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Condensation to Fractal Shapes Constructing

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Abstract*—Two properties must be available in order to construct a fractal set. The first is the self similarity of the elements. The second is the real fraction number dimension. In this paper, condensation principle is introduced to construct fractal sets. Condensation idea is represented in three types. The first is deduced from rotation –reflection linear transformation. The second is dealt with group action. The third is represented by graph function.*

Keywords—Fractals; FIF; Groups.

I. INTRODUCTION

When Mandelbrot started writing about fractals, he came up with the notion that they should be self-similar. Our way to formalize is through iterated function systems (IFS) which is due to analogous the dynamical systems. They have lots of similar properties, where the name IFS comes from, and reminds us of dynamical systems.

Fractal technique is a good representation of natural phenomena and shapes [1]. We can consider that the IFS is the generator of fractal set or shape.

Iterated Function Systems (IFS) play a good role for constructing fractal shapes. IFS was introduced by Hutchinson in [2], then studied by Mandelbrot in [3] and Falconer in [4], and popularized by Barnsley in [5-7].

IFS fractals work by transforming segments of a data set into smaller segments that are self-similar.

The resulting IFS parameterization has advantages over the raw raster : considerably less storage is required, the data may be graphically manipulated, and other operations are performed while the data are in a compressed form . .

An IFS is a set of contraction mappings $w=\{w_1, w_2, \ldots, w_N\}$ acting on a space X. Associated with this set of mappings w, is a set of probabilities $p=\{p_1, p_2, \ldots, p_N\}$.These probabilities are used to generate a random walk in the space X. If We start with any point (such as dot, flag ,or heart) in X and applies these maps iteratively , he will come arbitrarily close to a set of points A in X called the attractor of the IFS. These attractors are very often fractal. In order to represent the sets of fractals there are two techniques , the random iteration algorithm, and deterministic method .

Fractal set is famous with two properties. The first is the self similarity of the elements which belong to the set. The second is concerned with the real fraction number dimension. In this paper, condensation principle is introduced to construct fractal set of many shapes. Here condensation idea is represented in three types. The first is deduced from rotation –reflection linear transformation. The second is dealt with group action. The third is represented by graph function.

II. BASIC CONCEPTS OF FRACTALS

This section is concerned of writing the basic mathematical concepts of fractal sets.

Definition (1, [8]) *Let* X be a non-empty set .A real-valued function d is defined on $X \times X$, is called a metric or distance function on X if and only if it satisfies , for every $x, y, z \in X$, the following axioms :

1. $d(x,y) > 0$ and $d(x,x) = 0$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \le d(x,y) + d(y,z)$

The real number $d(x,y)$ is called the metric distance from x to y .

For example Barnesly metric [7] which is defined as
$X = R^2 : d((x_1, y_1),(x_2, y_2)) = |x_1 - x_2| + \vartheta |y_1 - y_2|$, $\vartheta \in R^+$, $\forall (x_1, y_1),(x_2, y_2) \in R^2$

Definition (2, [8]) Let d be a metric on a non-empty set X . The topology τ on X generated by the class of open spheres in X is called the metric topology (or ,the topology induced by the metric d) . Furthermore the set X together with the topology τ induced by the metric d is called a metric space and is denoted by (X,d) .

In this paper we used R^2 induced topological metric space.

Definition (3, [8]) Let V be a real vector space under an operation of vector addition and of scalar multiplication by real numbers . A function which assigns to each vector $v \in V$ the real number $\|v\|$ is a norm on V iff satisfies , for all $v, w \in V$ and $k \in R$, the following axioms :

1) $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$.
2) $\|v + w\| \leq \|v\| + \|w\|$.
3) $\|kv\| = |k| \|v\|$.

Indeed $d(v,w) = \|v - w\|$ is a metric on V) , where $v, w \in V$.

Definition (4, [8]) Let X be a metric space .A sequence $\{a_n\}_{n=1}^{\infty}$ in X is a Cauchy sequence iff for every $\varepsilon > 0, \exists n_0 \in N$ such that
$n, m > n_0 \Rightarrow d(a_n, a_m) < \varepsilon$.

Definition (5, [8]) A metric space (X,d) is complete if every Cauchy sequence in X converges to a point $p \in X$.

Definition (6, [8]) A complete normed vector space is called a Banach space

Definition (7, [2]) A continuous function $f : X \rightarrow X$ is a contracting if there is an $\alpha \, \varepsilon \, (0 , 1)$ such that
$d(f(x), f(y)) \leq \alpha d(x, y)$ for each x and y in X ,and α is called a counteractivity factor of f .

