IHJPAS. 36(1)2023

Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: jih.uobaghdad.edu.iq

Almost and Strongly Almost Approximately Nearly Quasi Compactly Packed Modules

Ali Sh. Ajeel
Department of Mathematics , College of Computer Sciences and Mathematics,Tikrit University , Iraq.
Ali.shebl@st.tu.edu.iq

Haibat K. Mohammadali
Department of Mathematics , College of Computer Sciences and Mathematics,Tikrit University , Iraq.
H.mohammadali@tu.edu.iq

Article history: Received 14 September 2022, Accepted 6 November 2022, Published in January 2023.

doi.org/10.30526/36.1.3013

Abstract

In this paper we present the almost approximately nearly quasi compactly packed (submodules) modules as an application of almost approximately nearly quasiprime submodule. We give some examples, remarks, and properties of this concept. Also, as the strong form of this concept, we introduce the strongly, almost approximately nearly quasi compactly packed (submodules) modules. Moreover, we present the definitions of almost approximately nearly quasiprime radical submodules and almost approximately nearly quasiprime radical submodules and give some basic properties of these concepts that will be needed in section four of this research. We study these two concepts extensively.

Keywords: Alappnq-prime submodules, Alappnq compactly packed, Strongly Alappnq compactly packed, Alappnq-prime radical of submodule, Alappnq-prime radical submodule.

1. Introduction

The concept of almost approximately nearly quasiprime was recently introduced by [1] as a generation of "quasiprime, nearly quasiprime and approximately quasiprime" submodules see [2-4]. The submodule F of Q is called almost approximately nearly quasiprime (simply Alappnq-prime) submodule, if for any $rsq \in F$, for $r, s \in R$, $q \in Q$, implying that either $rq \in F + (soc(Q) + J(Q))$ or $sq \in F + (soc(Q) + J(Q))$.

As an application of Alappnq-prime submodule, we introduce the concepts of [almost approximately nearly quasi compactly packed (submodules) modules, strongly almost approximately nearly quasi compactly packed (submodules) modules] and study some basic properties of these concepts. This paper consists of three sections. Section one covers some basic concepts, recalls some remarks and propositions needed in the sequel. Section two introduces and studies the concept of almost approximately nearly quasi compactly packed (submodules) modules and gives some basic properties. Section three, devoted to introducing the concept of strongly almost approximately nearly quasi compactly packed (submodules) modules. Also, we introduce the concepts of almost approximately nearly quasiprime radical submodules and almost approximately nearly quasiprime radical submodules and study this concept in detail. Finally, we remark that all rings in this paper are commutative with identity and all modules are unitary left R-module.

2. Preliminaries

This section includes some well-known definitions, remarks, and propositions needed in our study of the next sections.

Remark 2.1 [5]

In a finitely generated R-module, every proper submodule contend in maximal submodule.

Definition 2.2 [6]

An R-module Q is multiplication if every submodule F of Q is of the form $F = IQ$ for some ideal I of R.

Proposition 2.3 [7]

Let Q be a non-zero multiplication R-module, then every proper submodule of Q contend in a maximal submodule.

Definition 2.4 [8]

A submodule F of an R-module Q is called small if $F + K = Q$ implies that $K = Q$ for any proper submodule K of Q.

Proposition 2.5 [1]

Let $f: Q \rightarrow Q'$ be an R-epimorphism, and $\ker f$ is small submodule for Q. If F is an Alappnq-prime submodule for Q' then $f^{-1}(F)$ is Alappnq-prime submodule for Q.

Proposition 2.6 [1]

Let $f: Q \rightarrow Q'$ be an R-epimorphism, and $\ker f$ is small submodule for Q. If F be an Alappnq-prime submodule for Q with $Ker\ f \subseteq F$ then $f(F)$ is Alappnq-prime submodule for Q'.

Definition 2.7 [9]

A subset S of a ring R is called multiplicatively closed if $1 \in S$ and $ab \in S$ for every $a, b \in S$. Let T be the set of all order pairs (q, s) where $q \in Q$ and $s \in S$. The relation on T is defined by $(q, s) \sim (q', s')$ if there exists $t \in S$ such that $t(sq' - s'q) = 0$ is an equivalence relation. We denote the equivalence classes of (q, s) by $\frac{q}{s}$. Let Q_S denote the set of all equivalence classes T with respect to this relation. Q_S is an R-module.

