(te_j)W(se_i) = u^j(t)u^i(s)$

so
$$\begin{aligned} R(\lambda : H_i)\left(u^j(t)x\right) &= \int_0^\infty e^{-\lambda s} u^i(s) u^j(t) x ds \\ &= \int_0^\infty e^{-\lambda s} u^j(t) u^i(s) x ds \\ &= u^j(t) \int_0^\infty e^{-\lambda s} u^i(s) x ds \\ &= u^j(t) R(\lambda : H_i) x \end{aligned}$$

Now, suppose that $\lambda' \geq w$, so by boundedness of $R(\lambda : H_i)$, we have

$$\begin{aligned} R(\lambda : H_i) R(\lambda' : H_j) x &= R(\lambda : H_i) \int_0^\infty e^{-\lambda' t} u^j(t) x dt \\ &= \int_0^\infty e^{-\lambda' t} u^j(t) R(\lambda : H_i) x dt \\ &= R(\lambda' : H_j) R(\lambda : H_i) x \end{aligned}$$

$\Leftarrow)$ : suppose that there is an $w > 0$ such that for each $\lambda, \lambda' > 0$, $R(\lambda : H_i)$ and $R(\lambda' : H_j)$ exist and $R(\lambda' : H_j) R(\lambda : H_i) = R(\lambda : H_i) R(\lambda' : H_j)$. Suppose $H^i_\lambda = \lambda^2 R(\lambda : H_i) - \lambda I$ and $H^j_{\lambda'} = \lambda'^2 R(\lambda' : H_j) - \lambda' I$ are the Yosida approximation of $H_i$ , $H_j$ respectively then we have $H^i_\lambda \ H^j_{\lambda'} = H^j_{\lambda'} H^i_\lambda$ so by

By using [13, 1.3.5] we have $u^i(s)x = \lim_{\lambda \to \infty} e^{sH^i_\lambda} x$ and $u^j(t)x = \lim_{\lambda' \to \infty} e^{tH^j_{\lambda'}} x$ thus

$$\begin{aligned} u^i(s)u^j(t)x &= \lim_{\lambda \to \infty} e^{sH^i_\lambda} u^j(t) x \\ &= \lim_{\lambda \to \infty} e^{sH^i_\lambda} \left( \lim_{\lambda' \to \infty} e^{tH^j_{\lambda'}} x \right) \\ &= \lim_{\lambda \to \infty} \lim_{\lambda' \to \infty} e^{tH^j_{\lambda'}} e^{sH^i_\lambda} x \\ &= \lim_{\lambda \to \infty} u^j(t) e^{sH^i_\lambda} x \\ &= u^j(t) \lim_{\lambda \to \infty} e^{sH^i_\lambda} x \quad = u^j(t) u^i(s) x \end{aligned}$$

Hence $W(t_1,...,t_n) = u^1(t_1) \ ... \ u^n(t_n)$ is a $C_0$-n-parameter semigroup of operators.

■

**Theorem.2.9:**Suppose $A: D(A) \subseteq X \to X$ is a closed operator and $I_T$ , $I_{T'}$ be two n-cells such that $I_{T'} \subseteq I_T$ . Then for each $x \in D(A)$ in $I_T$ , $I_{T'}$ the initial value problem (7) cannot have a unique solution.

**Proof :** by contradiction, suppose for each $x \in D(A)$ the problem (7) has a unique solution in $I_T$ , $I_{T'}$. As in Theorem2.4. we are going to show that if for $x \in D(A)$ and $t \in I_T$ , $u(t,x)$ is the unique solution of (7), then $W_1(t)x = u(t,x)$ can be extended to a $C_0$-n-parameter semigroup of operators on Banach space $X_1 = \left(D(A), \|.\|_A\right)$ then by using previous lemma we shall get a contradiction.It is clear, uniqueness of solution shows that $W_1(t)x = u(t,x)$ is a well-defined linear operator on Banach space $X_1 = \left(D(A), \|.\|_A\right)$ where $\|.\|_A$ is the graph norm on $D(A)$.

We notice that $Y = \left( C(I_T, X_1), \|.\|' \right)$, where $\|f\|' = \|f\|_\infty + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial t_i} f \right\|_\infty$ is a Banach space.

Next, we claim that the mapping $\phi: X_1 \to Y$ by $\phi(x)(t) = W_1(t)x$ is closed linear operator, for this; suppose $x_m \to x$ in $X_1$ and $\phi(x_m) \to f$ in *Y*. By integrating of (7) for initial value $x_m$,for $t_1$ from 0 to $t_1$ , we have

$$W_1(t_1,...,t_n)x_m = W_1(0,t_2,...,t_n)x_m - \sum_{i=2}^{n} \int_0^{t_1} \frac{\partial}{\partial t_i} W_1(s,t_2,...,t_n)x_m\, ds + \int_0^{t_1} A W_1(s,t_2,...,t_n)x_m\, ds$$

and when $m \to \infty$ by our choosing of the norm on *Y* and the closeness of *A*, we get

$$\left\| \frac{\partial}{\partial t_i} W_1(.)x_m - \frac{\partial}{\partial t_i} f(.) \right\|_\infty \to 0 \quad : \quad i = 1,2,...,n$$

and $\|W_1(.)x_m - f(.)\|_\infty = \sup_{t \in I_T} \left( \|W_1(t)x_m - f(t)\|_A \right)$

$$= \sup_{t \in I_T} \left( \|W_1(t)x_m - f(t)\| + \|AW_1(t)x_m - Af(t)\| \right) \to 0$$

