Kirkuk Journal of Science Vol. 19, Iss. 1, p. 1-15, 2024 doi:10.32894/kujss.2024.144848.1122

Existence Solutions for a Singular Nonlinear Problem with Dirichlet Boundary Conditions on Exterior Domains

Mageed Ali [1*], Joseph Iaia [2]

[1*] Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.
[2] Department of Mathematics, University of North Texas, Denton, Texas, USA.
*Corresponding author : mageedali@uokirkuk.edu.iq

Article Information

Article Type:
Research Article

Keywords:
exterior domains, singular, Nonlinear, existence

History:
Received: 23 November 2023
Reviced: 26 January 2024
Accepted: 28 January 2024
Published Online: 20 February 2024
Published: 30 March 2024

Citation: Mageed Ali and Joseph Iaia, Existence Solutions for a Singular Nonlinear Problem with Dirichlet Boundary Conditions on Exterior Domains., Kirkuk Journal of Science, 19(1), p.1-15, 2024, https://doi.org/10.32894/kujss.2024.144848.1122

Abstract

This paper has proved the existence of solutions that solve the Nonlinear Partial differential equation. A study of dynamical systems has developed on the exterior of the ball centered at the origin in R^N with radius $R > 0$, with Dirichlet boundary conditions $u = 0$ on the boundary, and $u(x)$ approaches 0 as $|x|$ approaches infinity, where $f(u)$ is local Lipschitzian singular at zero, and grows superlinearly as u approaches infinity. by introducing Various scalings to elucidate the singular behavior near the center and at infinity. Also, $N > 2$, $f(u) \sim \frac{-1}{(|u|^{q-1}u}$ for small u with $0 < q < 1$, and $f(u) \sim |u|^{p-1}u$ for large $|u|$ with $p > 1$. In addition, $K(x) \sim |x|^{-\alpha}$ with $2 < \alpha < 2(N-1)$ for large $|x|$. The fixed point method and other techniques have been used to prove the existence.

1. Introduction:

Certainly, exploring solutions to partial differential equations is crucial in various scientific disciplines, especially in physical mathematics[1, 2]. The existence and uniqueness of solutions, particularly in second-order PDEs with specified initial conditions, form a fundamental aspect of this field [3, 4, 5]. The existence of a positive solution of (1) on R^N with $K(r) \equiv 1$ has been studied extensively [6, 7, 8, 9, 10, 11].

Recently the exterior domain $R^N \backslash B_R(0)$ has been studied in [12, 13, 14, 15, 16, 17]. Since we are interested in the topic, it comes from the recent papers [16, 18, 11] that have been studied to find the solutions to differential equation problems on exterior domains.

In [19], was studied (1)–(3) with $K(r)$ $r^{-\alpha}$, where f is singular at 0 and grows sublinearly at infinity, with different values of α. Also, in [20], the singular semilinear problem has infinitely many solutions on exterior domain. This article has proved the existence of solutions when f is singular at 0 and grows superlinearly at infinity.

This paper deals with the problem:

$$\Delta u + K(|x|) f(u) = 0, \quad x \in \mathbb{R}^N \backslash B_R \tag{1}$$

$$u = 0 \text{ on } \partial \left(\mathbb{R}^N \backslash B_R\right) \tag{2}$$

$$u \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty \tag{3}$$

where Δ is the Laplacian operator, $u : \mathbb{R}^N \longrightarrow \mathbb{R}$, with $N > 2$, B_R is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ and $K(x) > 0$.
In addition, we suppose:

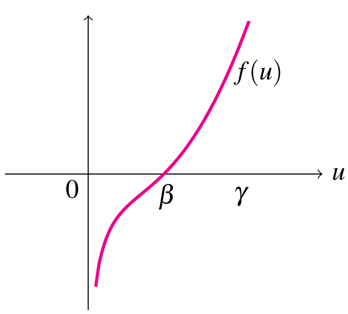
$$f \text{ is an odd function, increasing on } (0,\infty), \quad f \text{ is locally Lipschitz}, \exists \beta > 0 \text{ such that } f < 0 \text{ on } (0,\beta), \quad f > 0 \text{ on } (\beta,\infty). \tag{H1}$$

We assume:

$$f(u) = \frac{-1}{|u|^{q-1}u} + g_1(u) \quad \text{where } 0 < q < 1 \text{ for small } u \text{ and } g_1(0) = 0 \tag{H2}$$

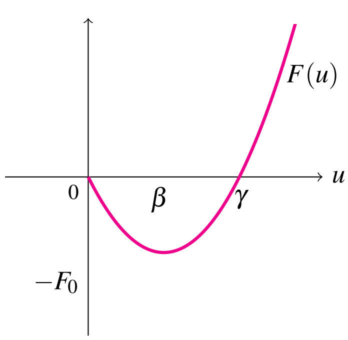
and:

$$f(u) = |u|^{p-1}u + g_2(u) \quad \text{where } p > 1 \text{for large } u \text{ and } \lim_{u\to+\infty} \frac{g_2(u)}{|u|^p} = 0. \tag{H3}$$



Also, we assume $F(u) = \int_0^u f(s)\,ds$. We know that f is odd it implies that F is even and from (H2) it follows that f is integrable near $u = 0$. Thus F is continuous and $F(0) = 0$. It also follows that F is bounded below and from (H1), $\exists\, \gamma$ with $0 < \beta < \gamma$ such that:

$$F < 0 \text{ on } (0,\gamma), F > 0 \text{ on } (\gamma,\infty), \text{ and } F > F_0 \text{ on } \mathbb{R}. \tag{H4}$$



We also suppose K and K' are continuous function on $[R,\infty)$ with:

$$K(r) > 0, \quad \exists \alpha \in (2, 2(N-1)) \text{ such that } \lim_{r\to\infty} \frac{rK'}{K} = -\alpha, \quad \text{and so } 2(N-1) + \frac{rK'}{K} > 0. \tag{H5}$$

In addition, we assume $\exists\, K_1 > 0,\ K_2 > 0$ such that:

$$\frac{K_1}{r^\alpha} \le K(r) \le \frac{K_2}{r^\alpha} \text{ on } [R,\infty). \tag{H6}$$

2. Preliminaries:

We are interested to study existence solutions of (1)–(3), we rewrite the equation with $r = |x|$, $u(r) = u(|x|)$ where u satisfies:

$$u''(r) + \frac{N-1}{r} u'(r) + K(r) f(u(r)) = 0 \quad \text{on } (R,\infty), \tag{4}$$

$$u(R) = 0, \quad u'(R) = a > 0. \tag{5}$$

To emphasize the dependence on the initial parameter a, we denote the solution by $u_a(r)$. Since $f(u)$ is not continuous at $u = 0$, here we can not apply the usual existence-uniqueness theorem for ordinary differential equations and so we have to prove the existence of a solution of equations (4)–(5) on $[R, R+\varepsilon)$ for some $\varepsilon > 0$ by using a different method.

First rewrite equation (4) as

$$\left(r^{N-1} u_a'(r)\right)' + r^{N-1} K(r) f(u_a(r)) = 0,$$

then integrate over $[R,r)$ and use $u_a'(R) = a$.
This gives:

$$r^{N-1} u_a'(r) - aR^{N-1} + \textstyle\int_R^r t^{N-1} K(r) f(u_a(r)\; dr = 0.$$

Multiply above by $r^{-(N-1)}$, integrate again over $[R,r]$ and use $u(R) = 0$ gives:

$$u_a(r) = aR^{N-1} \left[\frac{r^{2-N} - R^{2-N}}{2-N} \right] - \int_R^r \frac{1}{t^{N-1}} \int_R^t s^{N-1} K(s) f(u_a(s))\; ds\; dt \quad \text{for } r \in (R,\infty). \tag{6}$$

Now let $w(r) = \frac{u_a(r)}{r-R}$ so $u_a(r) = (r-R)w(r)$ and $w(R) = \lim_{r\to R^+} \frac{u_a(r)}{r-R} = u_a'(R) = a$.

Rewriting (6) we get:

$$w(r) = \frac{aR^{N-1}}{2-N} \left[\frac{r^{2-N} - R^{2-N}}{r-R} \right] - \frac{1}{r-R} \int_R^r \frac{1}{t^{N-1}} \int_R^t s^{N-1} K(s) f((s-R)w(s))\; ds\; dt.$$

(7)

We use the fixed point method to solve (7). Let define:

$$A = \Big\{ w \in C[R, R+\varepsilon] \text{ with } w(R) = a > 0 \text{ and } |w(r) - a| \leq \frac{a}{2} \text{ on } [R, R+\varepsilon] \Big\}$$

where $C[R, R+\varepsilon]$ is continuous functions on $[R, R+\varepsilon]$ with $\varepsilon > 0$.
Let:

$$||w|| = \sup_{x \in [R, R+\varepsilon]} |w(x)|.$$

Therefore $(A, ||.||)$ is a Banach space.

Now we define a map T on A by $Tw(R) = a$ and:

$$Tw(r) = \frac{aR^{N-1}}{2-N}\left[\frac{r^{2-N} - R^{2-N}}{r-R}\right] - \frac{1}{r-R}\int_R^r \frac{1}{t^{N-1}}\int_R^t s^{N-1}K(s) f((s-R)w(s)) \, ds \, dt \text{ for } r > R.$$

We will prove that T is a principle contraction mapping with $T(w) \in A$ for each $w \in A$ if $\varepsilon > 0$ is sufficiently small. By using L'Hôpital's rule it follows that $\lim_{r \to R^+} \frac{aR^{N-1}}{2-N}\left[\frac{r^{2-N} - R^{2-N}}{r-R}\right] = a.$

In addition, by (H2), by L'Hôpital's rule and $0 < q < 1$ we have:

$$\lim_{r \to R^+} \frac{\int_R^r \frac{1}{t^{N-1}}\int_R^t s^{N-1}K(s) f((s-R)w(s)) \, ds \, dt}{r-R} = 0.$$

Therefore $\lim_{r \to R^+} Tw(r) = a$, and it follows that: $|Tw(r) - a| \leq \frac{a}{2}$ on $[R, R+\varepsilon)$ if $\varepsilon > 0$ is sufficiently small.