Definition 2.8 [10]

An R-module Q is Z-regular if for each $q \in Q$ there exists $f \in Q' = Hom_R(Q, R)$ such that $q = f(q)q$.

Proposition 2.9 [5]

Let Q be an R-module then the following statements are equivalent:

1. Every proper submodule of Q is a semi prime.
2. Every proper submodule of Q is the intersection of prime submodule of Q.

Proposition 2.10 [5]

Let Q be a non-zero Z-regular R-module, then every proper submodule of Q is a semi prime.

From Propositions 2.9 and 2.10, we get the following corollary.

Corollary 2.11

Let Q be Z-regular R-module, then every proper submodule of Q is the intersection of a prime submodule of Q.

Remark 2.12 [1]

Every prime submodule F of an R-module Q is an Alappnq-prime submodule of Q.

Definition 2.13 [11]

An R-module Q is faithful if $ann_R(Q) = (0)$.

Proposition 2.14 [13]

A proper submodule F of faithful multiplication R-module Q is an Alappnq-prime submodule of Q if and only if $[F:_R Q]$ is an Alappnq-prime ideal of R.

Definition 2.15 [9]

"An R-module Q is called Bezout module if every finitely generated submodule of Q is cyclic".

3. Almost Approximately Nearly Quasi Compactly Packed Modules

Before we introduce the concept of almost approximately nearly quasi compactly packed modules, and study some properties, we need to define the concept of almost approximately nearly quasi compactly packed submodules.

Definition 3.1

A proper submodule F of an R-module Q is called almost approximately nearly quasi compactly packed (simply Alappnq compactly packed) if for each family $\{F_\alpha\}_{\alpha\in\Lambda}$ of Alappnq-prime submodules of Q with $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$ there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $F \subseteq \bigcup_{i=1}^{n} F_{\alpha_i}$.

Definition 3.2

An R-module Q is called Alappnq compactly packed if every proper submodule of Q is Alappnq compactly packed.

Remarks and Examples 3.3

1. Z_6 as Z-module Alappnq compactly packed Z-module.
2. Every module contains a finite number of Alappnq-prime submodules is Alappnq compactly packed.
3. Every proper finite submodule of an R-module Q is an Alappnq compactly packed.

Proposition 3.4

Let Q be Alappnq compactly packed R-module with $J(Q) \neq Q$, then Q satisfies the ascending chain condition for Alappnq-prime submodules.

Proof

Let $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ be ascending chain of Alappnq-prime submodules of Q. Let $L = \bigcup_i L_i$, we claim that $L \neq Q$. In fact if $L = Q$ and H is a maximal submodule of Q, then $H \subsetneq \bigcup_i L_i$, but Q is Alappnq compactly packed module, then there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $H \subseteq \bigcup_{i=1}^{n} L_{\alpha_i}$ and since $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ is ascending chain then there exists $q \in \{1, 2, \dots, n\}$ such that $\bigcup_{i=1}^{n} L_{\alpha_i} = L_{\alpha_q}$ then $H \subseteq L_{\alpha_q}$, and since H is maximal submodule then $H = L_{\alpha_q}$ and consequently $Q = \bigcup_i L_i = L_{\alpha_q}$ which is a contradiction. So L is a proper submodule of Q, thus there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $L \subseteq \bigcup_{i=1}^{n} L_{\alpha_i}$, and since $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ is an ascending chain then there exists $q \in \{1, 2, \dots, n\}$ such that $\bigcup_{i=1}^{n} L_{\alpha_i} = L_{\alpha_q}$ that is $\bigcup_i L_i \subseteq L_{\alpha_q}$, so$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_{\alpha_q}$. Therefore Q satisfies the ascending chain condition on Alappnq-prime submodules.

Since every proper submodule of finitely generated module contained in maximal submodule, so from the previous proposition, we have the following corollary.

Corollary 3.5

If Q is an Alappnq compactly packed finitely generated module, then Q satisfies the ascending chain condition for Alappnq-prime submodules.

Also since every proper submodule of multiplication module contained in maximal submodule, so from the previous proposition, we have the following corollary.

Corollary 3.6

If Q is Alappnq compactly packed multiplication module, then Q satisfies the ascending chain condition for Alappnq-prime submodules.