So when $m \to \infty$ we have

$$f(t_1,...,t_n) = f(0,t_2,...,t_n) - \sum_{i=2}^{n} \int_0^{t_1} \frac{\partial}{\partial t_i} f(s,t_2,...,t_n)\, ds + \int_0^{t_1} Af(s,t_2,...,t_n)\, ds$$

Hence,

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial t_i} f(t_1,...,t_n) = A\, f(t_1,...,t_n) \\ f(0) = \lim_{m \to \infty} W_1(0)x_m = x \end{cases}$$

So *f* is a solution of (5) and by the uniqueness of solution we conclude $f(t) = W_1(t)x$ or equivalently $\phi$ is closed . so, by closed graph theorem $\phi$ is bounded, thus $\sup_{t \in I_T} \|W_1(t)\| < \infty$.so, by theorem.2.4, we can extend $W_1(t)$ to the $C_0$-n-parameter semigroup $(X_1, R_+^n, W)$. Now, suppose

$(H_1,..., H_n)$ be the infinitesimal generator of *W*, then for $x \in \bigcap_{i=1}^{n} D(H_i) \subseteq D(A)$ we have $\frac{\partial}{\partial t_i} W(t)x = H_i W(t)x$. Thus, for $t \in I_T$ and because $W(t) = W_1(t)x$ ,we have $\sum_{i=1}^{n} \frac{\partial}{\partial t_i} W(t)x = \left( \sum_{i=1}^{n} H_i \right) W(t)x = AW(t)x$. But, we know that *W(t)x* is strongly continuous and $\frac{\partial}{\partial t_i} W(t)x$ is continuous and *A* is closed and $\sum_{i=1}^{n} H_i W(t)x = W(t) \sum_{i=1}^{n} H_i x$ . so, we can say if t tends to 0 then for $x \in \bigcap_{i=1}^{n} D(H_i) : \sum_{i=1}^{n} H_i x = Ax$. **(8)**

Now, by applying Lemma 2.8 there is $\omega > 0$ such that for each $\lambda, \lambda' \geq \omega$ we have $R(\lambda' : H_j) R(\lambda : H_i) = R(\lambda : H_i) R(\lambda' : H_j)$.

Now, let $H_1' = H_1 + I$ and $H_2' = H_2 - I$, if $\omega' = \omega + 1$ and $\lambda, \lambda' \geq \omega'$, then $\lambda + 1 , \lambda' - 1 \geq \omega$ and

$$\begin{aligned} R(\lambda' : H_1') R(\lambda : H_2') &= R(\lambda' - 1 : H_1) R(\lambda + 1 : H_2) \\ &= R(\lambda + 1 : H_2) R(\lambda' - 1 : H_2) \\ &= R(\lambda : H_2') R(\lambda' : H_1') \end{aligned}$$

Similarly, for $i = 1,2$ , $j = 3,4,..., n$ and $\lambda, \lambda' \geq \omega'$. we have $R(\lambda : H_i') R(\lambda' : H_j) = R(\lambda' : H_j) R(\lambda : H_i')$

By Lemma 2.7, $H_1', H_2'$ are infinitesimal generators of $C_0$-one parameter semigroup of operators . With the above equalities and Lemma 2.8, $(H_1', H_2', H_3..., H_n)$ is the infinitesimal generator of a $C_0$-n-parameter semigroup that we call it $(X_1, R_+^n, W')$. So for each $x \in X_1$,

$W'(te_1)x = W(te_1)x + \int_0^t W((t-\mu)e_1) W'(\mu e_1) x d\mu$ and

$W'(te_2)x = W(te_2)x - \int_0^t W((t-\nu)e_2) W'(\nu e_2) x d\nu$ and also for each $i > 2 : W'(te_i) = W(te_i)$.

Thus for $x \in \bigcap_{i=1}^{n} D(H_i)$

$$\frac{\partial}{\partial t_i} W'(t_1, t_2, ...., t_n)x = \begin{cases} H_i' W'(t_1, t_2, ..., t_n) & : \quad i = 1,2 \\ H_i W'(t_1, t_2, ..., t_n) & : \quad i > 2 \end{cases}$$ So by (8)

$$\begin{cases} \frac{\partial}{\partial t_i} W'(t) = (H_1' + H_2' + H_3 + ..... + H_n) W'(t) = \sum_{i=1}^{n} H_i W'(t) \\ \qquad = AW'(t)x \\ W'(0)x = x \end{cases}$$

But the solution of (7) is unique, and so for $i = 1,2,..., n$ and $0 \leq t \leq T_i$ : $W'(te_i) = W(te_i)$. This implies that

$$W(te_1)x = W'(te_1)x = W(te_1)x + \int_0^t W((t-\mu)e_1)W'(\mu e_1)x d\mu$$

$$= W(te_1)x + \int_0^t W(te_1)x d\mu = W(te_1)x + tW(te_1)x$$

So, $tW(te_1)x = 0$ or $W(te_1)x = 0$ and This is a contradiction,because $0 = \lim_{t\to 0} W(te_1)x = x \neq 0$ .Thus for each $x \in D(A)$, (7) cannot have a unique solution.o■

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