Thus We next show that T is a contraction from A into itself for sufficiently small ε.

For any $w_1, w_2 \in A$. we have:

$$Tw_1(r) - Tw_2(r) = -\frac{1}{r-R}\int_R^r \frac{1}{t^{N-1}}\int_R^t s^{N-1}K(s)\Big[f((s-R)w_1(s)) - f((s-R)w_2(s))\Big] \, ds \, dt. \tag{8}$$

For $u \geq 0$ and by (H2) we know that $f(u) = -u^{-q} + g_1(u)$ so $f((s-R)w(s)) = -(s-R)^{-q}w^{-q}(s) + g_1((s-R)w(s))$ where $0 < q < 1$.

Then we first estimate:

$$|f((s-R)w_1(s)) - f((s-R)w_2(s))| = \left|\frac{-1}{(s-R)^q}\left[\frac{1}{w_1^q} - \frac{1}{w_2^q}\right] + g_1((s-R)w_1(s)) - g_1((s-R)w_2(s))\right.$$

$$\leq \frac{1}{(s-R)^q}\left|\frac{1}{w_1{}^q} - \frac{1}{w_2{}^q}\right| + L|s-R||w_1 - w_2| \tag{9}$$

where L is the Lipschitz constant for g_1 near $u = 0$.

Applying the mean value theorem to the right-hand side of (9) we get: $\frac{1}{(s-R)^q}\left[\frac{q}{w_3^{q+1}}|w_1 - w_2|\right] + L|s-R||w_1 - w_2|$ where $w_3 \in (w_1, w_2)$. Since w_1 is in A and $|w_1 - a| < \frac{a}{2}$ then $\frac{a}{2} < w_1 < \frac{3a}{2}$. Similarly w_2 is between $\frac{a}{2} < w_2 < \frac{3a}{2}$ and since w_3 is between w_1 and w_2 then we have $\frac{a}{2} < w_3 < \frac{3a}{2}$. Thus it follows that $w_3{}^{q+1} \geq \left(\frac{a}{2}\right)^{q+1}$. Thus for $s \in [R, R+\varepsilon]$ we have:

$$|f((s-R)w_1(s)) - f((s-R)w_2(s))| \leq |w_1 - w_2|\left[\frac{q}{(s-R)^q}\left(\frac{2}{a}\right)^{q+1} + L\varepsilon\right]. \tag{10}$$

Using (10) in (8) and assuming $r \in [R, R+\varepsilon)$ gives:

$$|Tw_1 - Tw_2| \leq \frac{1}{r-R}\int_R^r \frac{1}{t^{N-1}}\int_R^t s^{N-1}K(s)|w_1 - w_2|\left[\frac{q}{(s-R)^q}\left(\frac{2}{a}\right)^{q+1} + L\varepsilon\right] ds \, dt$$

$$\leq \frac{K(R)}{r-R}||w_1 - w_2||\int_R^r\int_R^t \left[\frac{q}{(s-R)^q}\left(\frac{2}{a}\right)^{q+1} + L\varepsilon\right] ds \, dt$$

$$\leq K(R)||w_1 - w_2||\left[\frac{q\left(\frac{2}{a}\right)^{q+1}\varepsilon^{1-q}}{(2-q)(1-q)} + \frac{\varepsilon^2 L}{2}\right].$$

Since:

$$\lim_{\varepsilon \to 0} \frac{q\left(\frac{2}{a}\right)^{q+1}\varepsilon^{1-q}}{(2-q)(1-q)} + \frac{\varepsilon^2 L}{2} = 0$$

and $c = K(R)\left[\frac{q\left(\frac{2}{a}\right)^{q+1}\varepsilon^{1-q}}{(2-q)(1-q)} + \frac{\varepsilon^2 L}{2}\right]$, we can choose small enough $\varepsilon > 0$ satisfies that $0 < c < 1$ such that T is a contraction on $C[R, R+\varepsilon]$.

So there exists a unique solution $w \in A$ with $Tw = w$ on $[R, R+\varepsilon]$ for some $\varepsilon > 0$.

Thus $u_a(r) = (r-R)w(r)$ is a solution of (4)–(5) on $[R, R+\varepsilon]$ for some $\varepsilon > 0$.

Now let:

$$E_a(r)=\frac{1}{2}\frac{u_a'^2(r)}{K(r)}+F(u_a). \tag{11}$$

Using (4) and (H5) we get:

$$E_a'(r)=-\frac{u_a'^2(r)}{2K(r)}[2(N-1)+\frac{rK'}{K}]\leq 0. \tag{12}$$

It follows that E is non-increasing so:

$$E_a(r)=\frac{1}{2}\frac{u_a'^2(r)}{K(r)}+F(u_a)\leq \frac{1}{2}\frac{a^2}{K(R)}=E_a(R) \quad \text{for } r\geq R. \tag{13}$$

Since F is bounded from below by (H4), so from (13) it implies that u_a' and u_a are uniformly bounded on $[R,\infty)$ and so existence follows wherever they are defined. We know that $f(u)$ is undefined at $u=0$, so the solution of (4)–(5) exists as long as $u_a(r)>0$. In addition, if $u_a(r_0)=0$ but $u_a'(r_0)\neq 0$ we can use the same argument as on the previous page to establish existence of a solution of (4)–(5) in a neighborhood of r_0. If there is an r_0 such that $u_a(r_0)=0$ and $u_a'(r)=0$ then we show in the appendix that we can extend this solution to a neighborhood of r_0. Continuing this process we can find the existence of a solution of (4)–(5) on $[R,\infty)$.

Lemma 2.1: Let $u_a(r)$ solves (4)–(5) and assume that $2<\alpha<2(N-1)$. If a sufficiently small, then $u_a(r)>0\ \forall r\in(R,\infty)$,

Proof: From (5) we have $u_a(R)=0$ and $u_a'(R)=a>0$. If $u_a'(r)>0\ \ \forall r\in(R,\infty)$ then $u_a(r)>0\ \ \forall r\in(R,\infty)$. So we are done in this case.

If $u_a(r)$ is not always greater than zero on (R,∞), then u_a has a zero at z_a, and $u_a(r)>0$ on (R,z_a). In addition, $\exists$ M_a such that $R<M_a<z_a$, where M_a is a local maximum of u_a with $u_a(M_a)>0$ and $u_a'>0$ on (R,M_a). From (4) we then have $u_a'(M_a)=0$, $u_a''(M_a)\leq 0$ so $f(u_a(M_a)\geq 0$ so $u_a(M_a)\geq\beta>0$.

We now show $\lim\limits_{a\to 0^+} M_a=+\infty$. Assume by the way of contradiction $\lim\limits_{a\to 0^+} M_a\neq+\infty$. Then $\exists\, M^*>0$ and a subsequence (still labeled M_a) such that $\lim\limits_{a\to 0^+} M_a=M^*$.

Since $R\leq M_a\leq z_a$ then $0\leq E_a(z_a)\leq E_a(M_a)\leq E_a(R)$.

Thus $0\leq F(u_a(M_a))\leq \frac{1}{2}\frac{a^2}{K(R)}$ and so $\lim\limits_{a\to 0^+} F\left(u_a(M_a)\right)=0$. Since we know from earlier $u_a(M_a)\geq\beta>0$ it follows then that:

$$\lim_{a\to 0^+} u_a(M_a)=\gamma. \tag{14}$$

On the interval $[R,z_a]$ it follows from (13) that:

$$0\leq E_a(z_a)\leq E_a(r)=\frac{1}{2}\frac{u_a'^2(r)}{K(r)}+F(u_a(r))\leq\frac{1}{2}\frac{a^2}{K(R)}\longrightarrow 0 \text{ as } a\longrightarrow 0^+ \text{ on } [R,z_a], \tag{15}$$

and as we saw earlier u_a,u_a' are uniformly bounded on $[R,M^*+1]$. Thus there exists a subsequence still labeled u_a such that u_a is uniformly convergent on $[R,M^*+1]$ with $\lim\limits_{a\to 0^+} u_a(r)=u^*(r)$ on $[R,M^*+1]$ and $\lim\limits_{a\to 0^+} u_a(M_a)=u^*(M^*)$ on $[R,M^*+1]$. Then from (14) we get $u^*(M^*)=\gamma$. Also since u_a is increasing on $[R,M_a]$ it follows that u^* is increasing on $[R,M^*]$ and:

$$0\leq u^*\leq\gamma \text{ on } [R,M^*]. \tag{16}$$

Now consider the following identity which follows directly from (4):

$$\left(r^{2(N-1)}\left[\frac{1}{2}u_a'^2(r)+K(r)F(u_a)\right]\right)'=\left(r^{2(N-1)}K(r)\right)'F(u_a). \tag{17}$$

Integrating on $[R,r)$ gives:

$$r^{2(N-1)}\left[\frac{1}{2}u_a'^2(r)+K(r)F(u_a)\right]=R^{2(N-1)}\frac{1}{2}a^2+\int_R^r\left(t^{2(N-1)}K(t)\right)'F(u_a)\,dt. \tag{18}$$

Since $a\longrightarrow 0$ and $u_a\longrightarrow u^*$ uniformly on $[R,M^*+1]$ then taking the limit in (18) gives:

$$\lim_{a\to 0^+} r^{2(N-1)}\left[\frac{1}{2}u_a'^2(r)+K(r)F(u^*)\right]=\int_R^r\left(t^{2(N-1)}K(t)\right)'F(u^*)\,dt.$$

Dividing by $r^{2(N-1)}K(r)$ gives:

$$\lim_{a\to 0^+}\frac{1}{2}\frac{u_a'^2(r)}{K(r)}+F(u^*)=\frac{\int_R^r\left(t^{2(N-1)}K(t)\right)'F(u^*)\,dt}{r^{2(N-1)}K(r)}. \tag{19}$$

Thus $\lim\limits_{a\to 0^+} u_a'^2$ exists and since $u_a'\geq 0$ on $[R,M_a]$ then $\lim\limits_{a\to 0} u_a'$ exists and so $\lim\limits_{a\to 0^+} u_a'=u^{*\prime}$.