Proposition 3.7

Let $f: Q \to Q'$ be an R-epimorphism, and $\ker f$ is a small submodule of Q, such that $\ker f \subseteq P$ for each Alappnq-prime submodule P of Q. Then Q is an Alappnq compactly packed if and only if Q' is an Alappnq compactly packed.

Proof

($\Longrightarrow$) Suppose that Q is an Alappnq compactly packed R-module, and $F' \subseteq \bigcup_{\alpha\in\Lambda} P'_{\alpha}$, where F' is a proper submodule of Q' and P' is an Alappnq-prime submodule of Q' for all $\alpha \in \Lambda$. Then, $f^{-1}(F') \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} P'_{\alpha})$ and hence $f^{-1}(F') \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(P'_{\alpha})$. But by Proposition 2.5 we have $f^{-1}(P'_{\alpha})$ is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. Since Q is an Alappnq compactly packed then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $f^{-1}(F') \subseteq \bigcup_{i=1}^{n} f^{-1}(P'_{\alpha_i})$ implies that $f^{-1}(F') \subseteq f^{-1}(\bigcup_{i=1}^{n} P'_{\alpha_i})$. But f is an epimorphism then $F' \subseteq \bigcup_{i=1}^{n} P'_{\alpha_i}$. Thus Q' is an Alappnq compactly packed.

($\Longleftarrow$) Suppose that Q' is an Alappnq compactly packed R-module and $\ker f \subseteq P$ for each Alappnq-prime submodule P of Q. Let F be aproper submodule of Q such that $F \subseteq \bigcup_{\alpha\in\Lambda} P_{\alpha}$, where P_{α} is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. Then $f(F) \subseteq f(\bigcup_{\alpha\in\Lambda} P_{\alpha})$ implies that $f(F) \subseteq \bigcup_{\alpha\in\Lambda} f(P_{\alpha})$. But $\ker f \subseteq P_{\alpha}$ for each α. Then by Proposition 2.6 $f(P_{\alpha})$ is an Alappnq-prime submodule of Q' for all $\alpha \in \Lambda$. Since Q' is an Alappnq compactly packed R-module, then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $f(F) \subseteq \bigcup_{i=1}^{n} f(P_{\alpha_i})$. Now, let $x \in F$ then $f(x) \in f(F) \subseteq \bigcup_{i=1}^{n} f(P_{\alpha_i})$, then there exists $j \in \{1, 2, \dots, n\}$ such that $f(x) \in f(P_{\alpha_j})$, implies that there exists $b \in P_{\alpha_j}$ such that $f(x) = f(b)$, then $f(x) - f(b) = 0$, and $f(x - b) = 0$ so $x - b \in \ker f \subseteq P_{\alpha_j}$. That is $x \in P_{\alpha_j}$. Hence, $F \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$, that is F is an Alappnq compactly packed submodule. Therefore, Q is an Alappnq compactly packed R-module.

The following proposition gives a relation between an Alappnq compactly packed module Q and Q_S

Proposition 3.9

Let Q be an R-module, and S a multiplicatively closed set in R. If Q is an Alappnq compactly packed module, then Q_S is an Alappnq compactly packed module.

Proof

Let F be a proper submodule of Q_S, and $F \subseteq \bigcup_{\alpha\in\Lambda} P_{\alpha}$, where P_{α} is an Alappnq-prime submodule of Q_S for all $\alpha \in \Lambda$. Define $f: Q \to Q_S$ by $f(q) = \frac{q}{1}$ for every $q \in Q$. Thus f is an epimorphism. Therefore $f^{-1}(F) \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} P_{\alpha})$, implies that $f^{-1}(F) \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(P_{\alpha})$. Since P_{α} is an Alappnq-prime submodule of Q_S for all $\alpha \in \Lambda$ and f is an epimorphism, then by Proposition 2.5 we have $f^{-1}(P_{\alpha})$ is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. But Q is an Alappnq compactly packed then there exists $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $f^{-1}(F) \subseteq \bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})$. Hence,