Definition (8, [1-7]) Let $N_n(A)$ denote the number of boxes of length $\left(\frac{1}{2^n}\right)$ which intersect the attractor . If $D = \underset{n \to \infty}{Lim}\{\frac{Ln(N_n(A))}{Ln(2^n)}\}$ exist , then A has fractal dimension D.

Theorem (1, [1]) (Fixed Point Theorem) If f is a contracting function on a complete metric space X , then there exists a unique point $p \in X$ such that $f(p) = p$. And for each point $p_0 \in X$, the orbit $\{f^n(p_0)\}_{n=0}^{\infty}$ converges to that fixed point .

Definition(9,[9]) Let P be a set and G be a group .An action of G on P is a function $\circ : P \times G \rightarrow P$ such that :
$p \circ e = p$ for all $p \in P$.
$p \circ (g_1 * g_2) = (p \circ g_1) \circ g_2$ for all $p \in P$ and all $g_1, g_2 \in G$.
Under these conditions , P is a G-set.

Definition(10,[10]) A linear transformation
$w : R^2 \rightarrow R^2$ can be represented as of the form

$$w(x) = w\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ q \end{pmatrix}$$, where

$a,b,c,d,e,$ and q are real numbers.

III. Fractal Shapes Constructing Using Condensation

Let w be a function in a metric space (X,d), then $w(x)$ is a point in X , and the composition operation $w(w(x))$ is also in X .

In other words , We can write $w^2(x)$ for $w(w(x))$.
Similarly We can define
$w^{m+1}(x) = w(w^m(x)), x \in X$,
and m is natural number .
So for obtaining a infinite sequence
$\{x_m\}_{m=0}^{\infty}$ in X ,
We can write
$x_0 = x, x_1 = w(x), x_2 = w^2(x), \ldots, x_m = w^m(x), \ldots$

When w is contraction and using Fixed Point Theorem on a complete metric space R^2 , then there exists a unique point $p \in R^2$.

An IFS consists of a complete metric space (R^2, d) , together with a finite set of contraction mappings

$w_n : X \to X$, with respective contractility factors α_n , for $n = 1,2,\ldots,N$.

We can take { $H(X); w_1, w_2, \cdots, w_N$ } be an IFS with contractility factor $\alpha = \max\{\alpha_n\}_{n=1}^{N}$.

We can define $w(B) = \bigcup_{n=1}^{N} w_n(B)$

Then using fixed point theorem We can has:

1) $W(B)$ is a contracting mapping with counteractivity factor α .

2) Its unique fixed point $A \in H(X)$ obeys

$$A = W(A) = \bigcup_{n=1}^{N} w_n(A)$$

and is given by

$$A = \lim_{k\to\infty} W^k(B)$$

for any $B \in H(X)$.

Indeed any set $A \in H(X)$ such that $W(A) = A$ is considered as attractor or fractal set for the IFS $\{H(X); w_1, w_2, \cdots, w_N\}$.

A. Fractal Set Attractor via IFS

From above we have $A \in H(X)$ such that $W(A) = A$ is considered as attractor or fractal set for the IFS $\{H(X); w_1, w_2, \cdots, w_N\}$.

Following the Iteration Algorithm for constructing the attractor of IFS in complete metric space R^2.

Procedure 1 (Fractal IFS Shape)

1- Take $\{R^2; w_1, \cdots w_N\}$ be a IFS , where probability $p_n > 0$ has been assigned to w_n for $n = 1,2,\ldots,N$, where $\sum_{n=1}^{n} p_n = 1$

2- Choose initial $x_0 \in X$

3- Choose recursively , independently $x_i \in \{w_1(x_{i-1}), w_2(x_{i-1}), \cdots w_N(x_{i-1})\}$ for $i = 1,2,\ldots,$

4- The probability of the event $x_i = w_n(x_{i-1})$ is p_n

5- The sequence $\{x_i\}_{i=0}^{\infty}$ converges to the attractor of the IFS.

For implementation of procedure 1 we take the R^2 - Cantor triangle Set which is constructed by iteration of three R^2-functions,

$$f(x) = \frac{1}{2}x; g(x) = \frac{1}{2}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}; h(x) = \frac{1}{2}x + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the result is in Figure.1 with corresponding functions referred in Table.1

B. CONDENSATION via ROTATION-REFLECTION LINEAR TRANSFORMATON

We can take a linear transformation

$$w : R^2 \to R^2$$

$$w(x) = w\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$w(x) = Ax + t$$

Here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a two-dimensional , 2×2 real matrix and t is the column vector $\begin{pmatrix} e \\ f \end{pmatrix}$.