Combining this with (15) it follows that $\frac{1}{2}\frac{u^{*\prime 2}(r)}{K(r)}+F(u^*(r))\equiv 0$ on $[R,M^*]$ and then by (17) and (H5), $\left(t^{2(N-1)}K(r)\right)'F(u^*)\equiv 0$. Thus $F(u^*)\equiv 0$. Therefore $u^*=$ constant but since $u^*(M^*)=$

γ and $u^*(R) = 0 < \gamma$, we get a contradiction. Thus M_a cannot be bounded and therefore:

$$\lim_{a\to 0^+} M_a = \infty. \tag{20}$$

Next for $M_a < r < z_a$ we have $0 \le E_a(z_a) \le E_a(r) \le E_a(M_a) = F(u_a(M_a))$ thus $u_a(M_a) \ge \gamma$ and so:

$$\frac{1}{2}\frac{u_a'^2(r)}{K(r)} + F(u_a(r)) \le E(M_a) = F(u_a(M_a)) \quad r \ge M_a. \tag{21}$$

Rewriting and integrating (21) from M_a to z_a, and changing variable gives:

$$\int_0^{\gamma} \frac{dt}{\sqrt{2}\sqrt{F(u_a(M_a)) - F(t)}} \le \int_0^{u_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(u_a(M_a)) - F(t)}}$$
$$\le \int_{M_a}^{z_a} \frac{u_a'(r)dr}{\sqrt{2}\sqrt{F(u_a(M_a)) - F(u_a(r))}} \le \int_{M_a}^{z_a} \sqrt{K(r)}\, dr. \tag{22}$$

Now using (H5)–(H6) and that $\alpha > 2$ gives:

$$\int_{M_a}^{z_a} \sqrt{K(r)}\, dr \le \int_{M_a}^{z_a} \sqrt{K_2} r^{\frac{-\alpha}{2}}\, dr = \sqrt{K_2}\left(\frac{z_a^{1-\frac{\alpha}{2}} - M_a^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right)$$
$$\le \frac{2\sqrt{K_2}}{\alpha - 2} M_a^{1-\frac{\alpha}{2}}. \tag{23}$$

Thus combining (22) and (23) we obtain:

$$\int_0^{\gamma} \frac{dt}{\sqrt{2}\sqrt{F(u_a(M_a)) - F(t)}} \le \frac{2\sqrt{K_2}}{\alpha - 2} M_a^{1-\frac{\alpha}{2}}. \tag{24}$$

Now taking the limit as $a \longrightarrow 0^+$ in inequality (24) using (14), (20), and $\alpha > 2$ gives:

$$0 < \int_0^{\gamma} \frac{dt}{\sqrt{2}\sqrt{-F(t)}} \le \lim_{a\to 0^+} \frac{2\sqrt{K_2}}{\alpha - 2} M_a^{1-\frac{\alpha}{2}} = 0.$$

This is a contradiction. Thus $u_a(r) > 0$ on $[R, \infty)$ if $a > 0$ is sufficiently small. This completes the proof of Lemma 2.1.

Next we show that $u_a(r)$ has many zeros on (R, ∞) as $a \longrightarrow \infty$.

Lemma 2.2: Let $u_a(r)$ be the solution of (4)–(5) and suppose (H1)–(H6). Then $u_a(r)$ has a local maximum M_a if a is sufficiently large, $u_a(M_a) \longrightarrow \infty$ as $a \longrightarrow \infty$, and $M_a \longrightarrow R^+$ as $a \longrightarrow \infty$.

Proof: First, suppose M_a is a positive local maximum. Then $u_a'(M_a) = 0$, $u_a''(M_a) \le 0$ and from equation (4), we see $f(u_a(M_a)) \ge 0$ (since $K(M_a) > 0$) so $u_a(M_a) \ge \beta$. Thus u_a cannot have a local maximum before u_a reaches β.

Next, suppose by the way of contradiction that $0 \le u_a \le \beta$ for sufficiently large a and all $r \in [R, \infty)$. Then we see $f(u_a) \le 0$ and so $u_a'' + \frac{N-1}{r} u_a' \ge 0$. Hence $(r^{N-1}u_a')' \ge 0$ on $[R, r]$. Integrating on $[R, r]$ gives:

$$r^{N-1}u_a'(r) \ge R^{N-1}u_a'(R) = aR^{N-1} > 0 \tag{25}$$

Hence u_a is increasing on $[R, r]$. Rewriting (25) and integrating gives:

$$u_a(r) \ge aR^{N-1}\left[\frac{r^{2-N} - R^{2-N}}{2-N}\right] = \frac{aR}{N-2}\left[1 - \left(\frac{R}{r}\right)^{N-2}\right]$$
on $[R, r]$.
$$\tag{26}$$

Then from (26) we see $u_a(2R) \ge \frac{aR}{N-2}\left[1 - \frac{1}{2^{N-2}}\right]$ and $\lim\limits_{a\to\infty} \frac{aR}{N-2}\left[1 - \frac{1}{2^{N-2}}\right] = \infty$ which contradicts the assumption that $0 \le U_a \le \beta$. Thus if a is sufficiently large then $u_a(r)$ gets larger than β.

Next we show $\max\limits_{[R,2R]} u_a(r) \longrightarrow \infty$ as $a \longrightarrow \infty$. Suppose by way of contradiction that $\max\limits_{[R,2R]} u_a(r) \le B$ where B does not depend on a for a large.

Since $r^{2(N-1)}K(r)F(u_a)$ and $\left(r^{2(N-1)}K(r)\right)' F(u_a)$ are continuous on $[R, 2R]$ then $|r^{2(N-1)}K(r)F(u_a)| \le A_1$ with $A_1 > 0$ and $\left|\int_R^r \left(r^{2(N-1)}K(r)\right)' F(U_a)\right| \le A_2$ with $A_2 > 0$ so rewriting (18) we obtain:

$$r^{2(N-1)}\frac{1}{2}u_a'^2(r) \ge \frac{R^{2(N-1)}a^2}{2} - [A_1 + A_2]. \tag{27}$$

Since the right-hand side of (27) goes to ∞ as $a \longrightarrow \infty$ then we see there is a C_a with $C_a > 0$ such that $\lim\limits_{a\to\infty} C_a = \infty$ and:

$$\left|u_a'\right| \ge \frac{\sqrt{2C_a}}{r^{N-1}} > 0 \text{ on } [R, 2R] \tag{28}$$

thus $u_a' > 0$ for a sufficiently large $[R, 2R]$ and integrating (28) over $(R, 2R)$ we get:

$$B \ge u_a(2R) \ge \sqrt{2C_a}\left[\frac{1 - 2^{2-N}}{N-2}\right] R^{2-N}.$$

but $\lim\limits_{a\to\infty} \sqrt{2C_a}\left[\frac{1-2^{2-N}}{N-2}\right] R^{2-N} = \infty$ which is a contradiction to the fact that u_a was bounded by B on $[R, 2R]$. Thus

$$\max_{[R,2R]} u_a \longrightarrow \infty \text{ as } a \longrightarrow \infty. \tag{29}$$

Now let us show that $u_a(r)$ has a local maximum M_a if a is sufficiently large. Suppose by the way of contradiction that u_a is increasing for all $r > R$. Since it follows from (13) that u_a is bounded then we see $\lim_{r\to\infty} u_a(r) = L_a$ with $L_a > 0$. Also since E_a is non-increasing it follows that $\lim_{r\to\infty} \frac{1}{2}\frac{u_a'^2}{K(r)} + F(u_a(r))$ exists. Since $F(u_a) \longrightarrow F(L_a)$ as $r \longrightarrow \infty$ it then follows that $\lim_{r\to\infty} \frac{1}{2}\frac{u_a'^2}{K(r)}$ exists. Dividing (18) by $r^{2(N-1)}K(r)$ we have:

$$\frac{1}{2}\frac{u_a'^2}{K(r)} + F(u_a(r)) = \frac{R^{2(N-1)}a^2}{2r^{2(N-1)}K(r)} + \frac{\int_R^r \left(r^{2(N-1)}K(r)\right)' F(u_a)}{r^{2(N-1)}K(r)}. \tag{30}$$

By (H5)–(H6) it follows that $\frac{1}{r^{2(N-1)}K(r)} \longrightarrow 0$ as $r \longrightarrow \infty$.

Then taking limits as r goes to infinity and using L'Hopital's rule in (30) we get:

$$\lim_{r\to\infty} \frac{1}{2}\frac{u_a'^2}{K(r)} + F(L_a) = 0 + F(L_a). \tag{31}$$

And so $\lim_{r\to\infty} \frac{1}{2}\frac{u_a'^2}{K(r)} = 0$.