$(f^{-1}(F))_S \subseteq \left(\bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})\right)_S = \bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_S$. To prove the last equality, let $\frac{q}{s} \in \left(\bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})\right)_S$, where $s \in S$, $q \in \bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})$ so there exists $j \in \{1, 2, \dots, n\}$ such that $q \in f^{-1}\left(P_{\alpha_j}\right)$, thus $\frac{q}{s} \in \left(f^{-1}(P_{\alpha_j})\right)_S$, hence $\frac{q}{s} \in \bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_S$. It follows $\left(\bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})\right)_S \subseteq \bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_S$. Now, let $\frac{q}{s} \in \bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_S$, so $\frac{q}{s} \in \left(f^{-1}(P_{\alpha_j})\right)_S$ for some $j \in \{1, 2, \dots, n\}$, where $s \in S$, $q \in f^{-1}\left(P_{\alpha_j}\right)$. Hence, $q \in \bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})$, thus $\frac{q}{s} \in \left(\bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})\right)_S$. Therefore, $\bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_S \subseteq \left(\bigcup_{i=1}^{n} f^{-1}(P_{\alpha_i})\right)_S$. Now we prove that $(f^{-1}(F))_S = F$ for any submodule F of Q_S. Let $\frac{x}{s} \in (f^{-1}(F))_S$, where $x \in f^{-1}(F)$ and $s \in S$. Then $f(x) \in F$, therefore $\frac{x}{1} \in F$, hence $\frac{x}{s} = \frac{1}{s}\frac{x}{1} \in F$. Thus $(f^{-1}(F))_S \subseteq F$. Now, let $\frac{x}{s} \in F$, then, $\frac{1}{s}\frac{x}{1} \in F$ and hence $\frac{x}{1} \in F$, implies that $f(x) \in F$, therefore $x \in f^{-1}(F)$ and $\frac{x}{s} \in (f^{-1}(F))_S$. Thus, $F \subseteq (f^{-1}(F))_S$. Therefore $F = (f^{-1}(F))_S$ for any submodule F of Q_S. Since $(f^{-1}(F))_S \subseteq \bigcup_{i=1}^{n}\left(f^{-1}(P_{\alpha_i})\right)_{,S}$, we have $F \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$. Hence, F is an Alappnq compactly packed submodule of Q_S. Thus Q_S is an Alappnq compactly packed module.

4. Strongly Almost Approximately Nearly Quasi Compactly Packed Modules

In this section, we introduce the strongly Alappnq compactly packed modules and comprehensively study this concept. First, we must introduce the definitions of Alappnq-prime radical of submodules, Alappnq-prime radical submodules, and some propositions of these concepts needed in the sequel.

Definition 4.1

Let F be proper submodule of an R-module Q. if there exist an Alappn-prime submodules that contain F, then, the intersection of each Alappn-prime submodules containing F is called Alappnq-prime radical of F and denoted by $Alappnqrad_Q(F)$. If there exists no an Alappnq-prime submodule containing F, we put $Alappnqrad_Q(F) = Q$.

Definition 4.2

We say that a submodule F of Q is Alappnq-prime radical, if $Alappnqrad_Q(F) = F$.

Proposition 4.3

Let Q be an R-module and F, L are submodules of Q. Then:

1. $F \subseteq Alappnqrad_Q(F)$.
2. If $F \subseteq L$, then $Alappnqrad_Q(F) \subseteq Alappnqrad_Q(L)$.
3. $Alappnqrad_Q\left(Alappnqrad_Q(F)\right) = Alappnqrad_Q(F)$.

Proof

(1) and **(2)** direct from definition.

(3) By part (1) we have $Alappnqrad_Q(F) \subseteq Alappnqrad_Q\left(Alappnqrad_Q(F)\right)$. Now $Alappnqrad_Q(F) = \cap K$, where the intersection runs over all Alappnq-prime submodules K of Q with $F \subseteq K$. $Alappnqrad_Q\left(Alappnqrad_Q(F)\right) = Alappnqrad_Q(\cap K) \subseteq \cap Alappnqrad_Q(K) = \cap K$. Hence, $Alappnqrad_Q\left(Alappnqrad_Q(F)\right) \subseteq Alappnqrad_Q(F)$. Thus $Alappnqrad_Q\left(Alappnqrad_Q(F)\right) = Alappnqrad_Q(F)$.

Proposition 4.4
Let Q be an R-module. If Q is Z-regular, then $Alappnqrad_Q(F) = F$ for all submodule F of Q.

