Monotonicity of Some Functions Involving The Beta Function

Hanadi D. Saleem
hanadidawood@gmail.com
Department of Mathematics
College of Computers Sciences and Mathematics
University of Mosul, Iraq

Received on: 16/08/2010 **Accepted on: 25/10/2010**

Abstract

In this short paper the monotonicity of functions involving the Beta function is considered to be study by using a similar method given in [2] and [4].

Keywords: Euler gamma function, Digamma function, Beta function.
2000 Mathematics Subject Classification: 33B15.

رتيبة بعض الدوال المتضمنة دالة بيتا

هنادي داؤد سليم
قسم الرياضيات
كلية علوم الحاسوب والرياضيات

تاريخ استلام البحث: 2010/08/16 **تاريخ قبول البحث: 2010/10/25**

الملخص

في هذا البحث تم اثبات رتيبة دوال تتضمن دالة ألبيتا باستخدام طريقة مشابهة معطاة في [2] و[4].

الكلمات المفتاحية: دالة كاما–اويلر ، دالة دي كاما، دالة بيتا

1. Introduction

The Euler Gamma function Γ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \qquad \ldots(1.1)$$

The digamma function ψ is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \ ,$$

and has the following series representation

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}, \qquad \ldots(1.2)$$

where γ is the Euler constant (0.577215) (see [4]).

In [4], J. Sandor proved the following inequality

$$\frac{1}{\Gamma(a+1)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1, \quad x \in [0,1], \quad a \ge 1. \qquad \ldots(1.3)$$

Lator on, L. Bougoffa [2] generalized inequality (1.3) by giving the following

Theorem 1.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b}, \quad \forall x \ge 0,$$

in which $1+ax>0$ *and* $1+bx>0$*, then for all* $a \ge b > 0$ *or* $0 > a \ge b$ *(*$a>0$ *and* $b<0$*),* f *is decreasing (increasing) respectively on* $[0,\infty)$.

The aim of this paper is to present a new result concerning the beta function. In fact, we prove the following:

2. Result

Theorem 2.1. *Let* f *be a function defined by*

$$f(x) = \beta(1+ax, 1+x),$$

when $a>0$, $1+x>0$, $1+ax>0$. *Then f(x) is nonincreasing on* $(-1,\infty)$ *if* $a \le 1$ *and nonincreasing on* $\left(\frac{-1}{a},\infty\right)$ *if* $a \ge 1$.

Proof. We have

$$\beta(1+ax,1+x) = \frac{\Gamma(1+ax)\Gamma(1+x)}{\Gamma(2+ax+x)} = f(x), \text{ say.}$$

The above implies

$$\ln f(x) = \ln\Gamma(1+ax) + \ln\Gamma(1+x) - \ln\Gamma(2+ax+x)$$

By differentiation, we obtain

$$\frac{f'(x)}{f(x)} = \frac{a\Gamma'(1+ax)}{\Gamma(1+ax)} + \frac{\Gamma'(1+x)}{\Gamma(1+x)} - (1+a)\frac{\Gamma'(2+ax+x)}{\Gamma(2+ax+x)}$$
$$= a\psi(1+ax) + \psi(1+x) - (1+a)\psi(2+ax+x)$$

Now, by using equation (1.2) we get:

$$\frac{f'(x)}{f(x)} = a^2 x \sum_{k=0}^{\infty} \frac{1}{(k+1)(1+ax+k)} + x\sum_{k=0}^{\infty}\frac{1}{(k+1)(1+x+k)}$$
$$-(a+1+a^2x+2ax+x)\sum_{k=0}^{\infty}\frac{1}{(k+1)(2+ax+x+k)}$$

$$= -\sum_{k=0}^{\infty}\frac{(a^2x^2+1+4ax+ax^2+a)+(a^2x^2+ax^2+2+6ax+2a)k+(1+a+2ax)k^2}{(k+1)(1+ax+k)(1+x+k)(2+ax+x+k)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)[(1+ax)^2 + a(1+x)^2]}{(k+1)(1+ax+k)(1+x+k)(2+ax+x+k)} k^0$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)[(1+ax)(2+ax) + a(1+x)(2+x)]}{(k+1)(1+ax+k)(1+x+k)(2+ax+x+k)} k$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)[1+ax+a(1+x)]}{(k+1)(1+ax+k)(1+x+k)(2+ax+x+k)} k^2$$

$\leq 0.$

But $f(x) \geq 0,$ then $f'(x) \leq 0,$ and hence $f(x)$ is nonincreasing.

Corollary 2.2. *Let f be a function defined by*

$$f(x) = \beta(1+x, 1+x),$$

when $1+x>0$ *. Then f(x) is nonincreasing.*

Proof. The proof follows from theorem 2.1, by putting $a=1$.

REFERENCES

[1] E. Artin, *The Gamma function*, New York, Toronto, London,1964.

[2] L. Bougoffa, Some inequalities involving the gamma function, J. Ineq. Pure Appl. Math., 7(5)(2006), Art.179.

[3] A.McD.Mercer, Some new inequalities for the gamma, beta and zeta functions, J. Ineq. Pure Appl. Math., 7(1)(2006), Art.29.

[4] J.Sandor, A note on certain inequalities for the gamma function, J. Ineq. Pure Appl. Math., 6(3)(2005), Art.61.