Next by assumption $u_a(r)$ is increasing and s0 $L_a \geq \max_{[R,2R]} u_a(r)$. It follows then from (29) that

$$\lim_{a\to\infty} L_a = \infty. \tag{32}$$

Since E_a is non increasing and $\frac{1}{2}\frac{u_a'^2}{K(r)} \longrightarrow 0$ as $r \longrightarrow \infty$ then we see:

$$\frac{1}{2}\frac{u_a'^2}{K(r)} + F(u_a(r)) \geq F(u_a(L_a)) \; r \geq R. \tag{33}$$

Rewriting and integrating (33) over $[R,\infty)$ we get:

$$\int_0^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} = \int_R^\infty \frac{|u_a'(r)|dr}{\sqrt{2}\sqrt{F(L_a)-F(u_a(r))}} \geq \int_R^\infty \sqrt{K(r)}\,dr \tag{34}$$

From right-hand side of (34) since $\alpha > 2$ and using (H6) we get:

$$\int_R^\infty \sqrt{K(r)} \geq \int_R^\infty K_1 r^{\frac{\alpha}{2}} = \frac{2K_1}{\alpha-2}R^{1-\frac{\alpha}{2}}. \tag{35}$$

Thus we get:

$$\int_0^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} \geq \frac{2K_1}{\alpha-2}R^{1-\frac{\alpha}{2}}. \tag{36}$$

Finally let us show that $\lim_{a\to\infty}\int_0^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} = 0$ which contradicts and thus our assumption that u_a is increasing is false and therefore u_a must has a local maximum.

$$\int_0^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} = \int_0^{\frac{L_a}{2}} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} + \int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}}. \tag{37}$$

From (32) we know $L_a \longrightarrow \infty$ as $a \longrightarrow \infty$ and and so it follows from (H3) that $\lim_{a\to\infty} \frac{f(L_a)}{L_a} = \infty$ thus for a large $\frac{L_a}{2}$ is large then $F(t) < F(\frac{L_a}{2})$ also $F(L_a) - F(\frac{L_a}{2}) < F(L_a) - F(t)$ so

$$\int_0^{\frac{L_a}{2}} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} \leq \int_0^{\frac{L_a}{2}} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(\frac{L_a}{2})}} = \frac{\frac{L_a}{2}}{\sqrt{2}\sqrt{F(L_a)-F(\frac{L_a}{2})}}. \tag{38}$$

By the mean value theorem there is $d_1 > 0$ such that $\frac{L_a}{2} < d_1 < L_a$ then $F(L_a) - F(\frac{L_a}{2}) = f(d_1)\left[L_a - \frac{L_a}{2}\right] = f(d_1)\left[\frac{L_a}{2}\right]$ since f is increasing for a large then $f(\frac{L_a}{2}) \leq f(d_1)$ so

$$\frac{\frac{L_a}{2}}{\sqrt{2}\sqrt{F(L_a)-F(\frac{L_a}{2})}} \leq \frac{\sqrt{\frac{L_a}{2}}}{\sqrt{2}\sqrt{f(\frac{L_a}{2})}} \tag{39}$$

taking limit as a goes to infinity and by (H3) and (35)

$$\lim_{a\to\infty} \frac{1}{\sqrt{2}}\sqrt{\frac{\frac{L_a}{2}}{f(\frac{L_a}{2})}} = 0. \tag{40}$$

Thus by (38), (39), and (40) then:

$$\lim_{a\to\infty} \int_0^{\frac{L_a}{2}} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} = 0. \tag{41}$$

Second, we estimate $t \in [\frac{L_a}{2}, L_a]$ we have F is continuous and f is increasing so by the mean value theorem there is a $d_2 > 0$ with $\frac{L_a}{2} < d_2 < L_a$ so $F(L_a) - F(t) = f(d_2)[L_a - t] \geq f(\frac{L_a}{2})[L_a - t]$ rewrite the second part of (37) we get:

$$\int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}} \leq \int_{\frac{L_a}{2}}^{L_a} \frac{dt}{\sqrt{2}\sqrt{f(\frac{L_a}{2})(L_a-t)}} = \sqrt{2}\sqrt{\frac{\frac{L_a}{2}}{f(\frac{L_a}{2})}} \tag{42}$$

taking limit as a goes to infinity and by (H3) and (42)

$$\lim_{a\to\infty} \sqrt{2}\sqrt{\frac{\frac{L_a}{2}}{f(\frac{L_a}{2})}} = 0. \tag{43}$$

Thus (42) and (43) gives:

$$\lim_{a\to\infty}\int_{\frac{L_a}{2}}^{L_a}\frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}}=0. \tag{44}$$

Combining (41) and (44) with (37) we have:

$$\int_0^{L_a}\frac{dt}{\sqrt{2}\sqrt{F(L_a)-F(t)}}=0. \tag{45}$$

Now taking limits in (36) we get: $K_1\frac{R^{1-\frac{\alpha}{2}}}{\frac{\alpha}{2}-1}\leq 0$ which is false. Thus u_a must have a first local maximum M_a if a is sufficiently large.

Next we show that $u_a(M_a)\geq\max\limits_{[R,2R]}u_a$. Since u_a has a first local maximum M_a. Case 1: if $M_a>2R$. Since u_a is increasing on $[R,M_a]$ then $u_a(M_a)\geq u_a(2R)=\max\limits_{[R,2R]}u_a$ so we done this case. case 2: if $R<M_a<2R$. Suppose by way of contradiction there is t_0 with $M_a<t_0<2R$ such that $u_a(t_0)>u_a(M_a)$ then there is a smallest s_0 with $s_0>M_a$ such that $u_a(s_0)=u_a(M_a)$ then for $M_a<r<s_0$ we have $F(u_a(M_a))=E(s_0)\leq E(r)\leq E(M_a)=F(u_a(M_a))$ since $\frac{1}{2}\frac{u'^2{}_a(s_0)}{K(s_0)}=0$ and $F(u_a(M_a))=F(u_a(s_0))$ therefore $E(r)$ is a constant on $[M_a,s_0]$ thus $E'(r)=0$ then $u_a'(r)\equiv 0$ on $[M_a,s_0]$. By the uniqueness of the solution of the initial value problem we have $u_a(r)\equiv 0$ on $[R,\infty)$ but we know $u'(R)=a>0$ which is a Contradiction. so no t_0 exists. Thus $u_a(M_a)\geq\max\limits_{[R,2R]}u_a$ and $\max\limits_{[R,2R]}u_a\longrightarrow\infty$ as $a\longrightarrow\infty$. Thus $\lim\limits_{a\to\infty}u_a(M_a)=\infty$.

Now let us show $\lim\limits_{a\to\infty}M_a=R$. Since $E_a(r)$ is non-increasing it follows that if $R\leq r\leq M_a$ then:

$$\frac{1}{2}\frac{U_a'^2}{K(r)}+F(u_a(r))\geq F(u_a(M_a)) \text{ on } [R,M_a].$$

Rewriting, integrating over (R,M_a) and changing variables we get:

$$\begin{aligned}&\int_0^{u_a(M_a)}\frac{dt}{\sqrt{2}\sqrt{F(u_a(M_a))-F(t)}}=\\&\int_R^{M_a}\frac{u_a'(r)dr}{\sqrt{2}\sqrt{F(u_a(M_a))-F(u_a(r))}}\geq\int_R^{M_a}\sqrt{K(r)}\,dr.\end{aligned} \tag{46}$$

From the right-hand side of (46) using (H6) we get:

$$\int_R^{M_a}\sqrt{K(r)}\geq\int_R^{M_a}\sqrt{K_1r^{-\alpha}}=\sqrt{K_1}\left(\frac{M_a^{1-\frac{\alpha}{2}}-R^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right) \tag{47}$$

since $\alpha>2$. It follows from (45) that the left-hand side of (46) goes to 0 as $a\longrightarrow\infty$ therefore it follows from (47) that $M_a\longrightarrow R$ as $a\longrightarrow\infty$.

This completes the proof of lemma.

Lemma 2.3: Suppose (4)–(5) and $N\geq 2$. Let $u_a(r)$ be the solution of (4)–(5) and suppose $2<\alpha<2(N-1)$. Then $u_a(r)$ has at least n zeroes on $(0,\infty)$ if a sufficiently large.

Proof: Let $V(r)=u_a(r+M_a)$, then $V(0)=u_a(M_a)$, $V'(r)=u_a'(r+M_a)$ and $V''(r)=u_a''(r+M_a)$. Substituting in equation (4) we get:

$$u_a''(r+M_a)+\frac{N-1}{r+M_a}u_a'(r+M_a)+K(r+M_a)f(u_a(r+M_a))=0 \tag{48}$$

so $V''(r)+\frac{N-1}{r+M_a}V'(r)+K(r+M_a)f(V(r))=0$ with $V(0)=u(M_a)$ and $V'(0)=0$. Now if we replace r with $\frac{r}{\lambda}$ where $\lambda>0$ then we get:

$$V''\left(\frac{r}{\lambda}\right)+\frac{N-1}{\frac{r}{\lambda}+M_a}V'\left(\frac{r}{\lambda}\right)+K\left(\frac{r}{\lambda}+M_a\right)f\left(V\left(\frac{r}{\lambda}\right)\right)=0. \tag{49}$$

Now let:

$$W_\lambda(r)=\lambda^{\frac{-2}{P-1}}V\left(\frac{r}{\lambda}\right)=\lambda^{\frac{-2}{P-1}}u_a\left(\frac{r}{\lambda}+M_a\right). \tag{50}$$

Then:

$$W_\lambda'(r)=\lambda^{\frac{-2}{P-1}-1}V'\left(\frac{r}{\lambda}\right)=\lambda^{\frac{-2}{P-1}-1}u_a'\left(\frac{r}{\lambda}+M_a\right)$$

$$W_\lambda''(r)=\lambda^{\frac{-2}{P-1}-2}V''\left(\frac{r}{\lambda}\right)=\lambda^{\frac{-2}{P-1}-2}u_a''\left(\frac{r}{\lambda}+M_a\right)$$

and substituting above in (49) we get:

$$\begin{aligned}&W_\lambda''(r)+\frac{N-1}{r+\lambda M_a}W_\lambda'(r)+\lambda^{\frac{-2P}{P-1}}K\left(\frac{r}{\lambda}+M_a\right)\Big[|W_\lambda|^{P-1}W_\lambda\lambda^{\frac{2P}{P-1}}\\&+g_2\left(W_\lambda\lambda^{\frac{2}{P-1}}\right)\Big]=0.\end{aligned} \tag{51}$$

simplifying (51) we get:

$$\begin{aligned}&W_\lambda''(r)+\frac{N-1}{r+\lambda M_a}W_\lambda'(r)+Kbig(\frac{r}{\lambda}+M_a)\Big[|W_\lambda|^{P-1}(r)W_\lambda(r)\\&+\lambda^{\frac{-2P}{P-1}}g_2\left(\lambda^{\frac{2}{P-1}}W_\lambda(r)\right)\Big]=0.\end{aligned}$$