Indeed we take matrix A as a rotation or reflection which acts on the point $(x_1, x_2) \in R^2$ and denoted by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

In R^2 maps any domain with a vertex at the origin to another domain with a vertex at the origin. And that the domain may be turned over by the transformation $w(x) = Ax + t$ in R^2 , A which preserve the space relative to the origin , followed by a translation or shift specified by the vector t .

Now for $0 < r < 1$ and $0 \le \theta \le 2\pi$, We can take the rotation transformations,

$$w\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

and a reflection transformations,

$$u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Procedure. 2 (Reflection-Rotation Transformation)

1) take m=3 to be the number of copies
2) take the angle $\theta \in [0,2\pi]$,and the scaling $r \in [0,1]$
3) take initial point $(x_0, y_0) \in R^2$

4) for θ =0: $\frac{\pi}{6}$:2 π
for r=0: 0.1 :1
for j =1 to n
k = integer(2*random) +1

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = rw\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}_t \quad \text{for fixed k}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = ru\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}_t \quad \text{for fixed k}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = rA\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \quad \text{for fixed k}$$

end j
end r
end θ

Implementations of above procedure is showed in Figure 2 for some values written in Table 1.

C. FRACTAL SHAPES via GROUP ACTION

In this section, we study the action on a metric space R^2 by the finite group G.
We study the condensation of fractal set using the action on R^2 of G.
Let G has the eight elements $G = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$.
Table * represents all operations of the elements of G.

Table * The group G

*	g1	g2	g3	g4	g5	g6	g7	g8
g1	g1	g2	g4	g5	g6	g7	g8	g4
g1	g2	g3	g4	g1	g7	g8	g6	g5
g1	g3	g4	g1	g2	g6	g5	g8	g7
g1	g4	g1	g2	g3	g8	g7	g5	g6
g1	g5	g8	g6	g7	g1	g3	g4	g2
g1	g6	g7	g5	g8	g3	g1	g2	g4
g1	g7	g5	g8	g6	g2	g4	g1	g3
g1	g8	g6	g7	g5	g4	g2	g3	g1

For studying the group action of G on R^2 , we taking the following actions.

1) $g_1 p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the rotation of P about the origin , through 360^0 .

2) $g_2 p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the rotation of P about the origin , through 90^0 .

3) $g_3 p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the rotation of P about the origin , through 180^0 .

4) $g_4 p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the rotation of P about the origin , through 270^0 .

5) $g_5 p = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the reflection of P in the x-axis .

6) $g_6 p = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the reflection of P in the y-axis .

7) $g_7 p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ represents the reflection of P in the line $y = x$.

8) $g_8 p = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ represents the reflection of P in the $y = -x$.

Implementation of the action of G on a set F may be angles preserver as in figure.3 or angles distorter as in figure.4 .

When we have group of order n, then the number of action operations for a non commutative group is n^2, while the number of action operations for a non commutative group is $\frac{1}{2}n(n+1)$.

D. CONDENSATION VIA GRAPH FUNCTION

When we have a graph H of a continuous R^2-function $y = f(x)$, and a fractal set F, we can expand f for $Y = f(F)$ via plotting F under our function f . Implementation of our Condensation of fractal set F via Graph Function represented in figure. 5 with corresponding original functions referred in table.5.

E. IMPLEMENTATIONS and CONCLUSIONS

The idea of condensation is represented in many mathematical techniques. The first is the concept of rotation and reflection matrix with translation which is studying in procedure.2. Many values of counteractivity factor r and angle θ referred in Table.2.
The implementation of procedure.2 presented in Figure.2. Clearly there are many fantastic views.

The second is the concept of preserver and distorter group action which is studying in procedure 3. Preserver elements referred in Table 3 where its implementation appears in Figure 3. In the same way the distorter group action elements were written in Table 4 with its implementations were presented in Figure.4. All elements group act the fern F.
We can notice that there are many degrees of complexity of F deduced from distorter elements group.

The third is dealing with the graph function in R^2 space. The big shape named as fern(F) is dealt as dependent variable under continuous function via procedure 5. Many functions are used (as referred in Table.5) and its implementations were appeared in Figure.5 for many good landscapes of gardens.

Indeed procedure 1 concerned for constructing fractal shape via we initial point in R^2 space and finite number of functions under iterations. Table.1 referred to simple IFS and its implementation shows a good shapes as in Figure.5.

