Number of Spinal-Convex Polyominoes

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Abstract—: *In his paper we describe a restricted class of polyominoes called spinal-convex polyominoes. Spinal-convex polyominoes created by two columns such that column 1 (respectively, column2) with at most two set columns sequence of adjacent ominoes and column 2 (respectively, column1) with at least one set column sequence of adjacent ominoes. In addition, this study reveals new combinatorial method of enumerating spinal-convex polyominoes.*
Keywords—: Polyominoes, Spinal-convex, Set column sequence, enumerating.

I. INTRODUCTION

The connected shape has been made and used as a part of the mainstream riddles since the year 1907, but the first researcher to study it systematically is Solomon Golomb in 1954[1]. He proposed a connection of n squares adjacent edge to edge with a internal connection as polyominoes as well as a unit square as ominoes. In advancing in this field, Klarners Konstant (1966)2] defines a connectivity of finite number of unit square devoid of cat point as n-ominoes. This polyominoes is one of the most popular areas in the field of combinatory mathematics.It's long history begins right from 19th century. It's also has drawn the attention of researchers from other fields of chemistry and physics. Particularly, the former establishes a relationship with polyominoes by given a definition to equivalent objects namely animals [3] which is acquired by taking the canter of the cells of polyominoes. The polyominoes is a vital aspect in this study and is considered as the object of many problems. The number polyominoes that can be achieved from n unit squares is included in the list of unsolved problem which is known as enumeration problem. This enumeration problem is associated with an equivalent of polyominoes object known as cell growth which begins from one omino and continue to grow gradually by the addition of omino at every step to its periphery [4]. Usually, closed-form expressions for the size of most classes of polyominoes are unknown [10]. In dealing with this type of problem, the easiest thing to do is to solve a similar or simpler problem which will give us the understanding or idea that can be used to solve the original problem. In the study of polyominoes, the enumeration of simpler subsets of polyominoes has been the most effective line of research. The set of polyominoes can be reduced until it is solvable by enforcing additional restrictions, such as directedness or convexity as mentioned by [5], [8], [6], [7], [9], [11], [12], [13], [14]. In the present paper, our interested is the number of polyominoes with n cells and two columns knows as $\mathfrak{N}^2$. In the rest of this paper the construction of class has been discussed and hence one of the important properties for the discrete objects (objects can be count and classified).

DEFINITION AND TERMINOLOGIES

In this section, a definitions of the of connectedness of ominoes with respect two columns is introduced as illustrated by some examples. We begin by introducing the definition of the connected ominoes located in the same column.

Definition 2.1 *Let w_δ and $w_{\delta'}$ be two ominoes in the polyominoes with e columns and r rows. w_δ and $w_{\delta'}$ are adjacent if one of the following conditions are satisfied:*

- $|w_\delta - w_{\delta'}| = e$ *if w_δ and $w_{\delta'}$ are located in one column.*
- $|w_\delta - w_{\delta'}| = 1$ *if w_δ and $w_{\delta'}$ are located in one row.*

Definition 2.2 A sequence $\{w_{\lambda_1}, w_{\lambda_2}, \dots, w_{\lambda_b}\}$ of ominoes in the polyominoes with e columns and r rows is called set-column of connected ominoes and denoted by SC if they belong to the same column and *every two consecutive elements in the sequence are adjacent.*

Lemma 2.3 Let $w_{\lambda_1} < w_{\lambda_2} <\ldots< w_{\lambda_b}$ then,

$$w_{\lambda_2} - w_{\lambda_1} =\ldots= w_{\lambda_b} - w_{\lambda_{b-1}} = e$$

where $1 \leqslant b \leqslant n$.

Proof. Since the sequence $w_{\lambda_1} < w_{\lambda_2} <\ldots< w_{\lambda_b}$ is belonging to one column and from Definition 2.2, every two consecutive elements are adjacent, therefore, from Definition 2.1,

$$w_{\lambda_2} - w_{\lambda_1} =\ldots= w_{\lambda_b} - w_{\lambda_{b-1}} = e$$

where $1 \leqslant b \leqslant n$.

Based on Definition 2.2, the ominoes positions located in same column are connected if they belong to one Sc.