We choose λ so that $\lambda^{\frac{-2}{P-1}}u_a(M_a)=1$. Then we have $W_\lambda(0)=\lambda^{\frac{-2}{P-1}}u(M_a)=1$. So $u_a(M_a)\lambda^{\frac{-2}{P-1}}$ and since $u_a(M_a)\longrightarrow\infty$ as $a\longrightarrow\infty$ then $\lambda\longrightarrow\infty$ as $a\longrightarrow\infty$. Now let:

$$E_\lambda(r)=\frac{1}{2}\frac{W_\lambda'^2}{K\left(\frac{r}{\lambda}+M_a\right)}+\frac{|W_\lambda|^{P+1}}{P+1}+\frac{G(\lambda^{\frac{2}{P-1}}W_\lambda(r))}{\lambda^{\frac{2(P+1)}{P-1}}}, \tag{52}$$

where $G(u) = \int_0^u g_2(t)\,dt$. Then: (59)

$$E'_\lambda(r) = \frac{-W'^2_\lambda}{2\lambda\left(\frac{r}{\lambda}+M_a\right)K\left(\frac{r}{\lambda}+M_a\right)}[2(N-1)+ \frac{\left(\frac{r}{\lambda}+M_a\right)K'\left(\frac{r}{\lambda}+M_a\right)}{K\left(\frac{r}{\lambda}+M_a\right)} \leq 0. \quad (53)$$

where the bracketed term is greater than or equal to 0 by (H5). It follows from (53) that E_λ is non-increasing and so:

$$E_\lambda(r) = \frac{1}{2}\frac{W'^2_\lambda}{K\left(\frac{r}{\lambda}+M_a\right)} + \frac{|W_\lambda|^{P+1}}{P+1} + \frac{G\left(\lambda^{\frac{2}{P-1}}W_\lambda(r)\right)}{\lambda^{\frac{2(P+1)}{P-1}}} \leq \frac{1}{P+1} + \frac{G(\lambda^{\frac{2}{P-1}})}{\lambda^{\frac{2(P+1)}{P-1}}} = E_\lambda(0). \quad (54)$$

Using (H3):

$$\lim_{\lambda\to\infty} \frac{G(\lambda^{\frac{2}{P-1}})}{\lambda^{\frac{2(P+1)}{P-1}}} = 0 \quad (55)$$

so for λ sufficiently large we get:

$$\frac{1}{2}\frac{W'^2_\lambda}{K(\frac{r}{\lambda}+M_a)} + \frac{|W_\lambda|^{P+1}}{P+1} + \frac{G(\lambda^{\frac{2}{P-1}}W_\lambda(r))}{\lambda^{\frac{2(P+1)}{P-1}}} \leq \frac{1}{P+1} + \frac{1}{P+1} = \frac{2}{P+1} \quad (56)$$

so:

$$\frac{1}{2}\frac{W'^2_\lambda}{K(\frac{r}{\lambda}+M_a)} + \frac{|W_\lambda|^{P+1}}{P+1} \leq \frac{2}{P+1} - \frac{G(\lambda^{\frac{2}{P-1}}W_\lambda(r))}{\lambda^{\frac{2(P+1)}{P-1}}}. \quad (57)$$

Using (H3) it follows that $\lim_{u\to\infty} \frac{|G(u)|}{|u|^{P+1}} = 0$.

So $\left|\frac{G(u)}{u^{P+1}}\right| < \frac{1}{2(p+1)}$ if $|u| \geq C_0$.

Also since $G(u)$ is continuous when $|u| \leq C_0$ then there is D so that $|G(u)| \leq D$ when $|u| \leq C_0$ and so

$$|G(u)| \leq D + \frac{1}{2(p+1)}|u|^{P+1} \quad \forall u \in \mathbb{R}. \quad (58)$$

Thus:

$$\left|G\left(\lambda^{\frac{2}{P-1}}W_\lambda(r)\right)\right| \leq D + \frac{1}{2(P+1)}\left|\lambda^{\frac{2}{P-1}}W_\lambda(r)\right|^{P+1} = D + \frac{1}{2(P+1)}\lambda^{\frac{2(P+1)}{P-1}}|W_\lambda(r)|^{P+1}$$

Substituting (59) into (57) gives:

$$\frac{1}{2}\frac{W'^2_\lambda}{K(\frac{r}{\lambda}+M_a)} + \frac{|W_\lambda|^{P+1}}{P+1} \leq \frac{2}{P+1} + \frac{D}{\lambda^{\frac{2(P+1)}{P-1}}} + \frac{1}{2(P+1)}|W_\lambda(r)|^{P+1}$$

so

$$\frac{1}{2}\frac{W'^2_\lambda}{K(\frac{r}{\lambda}+M_a)} + \frac{|W_\lambda|^{P+1}}{2(P+1)} \leq \frac{2}{P+1} + \frac{D}{\lambda^{\frac{2(P-1)}{P+1}}} \leq \frac{2}{P+1} + 1$$

for λ sufficiently large. Thus W_λ and W'_λ are uniformly bounded on compact sets. So by Arzela-Ascoli, there is a subsequence still labeled w_λ such that $W_\lambda \longrightarrow W^*$ uniformly on compact sets and so W^* is continuous. It can be shown in a similar argument as (59) that:

$$\lim_{a\to\infty} K\left(\frac{r}{\lambda}+M_a\right)\lambda^{\frac{-2P}{P-1}} g_2\left(\lambda^{\frac{2}{P-1}}W_\lambda(r)\right) = 0$$

since $\frac{g_2(u)}{u^P} \longrightarrow 0$ as $u \longrightarrow \infty$ so $\frac{g_2(u)}{u^P} < \varepsilon$ if $u \geq L$ then

$g_2(u) < \varepsilon|u|^P$ if $u \geq L$ thus $g_2(u) \leq D_1 + \varepsilon|u|^P$

so

$$\left|K\left(\frac{r}{\lambda}+M_a\right)\lambda^{\frac{-2P}{P-1}} g_2\left(\lambda^{\frac{2}{P-1}}W_\lambda(r)\right)\right| \leq K\left(\frac{r}{\lambda}+M_a\right)\lambda^{\frac{-2P}{P-1}} \left[D_1 + \varepsilon\lambda^{\frac{2P}{P-1}}|W_\lambda|^P\right] = K\left(\frac{r}{\lambda}+M_a\right)D_1\lambda^{\frac{-2P}{P-1}} + \varepsilon K\left(\frac{r}{\lambda}+M_a\right)|W_\lambda|^P$$

is also uniformly bounded. Then it follows from (51) that W''_λ is also uniformly bounded. Thus $W'_\lambda \longrightarrow W^{*'}$ uniformly on compact sets. Then taking limits in (51) we get:

$$(W^*)'' + K(R)|W^*|^{P-1}W^* = 0 \quad (60)$$

with $W^*(0) = 1$, $W^{*'}(0) = 0$. Thus:

$$\frac{1}{2}(W^*)'^2 + K(R)\frac{|W^*|^{P+1}}{P+1} = \frac{K(R)}{P+1}. \quad (61)$$

It follows from (61) that $|W^*| \leq 1$. We now show W^* has an infinite number of zeros on $[0,\infty)$. Suppose $(W^*)' \leq 0$ for all $r \geq R$. Then W^* is bounded and decreasing so:

$$\lim_{r\to\infty} W^*(r) = L. \quad (62)$$

Taking limits in (61) gives:

$$\lim_{r\to\infty} \frac{1}{2}W^{*\prime 2}(r) + K(R)\frac{|L|^{P+1}}{P+1} = \frac{K(R)}{P+1} \quad (63)$$

so:

$$\lim_{r\to\infty} W^{*\prime 2}(r) = \frac{2K(R)}{P+1}\left[1-|L|^{P+1}\right] \text{ Thus } |L| \le 1. \tag{64}$$

Now suppose $|L| < 1$

$$\lim_{r\to\infty}\left|(W^*)'(r)\right| = \sqrt{\frac{2K(R)}{P+1}\left[1-|L|^{P+1}\right]}. \tag{65}$$

Thus for large r and r_0

$$\int_{r_0}^{r} -(W^*)'(r)\,dr = \int_{r_0}^{r}\left|(W^*)'(r)\right|\,dr \ge \frac{1}{2}\int_{r_0}^{r}\sqrt{\frac{2K(R)}{P+1}\left[1-|L|^{P+1}\right]}\,dr \tag{66}$$

we get:

$$-W^*(r)+W^*(r_0) \ge \frac{1}{2}\sqrt{\frac{2K(R)}{P+1}\left[1-|L|^{P+1}\right]}(r-r_0) \longrightarrow \infty \text{ as } r\longrightarrow\infty \tag{67}$$

on the left-hand side of (67) is bounded which contradicts that W^* is bounded. Thus $|L|=1$ and since $W^*(0)=1$ and since $W^{*\prime\prime}<0$ then W^* is decreasing near $r=0$ also $W^{*\prime\prime}(0) = -K(R) < 0$ so W^* is not constant so $L \ne 1$ and thus $W^{*\prime} \le 0$ then $L=-1$.

$$\frac{1}{2}(W^*)'^2 + \frac{K(R)}{P+1}|W^*|^{P+1} = \frac{K(R)}{P+1} \tag{68}$$

$$(W^*)' = \sqrt{\frac{2K(R)}{P+1}\left[1-(W^*)^{P+1}\right]} \tag{69}$$

$$\int_0^r \frac{-W^{*\prime}(r)\,dr}{\sqrt{1-|W^*(r)|^{P+1}}} = \int_0^r \frac{\left|W^{*\prime}\right|(r)\,dr}{\sqrt{1-W^{*P+1}}} = \int_0^r\sqrt{\frac{2K(R)}{P+1}} = \sqrt{\frac{2K(R)}{P+1}}r \tag{70}$$

if we make change of variable $t=W^*(r)$ and $dt = W^*(r)\,dr$ we get:

$$\int_{W^*(r)}^{1}\frac{dt}{\sqrt{1-t^{P+1}}} = \sqrt{\frac{2K(R)}{P+1}}\,r \longrightarrow \infty \text{ as } r\longrightarrow\infty \tag{71}$$

if $W^*(r)\ge 0$ and $(W^*)'(r) \le 0$ since $|W^*(r)|$ bounded by 1, so $W^*(r) \longrightarrow -1$ as $r\longrightarrow\infty$.

$$\int_{-1}^{1}\frac{dt}{\sqrt{1-t^{P+1}}} = \infty \tag{72}$$

but left-hand side is finite. This is a contradiction. Thus W^* must have a first local minimum m^*. Let $r=m^*$ in (65) so $|W^*|^{p+1}(m^*)=1$ so $W^*(m^*)=\pm 1$ but since $W^*(0)=1$ and W^* is initially decreasing then it follows that $W^*(m^*)=-1$ so W^* has a first zero Z_1 and we can show $W(m^*+t) = W(m^*-t)$ therefore W is periodic with period $2m^*$ so W has infinite many zeros. This completes the proof.