REFERENCES

[1] Barnsley, M. F. "Fractals Everywhere", Academic Press, 1988.
[2] Hutchinson J. " Fractals And Self Similarity"; Indian University Mathematics J. , VOL.30, PP.713-740, 1981 .
[3] Mandelbrot, B. B."The Fractal Geometry of Nature ", San Francisco: W.H. Freeman, 1882.
[4] Falconer K., " Fractal Geometry: Mathematics Foundations and Applications", John Wiley & Sons, Chichester, England,1990.l
[5] Barnsley, M. F." Fractal Functions and Interpolation Constructive " Approximation, 2, 303– 329, 1986.
[6] Barnsley, M. F., "Fractals Everywhere", Academic Press, 1988.
[7] Barnsley, M. F. and Harrington A.N." The Calculus of Fractal Interpolation Functions ", J. Approximated Theory 57, PP.14-34, 1989.
[8] Berberian S.K. ," Introduction to Spectral Theory", Oxford Un. Press, 1961.
[9] Fraleigh J. B. ," A First Course in Abstract Algbra ", 3 e, Addison Wesley, 1984.
[10] Kolman B. " Introductory Linear Algebra with Applications ", Macmilan , 1984.

APPENDIX- Tables and Figures

Table 1 The 3-IFS

$f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$h(x) = \frac{1}{2}x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

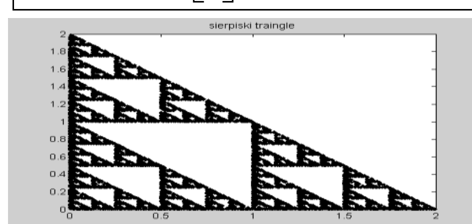

Fig.1 Fractal Set Deduced from IFS in Table1 of Procedure1

Table 2 parameters of procedure 2

Left figure	Right figure
1) $r = 0.5$, $\theta = 90^0$	1) $r = 0.5$, $\theta = 60^0$
2) $r = 0.1$, $\theta = 30^0$	2) $r = 0.25$, $\theta = 90^0$
3) $r = 0.5$, $\theta = 180^0$	3) $r = 0.9$, $\theta = 90^0$
4) $r = 0.3$, $\theta = 120^0$	4) $r = 0.8$, $\theta = 30^0$

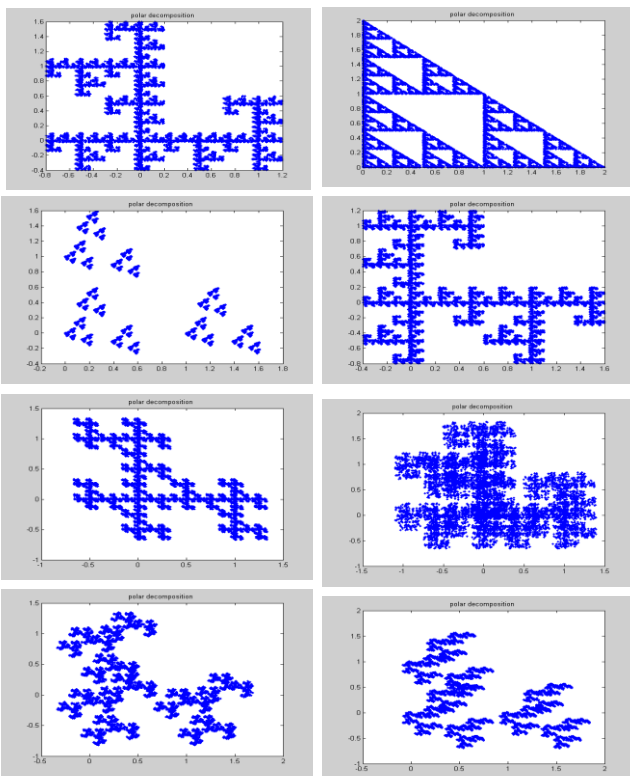

Fig. 2 Implementation of Procedure 2 for Values in Table.2

Table3 Preserver Group action of Procedure.3

Original Serpinisky triangle(ST)	ST & action g_1
ST & action g3	ST & action g4

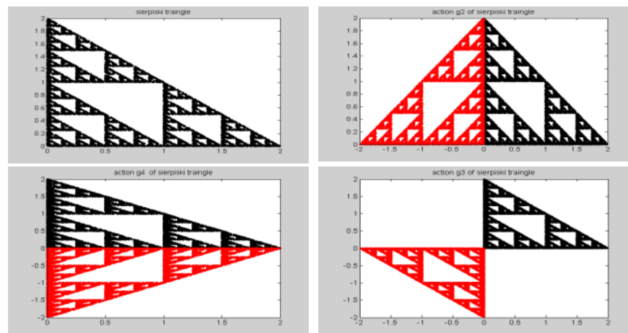

Fig. 3 implementation of procedure 3 for Detailed Referred in Table 3

Table 4 distorter group Action of Procedure 3

Original fern(F)	Action on F by g_8 g_6 g_7 g_4
Action on F by g_8 g_4 g_5 g_5	Action on F by g_5 g_6 g_7 g_4
Action on F by g_3 g_4 g_7 g_5	Action on F by g_8 g_6 g_1 g_5
Action on F by g_2 g_4 g_7 g_3	Action on F by g_3 g_8 g_7 g_5