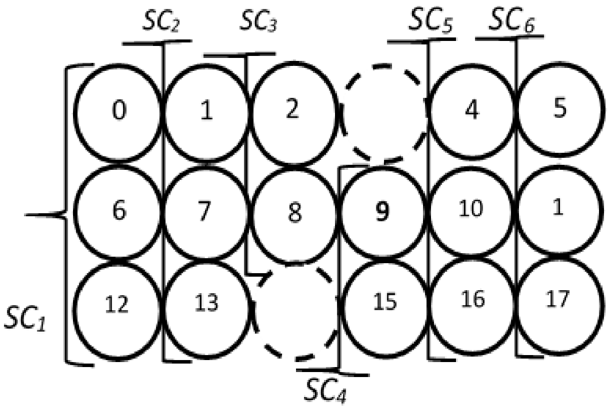

Figure 1: Nested chain abacus of 16-connected beads with 6 columns and 3 rows

In the rest of this section, the connectedness of any two bead positions located in different columns is discussed.

Let us suppose that w_δ and $w_{\delta'}$ are two ominoes positions belonging to SC_j and $SC_{j'}$ respectively, such that SC_j and $SC_{j'}$ are two set-columns located in column j and j' respectively. First, the connectedness between w_δ and $w_{\delta'}$ is defined when j and j' are consecutive numbers.

Definition 2.4 Let SC_j and $SC_{j'}$ be set-columns of connected beads located in columns j and j' respectively in the polyominoes such that $j = j' + 1$. Then SC_j is connected with $SC_{j'}$ if at least one of the ominoes in SC_j is adjacent to a bead in $SC_{j'}$.

Lemma 2.5 Let $SC_{j'}$ and $SC_{j'}$ be set-columns of connected ominoes located in columns j and $j+1$ respectively, in the polyominoes. Then $SC_{j'}$, $SC_{j'}$ are connected if $|w_a - w_b| = 1$ where $w_a \in SC_j$ and $w_b \in SC_{j'}$.

Proof. The two set-columns belong to two consecutive columns and from Definition 2.4, $\exists$ $w_a \in SC_j$ and $\exists w_b \in SC_{j'}$ are adjacent; therefore, based on Definition 2.1,

$$|w_a - w_b| = 1.$$

Based on Definition 2.4, then w_δ and $w_{\delta'}$ are connected if SC_j and $SC_{j'}$ are connected.

Next, the connectedness between w_δ and $w_{\delta'}$ if j and j' are not consecutive numbers is defined.

Definition 2.6 Let SC_j and $SC_{j'}$ be two set-columns of connected beads located in columns j and j' respectively, in the nested chain abacus such that $j > j' + 1$. SC_j is connected with $SC_{j'}$ if there exists $SC_{k_1}, SC_{k_2}, \ldots, SC_{k_z}$ which satisfy the following conditions:

1. SC_j is connected with SC_{k_1}
2. SC_{k_z} is connected with $SC_{j'}$,

where $SC_{k_{z'}}$ set-columns of connected beads and $SC_{k_{z'}}$ is directly linked to $SC_{k_{z'+1}}$ for

$$1 \leqslant z \leqslant j' - j - 1, 1 \leqslant z' \leqslant z - 1$$

and $k_1, k_2, \ldots, k_{z-1}$ are consecutive numbers.

II. Enumeration of $\mathfrak{N}^2$ Polyominoes

We begin this section by introducing the definition of $\mathfrak{N}^2$ polyominoes

Definition 3.1 A polyominoes is knows as $\mathfrak{N}^2$ if

1. consists of two columns
2. column 1 (respectively, column 2) with at most two SC sequences of adjacent beads.
3. column 2 (respectively, column 1) with at least one SC sequence of adjacent beads.
4. all SC sequences are connected.

Figure 2 illustrate the construction of $\mathfrak{N}^2$ polyominoes for 4-connected beads.

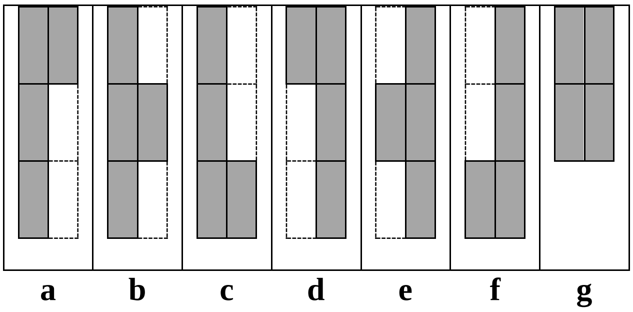

Figure 2: The 7 distinct forms of $\mathfrak{N}^2$ for 4-connected squar

Consider Figure 2 there are 7 distinct forms of $\mathfrak{N}^2$ polyominoes. In Theorem 3.6 the number of $\mathfrak{N}^2$ polyominoes with one SC where $\binom{a}{b} = 0$ if $b > a$ is enumerate.