In this paper, we prove the following:

Theorem 1: Assuming (H1)–(H6) then there exists a solutions of (1)–(3).

3. The Main Results:

Let $S_0 = \{a>0|u_a(r)>0\,\forall r>R\}$. By Lemma 2.1 we know that if $a>0$ and a is sufficiently small then $u_a>0$ for all $r>R$. Thus S_0 is nonempty. By Lemma 2.3 we see that if a sufficiently large then u_a has a zero. Hence S_0 is bounded from above. So the supremum of S_0 exists and let $a_0 = \sup\ S_0 > 0$.

Lemma 3.1: $u_{a_0}(r)>0$ for $r>R$ and $\lim_{r\to\infty} u_{a_0}(r) = 0$.

Proof: Suppose first by the way of contradiction that $u_{a_0}(r)$ is not positive for $r>R$. So there exists $Z_0>R$ such that $u_{a_0}(Z_0)=0$ and $u_{a_0}(r)>0$ on (R,Z_0).

Assume $u'_{a_0}(Z_0)<0$ So there is $r_1>Z_0$ such that $u_{a_0}(r_1)<0$. We also know that $u'_a(r)$ varies continuously with a. Thus on any compact set K_0 , $\lim_{a\to a_0} u_a(r) = u_{a_0}(r)$ uniformly on K_0.

So if a is close enough to a_0 then $u_a(r_1)<0$.

In particular if $0<a<a_0$ then $u_a(r_1)<0$, but this contradicts that then $u_a(r)>0$ for $r>R$ and $0<a<a_0$.

Therefore $u_{a_0}(r)$ does not have a zero. So $u_{a_0}(r)>0$ for $r>R$.

For $a>a_0$ then $u_a(r)$ has a zero z_a. We now show $\lim_{a\to a_0^+} Z_a(r) = \infty$, because otherwise if there is a $B>0$ such that $z_a \le B$ for all a close to a_0 then there is a subsequence still labeled a such that $Z_a \longrightarrow Z^*$.

Also since $E_a(r) \le \frac{1}{2}\frac{a^2}{K(R)} \le \frac{1}{2}\frac{(a_0+1)^2}{K(R)}$ for all $r\ge R$ then u_a and u'_a are uniformly bounded on $[R, a_0+1]$ and so for further subsequence still labeled u_a we have $u_a \longrightarrow u_{a_0}$ uniformly on compact sets so $0 = \lim_{a\to a_0^+} u_a(Z_a) = u_{a_0}(Z^*)$.

So $u_{a_0}(Z^*)=0$ but we showed earlier $u_{a_0}(r)>0$ for $r>R$. This is a contradiction. Thus $\lim_{a\to a_0^+} Z_a(r)=+\infty$.

In addition, we now show $E_{a_0}(r)\geq 0$ for all $r>R$. Let us integrate the identity over (r_0,r) we get:

$$\int_{r_0}^{r}\left[\left((r^{2(N-1)}\left[\frac{1}{2}u_{a_0}'^2(r)+K(r)F(u_{a_0}(r))\right]\right)'=\left(r^{2(N-1)}K(r)\right)'F(u_{a_0}(r))$$

we rewriting

$$=r^{2(N-1)}\left[\frac{1}{2}u_{a_0}'^2(r)+K(r)F(u_{a_0}(r))\right]'=\left(r^{2(N-1)}K(r)\right)'F(u_{a_0}(r))$$

Suppose by the way of contradiction suppose there is $r_1>R$ such that $E_{a_0}(r_1)<0$. Again by continuous dependence of the $u_a(r)$ and $u_a'(r)$ on the parameter a we get $E_a(r_1)<0$ if a is close enough to a_0. On the other hand, if $a>a_0$ then u_a has a first zero z_a and $U_a>0$ for $R<r<z_a$ and since $E_{a_0}(r_1)<0$ and E_{a_0} is non-increasing then $z_a\leq r$, thus $0\leq E_a(z_a)\leq E_a(r_1)<0$ where $z_a<r_1$. But $z_a\longrightarrow\infty$ as $a\longrightarrow a_0$ therefore $E_{a_0}(r)\geq 0\ \forall r\geq R$.

Lemma 3.2: $u_{a_0}(r)$ has a local maximum $M_{a_0}>R$.

Proof: Suppose not. Then $u_{a_0}'(r)\geq 0\ \forall r>R$. Also $\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}+F(u_{a_0}(r))=E_{a_0}(r)\leq E_{a_0}(R)=\frac{a_0^2}{2K(R)}$. It follows from this that u_{a_0} is bounded and since $u_{a_0}'\geq 0$ then $\lim_{r\to\infty}u_{a_0}(r)=L>0$.

Since E_{a_0} is non-increasing then for all $r>R$ then from (H4) it follows that $F(u_{a_0})$ is bounded from below and since $\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}\geq 0$ then $\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}+F(u_{a_0})$ is bounded from below and thus

$$\lim_{r\to\infty}\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}+F(u_{a_0}(r))\text{ exists.}$$

Also since $u_{a_0}\longrightarrow L$ it follows that $\lim_{r\to\infty}F(u_{a_0}(r))=F(L)$ and so it follows that $\lim_{r\to\infty}\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}$ exists. Now let us show $\lim_{r\to\infty}\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}=0$. Consider the following identity which follows from (4) and integrating over (r,r_0) we get:

$$\int_{r_0}^{r}\left(r^{2(N-1)}\left[\frac{1}{2}u_{a_0}'^2(r)+K(r)F(u_{a_0}\right]\right)'dr=\int_{r_0}^{r}\left(r^{2(N-1)}K(r)\right)'F(u_{a_0})\,dr$$

so

$$\frac{1}{2}\frac{U_{a_0}'^2(r)}{K(r)}+F(u_{a_0}(r))=\frac{C_0}{K(r)r^{2(N-1)}}+\frac{\int_{r_0}^{r}\left(r^{2(N-1)}K(r)\right)'F(u_{a_0})}{K(r)r^{2(N-1)}}$$

for some constant C_0. Taking the limit as r goes to infinity and using (H6) then $\lim_{r\to\infty}\frac{C_0}{K(r)r^{2(N-1)}}=0$ so using L'Hopital rule

$$\lim_{r\to\infty}\left[\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}+F(u_{a_0}(r))\right]=\lim_{r\to\infty}\frac{\int_{r_0}^{r}\left(r^{2(N-1)}K(r)\right)'F(u_{a_0})}{K(r)r^{2(N-1)}}=F(L)$$

so $\lim_{r\to\infty}\frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)}=0$. Also from lemma 3.1, $E_{a_0}\geq 0$ and since $0\leq E_{a_0}\longrightarrow F(L)$ it follows $L\geq\gamma$.

Next we return to (4) $-\left(r^{N-1}u_{a_0}'(r)\right)'=r^{N-1}K(r)f(u_{a_0}(r))$ since $L\geq\gamma$ and $f(u_{a_0})\geq 0$. Since u_{a_0} is increasing and $u_{a_0}(r)\longrightarrow L\geq\gamma$ as $r\longrightarrow\infty$ then for large $u_{a_0}(r)\geq\frac{\gamma+\beta}{2}>\beta$ then there exists $C_1>0$ such that $f(u_{a_0})\geq C_1>0$ for r sufficiently large we get:$-\left(r^{(N-1)}U_{a_0}'(r)\right)'\geq C_1r^{(N-1)}K(r)$. Integrating on (r_0,r) where r_0,r are sufficiently large then we get:

$$\int_{r_0}^{r}\left[\left(r^{(N-1)}u_{a_0}'(r)\right)'+C_1r^{(N-1)}K(r)\right]\leq 0$$

so

$$r^{(N-1)}u_{a_0}'(r)-r_0^{(N-1)}u_{a_0}'(r_0)+C_1\frac{r^{N-\alpha}-r_0{}^{N-\alpha}}{N-\alpha}\leq 0\text{ if }2<\alpha<N\text{ then }r^{(N-1)}u_{a_0}'(r)\leq r_0{}^{(N-1)}u_{a_0}'(r_0)+C_1\frac{r^{N-\alpha}-r_0{}^{N-\alpha}}{N-\alpha}\longrightarrow-\infty$$

Since $r_0^{N-1}u_{a_0}'(r_0)=$ constant and $\lim_{r\to\infty}r^{N-\alpha}=+\infty$,then $\lim_{r\to\infty}r^{N-1}u_{a_0}'(r)=-\infty$ so u_{a_0}' must get negative. Thus u_{a_0} has a local max M_{a_0}.