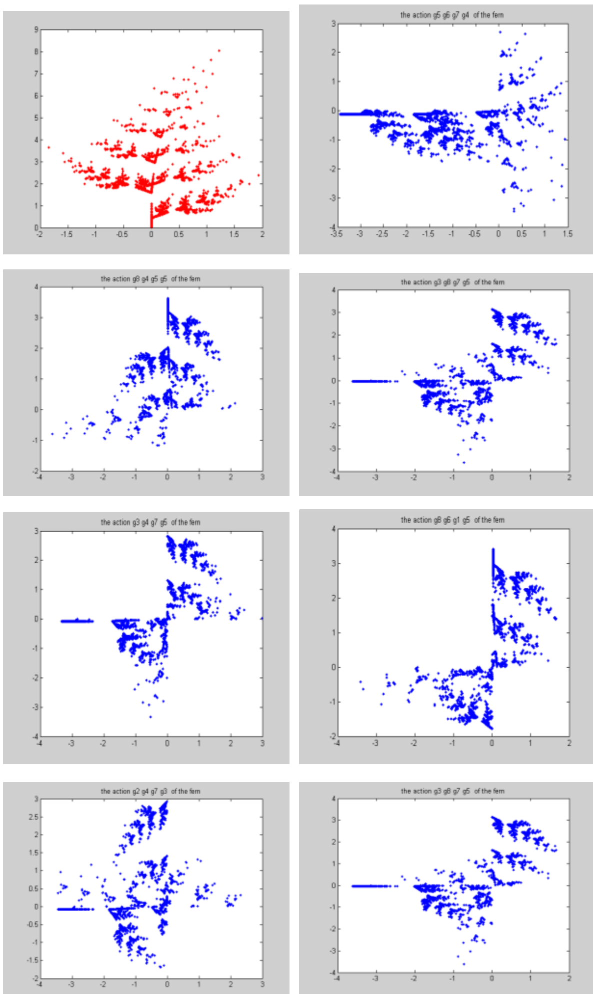

Fig.4 Implementation of Procedure 3 for Detailed Referred in Table 4

Table 5 of Procedure 4 for Function Depend On The Fern F

Circle equation	absolute function
Sine function	Cubic function
Reduction function	3- origin Circles
Minus Square function	Square function

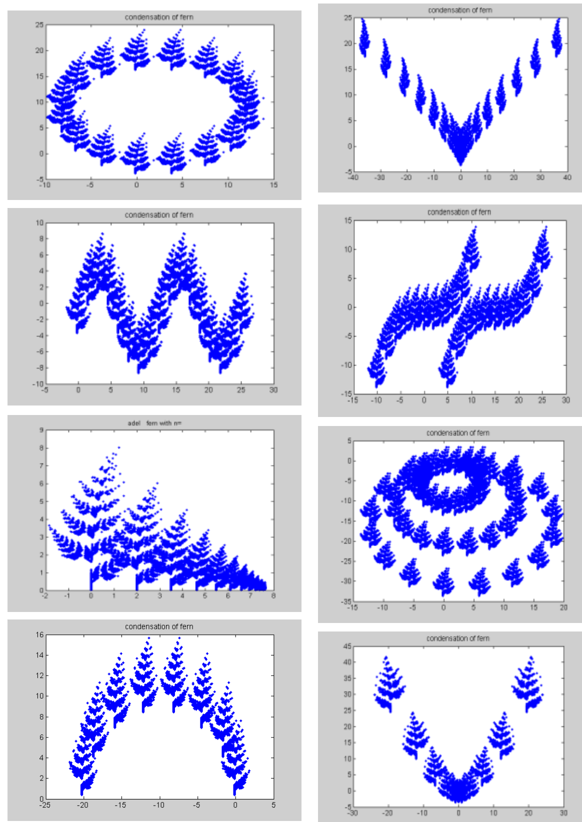

Fig. 5 Implementation of Procedure 4 for Detailed Referred in Table 5