Lemma 3.2 Let $\mathfrak{N}^2$ be polyominoes with n beads and two columns such that column 1 (respectively, column 2) with one SC sequence of adjacent beads and r beads. If bijection from column 1 (respectively, column 2) with r beads to column 2 (respectively, column 1) with k beads then, the number of $\mathfrak{N}^2$ is

Where k is the number of beads in column 2 (respectively, column 1), $k \leqslant r$ and

$$\delta = \begin{cases} \frac{n}{2} & if \quad n \text{ is even number} \\ \frac{n-1}{2} & if \quad n \text{ is odd number}. \end{cases}$$

Proof. Since $k \leqslant r$ then, k beads will connected with r beads in $\binom{r}{k}$ different ways. Since $1 \leqslant k \leqslant \delta$ then, there are

$$\sum_{k=1}^{\delta} \binom{r}{k}$$

different ways to connected column 1 with column 2. Based on Definition 3.1 the number of $\mathfrak{N}^2$ is

$$\sum_{k=1}^{\delta} \binom{r}{k}.$$

Now, if column 2 with r and based on previous result the number of $\mathfrak{N}^2$ is

$$2\sum_{k=1}^{\delta} \binom{r}{k}.$$

Lemma 3.3 Let $\mathfrak{N}^2$ be polyominoes with n beads and two columns such that column 1 (respectively, column 2) with one SC sequence of adjacent beads and r beads. If the r beads in column 1 connected with S beads in column 2 and $k - S$ beads connected with columns 1 by bead number $q + 1$ or $r + 1$ then, the number of $\mathfrak{N}^2$ is

$$4\left[1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}\right]$$

Where k is the number of beads in column 2 (respectively, column 1), $k \leqslant r$ and

$$\delta = \begin{cases} \frac{n}{2} & if \quad n \text{ is even number} \\ \frac{n-1}{2} & if \quad n \text{ is odd number}. \end{cases}$$

Proof. If the r beads in column 1 will be connected with S beads in column 2 and $k - S$ beads connected with columns 1 by bead number $q + 1$ or $r + 1$ as shown in next figure,

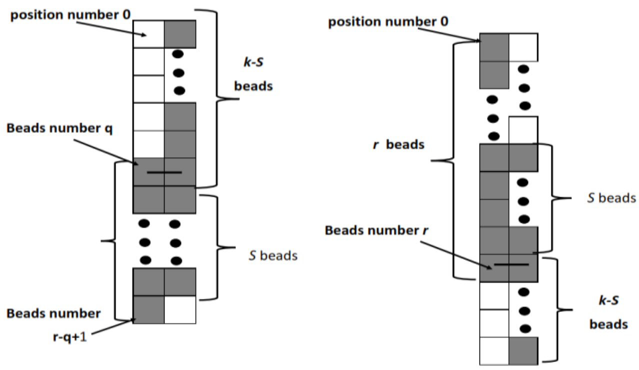

Figure 3: $\mathfrak{N}^2$ with S beads in column 2 and $k - S$ beads connected with columns 1where $1 \leqslant S \leqslant k - 1$.

First, if $\{q + 1, q - 1, q - 3, \dots\}$ are beads positions as shown in Figure 4,

- $S = 0$ then, there is one way to connected column 1 with column 2. Thus there is a polyominoes.

- $S = 1$, Since $\mathrm{k} - \mathrm{s}$ beads connected with column 1 by a bead then, there are $\binom{r-1}{1}$ ways to connected column 1 with column 2.
- $S = 2$, Since $k - s$ beads connected with column 1 by a bead then, there are $\binom{r-1}{2}$ ways to connected column 1 with column 2.

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Thus there are

$$\binom{r-1}{s}$$

way to connected column 1 with column 2. Since $1 \leqslant \mathrm{S} \leqslant \mathrm{k} - 1$ and $1 \leqslant \mathrm{k} \leqslant \delta$ there are

$$1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}$$

ways to connected column 1 with column 2. Based on Definition 3.7 there are

$$1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}$$

of $\mathfrak{R}^2$ polyominoes.

Second, if $\{r+1, r+3, r+5, \dots\}$ are beads positions as shown in above figure, then there are

$$1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}$$

ways to connected column 1 with column 2. Similarity , if the r beads in column 2. Thus the number of $\mathfrak{R}^2$ polyominoes is

$$4\left[1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}\right].$$

Lemma 3.5 Let $\mathfrak{R}^2$ be polyominoes with n beads and two columns such that column 1 (respectively, column 2) with one SC sequence of adjacent beads and r beads. If column 2 with two set-sequences $\{SC_1, SC_2\}$ satisfies the following conditions:

1. SC_1 and SC_2 have at least two beads.
2. SC_1 connected only with the first beads in column 1 and SC_2 connected only with the last beads in column 1.