Now we show that $u_{a_0}'(r)\leq 0$ for $r>M_{a_0}$. If not then there is $r_1>M_{a_0}$ such that $u_{a_0}'(r_1)>0$ so u_{a_0} has a local min $m_{a_0}>M_{a_0}$ such that $u_{a_0}'(m_{a_0}))=0$ and $u_{a_0}''(m_{a_0}))\geq 0$

so $f(u_{a_0}(m_{a_0}))\leq 0$, but $0<u_{a_0}(m_{a_0})\leq\beta$. From lemma 3.1 we have $0\leq E_{a0}(m_{a_0})=F(u_{a_0}(m_{a_0}))$

$u_{a_0}(m_{a_0})\geq\gamma$, but this is a contradiction. Since $0<u_{a_0}(m_{a_0})\leq\beta<\gamma$. Thus $u_{a_0}'<0$ for all $r>M_{a_0}$. Since $u_{a_0}>0$ then $\lim_{r\to\infty}u_{a_0}(r)=A$ with $A\geq 0$ for $r>R$.

We will show that $A=0$. We know $E_{a_0}(r)$ is non-increasing and bounded below so: $\lim_{r\to\infty} E_{a_0}(r)$ exists, and $\lim_{r\to\infty} u_{a_0}'^2(r) = 0$ exists $0 \le E_{a_0}(r) = \frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)} + F(u_{a_0}(r))$. Taking limit of $E_{a_0}(r)$ as r goes to infinity.

$$\lim_{r\to\infty} E_{a_0}(r) = F(A)$$

so $0 \le F(A)$ so since $A \ge 0$ then either $A=0$ or $A \ge \gamma$. Let us assume $A \ge \gamma$ by above we get: $0 \le \lim_{r\to\infty} E_{a_0}(r) = \lim_{r\to\infty} \frac{1}{2}\frac{u_{a_0}'^2(r)}{K(r)} + F(A)$.

So $A=0$ and thus $\lim_{r\to\infty} u_{a_0}(r) = 0$ so u_{a_0} is solution of (4)–(5).

4. Conclusions:

Through this work, We have been able to prove the existence of a solution to the singular superlinear Dirichlet problem (1) on the exterior domain in R^N. When f is singular at zero and f grows superlinear at infinity, the proof we presented here seems to have some techniques for localized solutions. Also, we show that the energy is strictly decreasing.

A. Appendix

Lemma 1: Let $z>0$. There is a solution U_a of equation (4) if $u_a(z) = u_a'(z) = 0$ on $(z, z+\varepsilon)$ for some $\varepsilon > 0$.

Proof: Suppose first that u_a is a positive solution to (4) on (R,z) with $u_a(z)=0$ and $u_a'(z)=0$ with $u_a \in C^2(R, z-\varepsilon)\bigcap C^0[R, z-\varepsilon)$. Let us determine the behavior of $u_a(r)$ on $(z-\varepsilon, z)$.

Using the fact that $f(u_a) = \frac{-1}{|u_a|^{q-1}u_a} + g_1(u_a)$ where $0 < q < 1$, $g_1(0) = 0$ and g_1 is continuous at $u_a = 0$ then multiplying (4) by $|u_a|^{q-1}u_a$ we obtain:

$$|u_a|^{q-1} u_a u_a''(r) + \frac{N-1}{r}|u_a|^{q-1} u_a u_a'(r) + K(r)\left(-1 + g_1(u_a)\right| {u_a}^{q-1} u_a = 0. \tag{73}$$

Since g_1 is continuous at $u_a = 0$ with $0<q<1$ then

$$\lim_{r\to z^-} K(r) g_1(u_a)|u_a|^{q-1} u_a = 0.$$

Also since u_a' is continuous with $u_a'(z)=0$ and $0<q<1$ then $\lim_{r\to z^-} \frac{1}{r}|u_a|^{q-1} u_a u_a' = 0$ therefore from (73) this implies $\lim_{r\to z^-} |u_a|^{q-1} u_a u_a''(r) = K(z)$. In addition, since $\lim_{r\to z^-} \frac{1}{2}u_a'^2 = 0$ and $\lim_{r\to z^-} \frac{1}{1-q}|u_a|^{1-q} = 0$ then by L'Hopital's rule we have:

$$\begin{aligned} K(z) &= \lim_{r\to z^-} |u_a|^{q-1} u_a u_a''(r) \\ &= \lim_{r\to z^-} \frac{\left(\frac{1}{2}u_a'^2\right)'}{\left(\frac{1}{1-q}|u_a|^{1-q}\right)'} \\ &= \lim_{r\to z^-} \frac{\frac{1}{2}u_a'^2}{\frac{1}{1-q}|u_a|^{1-q}}. \end{aligned}$$

Thus $\lim_{r\to z^-} \frac{|u_a'|}{|u_a|^{\frac{1-q}{2}}} = \sqrt{\frac{2}{1-q}K(z)} > 0$. Therefore $u_a' \neq 0$ on $(z-\varepsilon, z)$ (for some perhaps small ε) and since $u_a > 0$ on $(z-\varepsilon, z)$ it follows that $u_a' < 0$ on $(z-\varepsilon, z)$. Thus $\lim_{r\to z^-} \frac{-u_a'}{u_a^{\frac{1-q}{2}}} = \sqrt{\frac{2}{1-q}K(z)}$, and so on the interval $(z-\varepsilon, z)$ with $\varepsilon > 0$ sufficiently small then there is $\delta > 0$ so that: $\sqrt{\frac{2}{1-q}K(z)} - \delta < \frac{-u_a'}{u_a^{\frac{1-q}{2}}} < \sqrt{\frac{2}{1-q}K(z)} + \delta$. Integrating on (r,z) for r sufficiently close to z gives:

$$\int_r^z \left(\sqrt{\frac{2}{1-q}K(z)} - \delta\right) ds < \int_r^z \frac{-u_a'\, ds}{u_a^{\frac{1-q}{2}}} < \int_r^z \left(\sqrt{\frac{2}{1-q}K(z)} + \delta\right) ds$$

$$\left(\sqrt{\frac{2}{1-q}K(z)} - \delta\right)(z-r) < \frac{2}{q+1} u_a^{\frac{q+1}{2}} < \left(\sqrt{\frac{2}{1-q}K(z)} + \delta\right)(z-r)$$

so

$$\left(\sqrt{\frac{2}{1-q}K(z)} - \delta\right) \le \frac{2}{q+1}\frac{u_a^{\frac{q+1}{2}}}{(z-r)} \le \left(\sqrt{\frac{2}{1-q}K(z)} + \delta\right) \text{ on } (z-\varepsilon, z)$$

Thus:

$$\lim_{r\to z^-} \frac{u_a^{\frac{q+1}{2}}}{(z-r)} = \frac{q+1}{2}\sqrt{\frac{2}{1-q}K(z)}.$$

Let

$$W(r) = \frac{u_a(r)}{(z-r)^{\frac{2}{q+1}}} \text{ where } r \neq z$$

so $\lim_{r\to z^-} W(r) = \left[\frac{q+1}{2}\sqrt{\frac{2}{1-q}K(z)}\right]^{\frac{2}{q+1}}$ so we define

$$W(z) = \lim_{r\to z^-} W(r) = \lim_{r\to z^-} \frac{u_a(r)}{(z-r)^{\frac{2}{q+1}}} = \left[\frac{q+1}{2}\sqrt{\frac{2}{1-q}K(z)}\right]^{\frac{2}{q+1}}.$$

This tells us how u_a behaves on $(z-\varepsilon, z)$ so we expect U to behave similarly on $(z, z+\varepsilon)$ so we will try now to prove the existence of a solution on $(z, z+\varepsilon)$ so that:

$$\lim_{r\to z^+} \frac{u_a}{(z-r)^{\frac{2}{q+1}}} = -\left[\frac{q+1}{2}\sqrt{\frac{2}{1-q}K(z)}\right]^{\frac{2}{q+1}}. \tag{74}$$

Now assuming such a solution exists, Rewriting (4), integrating over (z, r), and using $u_a'(z) = 0$ we get:

$$r^{N-1}u_a'(r) = -\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)f(u_a)\, ds\, dt \tag{75}$$

Multiplying (75) by $\frac{1}{r^{N-1}}$, integrating over (z, r) and using (H1) gives:

$$u_a(r) = -\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{|u_a|^{q-1}u_a(s)} + g_1(u_a)\right] ds\, dt. \tag{76}$$

Making the change of variables of (76):

$$u_a(r) = -(r-z)^{\frac{2}{q+1}}W(r).$$

Then (76) becomes :

$$(r-z)^{\frac{2}{q+1}}W(r) = -\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}|W|^{q-1}W}\right.$$

$$\left. + g_1\left(-(s-z)^{\frac{2}{q+1}}W(s)\right) ds\, dt\right.$$

So:

$$W(r) = \frac{-1}{(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}|W|^{q-1}W}\right.$$

$$\left. + g_1\left(-(s-z)^{\frac{p}{q+1}}W\right) ds\, dt.\right. \tag{77}$$

Assuming $W(r)$ is continuous at z then taking limits in (77) and using L'Hopital's rule we get:

$$W(z) = \lim_{r\to z^+} \frac{\frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}|W|^{q-1}W} + g_1\left(-(s-z)^{\frac{2}{q+1}}W\right)\right] ds}{\frac{2}{q+1}(s-z)^{\frac{1-q}{q+1}}}.$$

Using L'Hopital's rule again we get:

$$W(z) = \frac{1}{z^{N-1}}\lim_{r\to z^+} \frac{r^{N-1}K(r)\left[\frac{-1}{(r-z)^{\frac{2q}{q+1}}|W|^{q-1}W} + g_1\left(-(r-z)^{\frac{2}{q+1}}W\right)\right]}{\frac{2}{q+1}\frac{1-q}{q+1}(r-z)^{\frac{-2q}{q+1}}}.$$

simplifying above we get:

$$W(z) = \frac{(q+1)^2K(z)}{2(1-q)|W(z)|^{q-1}W(z)}$$

and thus:

$$|W(z)| = \left[\frac{(q+1)^2K(z)}{2(1-q)}\right]^{\frac{1}{q+1}}.$$

Let $W(r) = CY(r)$ where $C = -\left(\frac{(q+1)^2K(z)}{2(1-q)}\right)^{\frac{1}{q+1}}$.
Then $Y(z) = 1$
so

$$Y(r) = \frac{-1}{C(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}C^qY^q}\right.$$

$$\left. + g_1\left((s-z)^{\frac{2}{q+1}}CY\right)\right] ds\, dt. \tag{78}$$

Now we attempt to can solve (78) by using the contraction mapping principle theorem. We define the set:

$B = \{Y \in C[z, z+\varepsilon] \mid Y(z) = 1 \text{ and } ||Y(r) - 1|| < \delta\}$

where δ is sufficiently small.

Let:

$$||Y|| = \sup_{x\in[z,z+\varepsilon]} |Y(x)|$$

Now define $T : B \longrightarrow C[z, z+\varepsilon]$ by:

$$TY(r) = \frac{-1}{C(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}C^qY^q}\right.$$

$$\left. + g_1\left((s-z)^{\frac{2}{q+1}}CY\right)\right] ds\, dt.$$

Let us suppose $Y_1, Y_2 \in B$ then:

$$TY_1(r) - TY_2(r) = \frac{-1}{C(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}C^q}\right.$$

$$\left[\frac{1}{Y_1{}^q} - \frac{1}{Y_2{}^q}\right] + g_1\left((s-z)^{\frac{2}{q+1}}CY_1\right) - g_1\left((s-z)^{\frac{2}{q+1}}\right.$$

$$\left.\left. CY_2\right)\right] ds\, dt. \tag{79}$$

divided the integration in two parts:

For the first part of the integral since $Y_1, Y_2 \in B$.

Then by the mean value theorem there is Y_3 between Y_1, Y_2 also since $0 < Y_2 < Y_3 < Y_1 < \delta + 1$ where $|Y_i - 1| < \delta$ for $i = 1,2,3$ then $1 - \delta < Y_3 < 1 + \delta$ then $\left|\left[\frac{1}{Y_1^q} - \frac{1}{Y_2^q}\right]\right| = \frac{q}{{Y_3}^{q+1}}|Y_1 - Y_2| \leq \frac{q}{(1-\delta)^{q+1}}|Y_1 - Y_2|$.

Then the first part of the integral becomes:

$$\left|\frac{-1}{C(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\left[\frac{-1}{(s-z)^{\frac{2q}{q+1}}C^q}[\frac{1}{{Y_1}^q} - \frac{1}{{Y_2}^q}]\right]\right| ds\, dt$$

$$\leq \frac{q}{(1+\delta)^{1+q}}\frac{|Y_1 - Y_2|}{C^{q+1}(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t \frac{s^{N-1}K(s)}{(s-z)^{\frac{2q}{q+1}}} ds\, dt.$$

$$\leq \frac{q}{(1+\delta)^{1+q}}\frac{|Y_1 - Y_2|}{C^{q+1}(r-z)^{\frac{2}{q+1}}}\int_z^r \int_z^t \frac{K(s)}{(s-z)^{\frac{2q}{q+1}}} ds\, dt.$$

$$\leq \frac{q}{(1+\delta)^{1+q}}\max_{[z,z+\varepsilon]}K(r)\frac{|Y_1 - Y_2|}{C^{q+1}(r-z)^{\frac{2}{q+1}}}\int_z^r \int_z^t \frac{1}{(s-z)^{\frac{2q}{q+1}}} ds\, dt.$$

Carrying out the integration and recalling $C^{q+1} = \frac{(q+1)^2K(z)}{2(1-q)}$ we obtain:

$$\leq \frac{q}{(1+\delta)^{1+q}}\max_{[z,z+\varepsilon]}K(r)\frac{|Y_1 - Y_2|}{\frac{(q+1)^2K(z)}{2(1-q)}}\frac{(q+1)^2}{2(1-q)}$$

$$= \frac{q}{(1+\delta)^{1+q}}\frac{\max_{[z,z+\varepsilon]}K(r)}{K(z)}|Y_1 - Y_2|.$$

Since $K(z) \neq 0$ and K is continuous then $\frac{\max_{[z,z+\varepsilon]}K(r)}{K(z)} \longrightarrow 1$ as $\varepsilon \longrightarrow 0$. Also since $\delta > 0$ and $q < 1$ we see that for $\varepsilon > 0$ sufficiently small then $\frac{q}{(1+\delta)^{1+q}}\frac{\max_{[z,z+\varepsilon]}K(r)}{K(z)} \leq d \leq 1$.

For the second part of the integral since g_1 is locally Lipschitz at W near 0 then:

$$\left|g_1\left(-(s-z)^{\frac{2}{q+1}}CY_1\right) - g_1\left(-(s-z)^{\frac{2}{q+1}}CY_2\right)\right| \leq L|s-z|^{\frac{2}{q+1}}C\, ||Y_1 - Y_2||$$

so substituting into the second part of (78) gives:

$$\frac{-1}{C(r-z)^{\frac{2}{q+1}}}\int_z^r \frac{1}{t^{N-1}}\int_z^t s^{N-1}K(s)\Big[g_1\left((s-z)^{\frac{2}{q+1}}CY_1\right) - g_1\left((s-z)^{\frac{2}{q+1}}CY_2\right)\Big] ds\, dt.$$

$$\leq \frac{|Y_1 - Y_2|CL}{C(r-z)^{\frac{2}{q+1}}}\max_{[z,z+\varepsilon]}K(r)\int_z^r \int_z^t |s-z|^{\frac{2}{q+1}} ds\, dt.$$

$$\leq \frac{|Y_1 - Y_2|L}{(r-z)^{\frac{2}{q+1}}}\max_{[z,z+\varepsilon]}K(r)(r-z)^{\frac{2}{q+1}}(r-z)^2 = |Y_1 - Y_2|L\max_{[z,z+\varepsilon]}K(r)(r-z)^2 \leq \frac{1-d}{2}|Y_1 - Y_2|.$$

since $\lim_{r\to z} L\max K(r-z)^2 = 0$ we can choose ε small enough so that $L\max K(r-z)^2 < \frac{(1-d)}{2}$ so $d + \frac{1-d}{2} = \frac{1+d}{2} < 1$ and so combining these two part we get

$$|TY_1(r) - TY_2|(r) \leq \frac{1+d}{2}|Y_1 - Y_2|$$

Thus T is a contraction mapping if $0 < \frac{1+d}{2} < 1$ is sufficiently small, so there is a unique solution $Y \in B$ to $T(Y) = Y$ on $[z, z+\varepsilon]$. Then $u_a(r) = -(r-z)^{\frac{2}{q+1}}W(r)$ is a solution of (4)–(5)on $[z-\varepsilon, z+\varepsilon]$ for some $\varepsilon > 0$.

Lemma 2: The energy equation $E(r)$ is strictly decreasing.

Proof: From (12) we know that $E'(r) \leq 0$ so $E(r)$ is non-increasing. Suppose by way of contradiction that E is not strictly decreasing then there are r_1, r_2 with $r_1 < r_2$ such that $E(r_1) = E(r_2)$ so $E(r)$ is constant on $[r_1, r_2]$ so $E'(r) \equiv 0$ on $[r_1, r_2]$ so $U_a'(r) \equiv 0$ on $[r_1, r_2]$ then by the uniqueness of solution of initial value problem $u_a \equiv 0$ on $[R, \infty]$ but $u_a'(R) = a > 0$ contradiction so E must be strictly decreasing. this proofs lemma 2.

Funding: None.

Data Availability Statement: All of the data supporting the findings of the presented study are available from corresponding author on request.

Declarations:
Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: The manuscript has not been published or submitted to another journal, nor is it under review.

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وجودية الحلول لمسائلة منفردة غيرخطية مع الشروط الحدودية لدرجلت في المجال الخارجي

مجيد علي [1*] ، جوزيف اية [2]

[1*] قسم الرياضيات، كلية العلوم، جامعة كركوك، كركوك، العراق.

[2] قسم الرياضيات، جامعة شمال تكساس، دينتون، تكساس، الولايات المتحدة.

* الباحث المسؤول: mageedali@uokirkuk.edu.iq

الخلاصة

لقد أثبت هذا البحث وجود الحلول التي تحل المعادلة التفاضلية الجزئية غير الخطية. لقد تم تطوير دراسة للأنظمة الديناميكية على الجزء الخارجي من الكرة المتمركزة في الأصل في R^N مع نصف القطر $R > 0$ ، مع شروط حدود من النوع الاول عندما $u = 0$ على الحدود، و $u(x)$ تقترب من. عندما $|x|$ تقترب من اللانهاية، حيث الدالة $f(u)$ هي المنفرد الليبشيتزي المحلي عند الصفر، وتنمو الدالة بشكل فائق عندما تقترب من اللانهاية. من خلال إدخال مقاييس مختلفة لتوضيح سلوك الدالة المنفرد بالقرب من المركز وفي اللانهاية. وأيضا، $N > 2$ ، والدالة تسلك ك $|u|^{q1}|u| - |$ عندما u صغيرة مع $0 < q < 1$ ، والدالة تسلك $|u|^p|u|$ عندما u كبيرة مع $p > 1$ بالإضافة إلى ذلك، $K(x)$ $|x|^-$ مع $2(N-1) << 2$ عندما لـ $|x|$ كبيرة وقد تم استخدام طريقة النقطة الثابتة وغيرها من التقنيات لإثبات الوجودية.

الكلمات الدالة : المجالات الخارجية، المنفرد، اللاخطية، الوجودية.

التمويل: لايوجد.

بيان توفر البيانات: جميع البيانات الداعمة لنتائج الدراسة المقدمة يمكن طلبها من المؤلف المسؤول.

اقرارات:

تضارب المصالح: يقر المؤلفون أنه ليس لديهم تضارب في المصالح.

الموافقة الأخلاقية: لم يتم نشر المخطوطة أو تقديمها لمجلة أخرى، كما أنها ليست قيد المراجعة.