Then, the number of $\mathfrak{R}^2$ is

$$2\sum_{k=4}^{\delta} k - 4 + 1$$

Where k is the number of beads in column 2 (respectively, column 1), $k \leqslant r$ and

$$\delta = \begin{cases} \dfrac{n}{2} & if \quad n \text{ is even number} \\ \dfrac{n-1}{2} & if \quad n \text{ is odd number}. \end{cases}$$

Proof.

1. If $k = 4$, then there is only one $\mathfrak{R}^2$ in this case.
2. If $k = 5$,
   - SC_1 has two beads and SC_2 has three beads.
   - SC_1 has three beads and SC_2 has two beads.

   .
   .
3. If column 2 with k beads,
   - SC_1 has 2 beads and SC_2 has $\mathrm{k} - 2$ beads.
   - SC_1 has 3 beads and SC_2 has $\mathrm{k} - 3$ beads.

   .
   .
   .
   - SC_{k-2} has 2 beads and SC_2 has 2 beads.

Thus there are $k - 3$ polyominoes and based on Definition 3.7 there are $\mathrm{k} - 3$ of $\mathfrak{R}^2$. Since $1 \leqslant \mathrm{k} \leqslant \delta$ then, the number of $\mathfrak{R}^2$ is

$$\sum_{k=4}^{\delta} k - 3.$$

Theorem 3.6 Let $\mathfrak{R}^2$ be polyominoes with n beads and two columns such that column 1 (respectively, column 2) with one SC sequence of adjacent beads and r beads. Then the number of $\mathfrak{R}^2$ is

$$2\sum_{k=1}^{\delta} \binom{r}{k} + 4\left[1 + \sum_{s=1}^{k-1} \sum_{k=1}^{\delta} \binom{r-1}{s}\right] + 2\sum_{k=4}^{\delta} k - 4 + 1$$

Where k is the number of beads in column 2 (respectively, column 1), $k \leqslant r$ and

$$\delta = \begin{cases} \frac{n}{2} & if \quad n \text{ is even number} \\ \frac{n-1}{2} & if \quad n \text{ is odd number}. \end{cases}$$

Proof. Based on Lemmas 3.3, 3.4, 3.5 the number of $\mathfrak{N}^2$ is

$$2\sum_{k=1}^{\delta}\binom{r}{k} + 4\left[1 + \sum_{s=1}^{k-1}\sum_{k=1}^{\delta}\binom{r-1}{s}\right] + 2\sum_{k=4}^{\delta} k - 4 + 1.$$

Theorem 3.7 Let $\mathfrak{N}^2$ be polyominoes with n beads and two columns such that column 1 (respectively, column 2) with two SC sequence of adjacent beads and r beads. Then, the number of $\mathfrak{N}^2$ is

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta}\binom{r-h-2}{M} + 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r-h-2}{M}$$

Where k is the number of beads in column 2 (respectively, column 1), $k \leqslant r$ and

$$\delta = \begin{cases} \frac{n}{2} & if \quad n \text{ is even number} \\ \frac{n-1}{2} & if \quad n \text{ is odd number}. \end{cases}$$

Proof. We shall prove this result by Principle of Mathematical Induction on r.

Let $r = 1$ then, we shall show that $k = 1$ becomes $k \leqslant r$. Thus, the number of $\mathfrak{N}^2$ is 1. Therefore this result is true for $r = 1$.

Now let us assume that the result is true for for $r = r'$. Then,

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta}\binom{r'-h-2}{M}$$

$$+2\sum_{M=0}^{a}\sum_{a=0}^{\delta+1-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\binom{r'-h-1}{M}$$

$$+2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r'-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-2}{M}$$

$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{\delta+1-g-a}\sum_{a=0}^{\delta+1-g}\sum_{g=3}^{\delta+1}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

Then,

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta+1}\binom{r'+1-h-2}{M}$$

$$+ 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r'-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

Then

$$1 + 2\sum_{M=0}^{a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=1}^{2}\sum_{k=3}^{\delta}\binom{r'-h-1}{M} + 2\sum_{M=0}^{a}\sum_{N=0}^{k-g-a}\sum_{a=0}^{k-g}\sum_{g=3}^{k}\sum_{h=3}^{r-1}\sum_{k=3}^{\delta}\binom{h-2}{N}\binom{r'-h-1}{M}.$$

CONCLUSION

This paper is devote to generate class of polyominoes by using spinal convex polyominoes desigen method. Further, property of the class are proposed. We also enumerate the class of spinal convex polyominoes.

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