Proof

Let $F \subset Q$. Then by Proposition 4.3(1) we have $F \subseteq Alappnqrad_Q(F)$. Since Q is Z-regular and F be aproper submodule of Q then by Corollary 2.11 we have F is the intersection of prime submodules. Hence $F = \bigcap_{\alpha\in\Lambda} P_\alpha$ where P_α is a prime submodule of Q for each $\alpha \in \Lambda$. Therefore $\bigcap_{\alpha\in\Lambda} P_\alpha \subseteq F$, where P_α is a prime submodule of Q such that $F \subseteq P_\alpha$. Since by Remark 2.12 every prime submodule of Q is an Alappnq-prime then $Alappnqrad_Q(F) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha$ implies that $Alappnqrad_Q(F) = F$.

Proposition 4.5
Let Q be an R-module. If Q satisfies the ascending chain condition for Alappnq-prime radical submodules, then every proper submodule of Q is an Alappnq-prime radical of a finitely generated submodule of it.

Proof

Assume that there exists a proper submodule F of Q which is not the Alappnq-prime radical of a finitely generated submodule of it. Let $q_1 \in F$ and $F_1 = Alappnqrad_Q(Rq_1)$, so $F_1 \subset F$. Thus, there exists $q_2 \in F - F_1$. Let $F_2 = Alappnqrad_Q(Rq_1 + Rq_2)$, then, $F_1 \subset F_2 \subset F$, hence there exists $q_3 \in F - F_3$. This implies an ascending chain of Alappnq-prime radical submodules $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$, which does not terminate and this contradicts with hypothesis.

Now, we introduce the concept of strongly Alappnq compactly packed modules, and study some properties.

Definition 4.6
A proper submodule F of an R-module Q is called strongly Alappnq compactly packed if for each family $\{F_\alpha\}_{\alpha\in\Lambda}$ of Alappnq-prime submodules of Q with $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$ there exists $\beta \in \Lambda$ such that $F \subseteq F_\beta$.

Definition 4.7
An R-module Q is called strongly Alappnq compactly packed if every proper submodule of Q is strongly Alappnq compactly packed.

Remark 4.8
Every strongly Alappnq compactly packed submodule is Alappnq compactly packed, but the convers is not true as explain in the following example:
Let $Q = Z_2[x]$ be a module over Z_2. Let $L = \{\bar{0}, \bar{1}, x, \bar{1} + x\}$ is a Z_2-submodule of Q. $L \subseteq \bar{1}Z_2 \cup xZ_2 \cup x^2Z_2$, where $\bar{1}Z_2, xZ_2, x^2Z_2$ are prime submodules of Q. But $L \not\subset \bar{1}Z_2 \cup xZ_2 \cup x^2Z_2$, that L is Alappnq compactly packed, but it is not strongly Alappnq compactly packed.

The following proposition gives a characterization of strongly Alappnq compactly packed modules.
Proposition 4.9
Let Q be an R-module. Then Q is strongly Alappnq compactly packed if and only if every proper submodule of Q is Alappnq-prime radical of a cyclic submodule of it.

Proof

($\Longmapsto$) Let F be a proper submodule of Q such that F is not Alappnq-prime radical of a cyclic submodule of it, thus for each $q \in F$, $F \neq Alappnqrad_Q((q))$. So there exists an Alappnq-prime submodule $L_q \supseteq (q)$ for each $q \in F$ and $F \not\subset L_q$. Thus $F = \bigcup_{q\in F}(q) \subseteq \bigcup_{q\in F} L_q$. Since Q is strongly Alappnq compactly packed, then there exists $q_0 \in F$ such that $F \subseteq L_{q_0}$ which is a contradiction. Hence F is Alappnq-prime radical of a cyclic submodule of it.

($\Leftarrow$) Let $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, where F_α is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$ and $F = Alappnqrad_Q((q))$ for some $q \in F$. since $q \in F$, thus $q \in \bigcup_{\alpha\in\Lambda} F_\alpha$. Hence there exists $\beta \in \Lambda$ such that $q \in F_\beta$. Thus implies that $Alappnqrad_Q((q)) \subseteq F_\beta$ and consequently $F \subseteq F_\beta$ which prove that Q is strongly Alappnq compactly packed.

The following theorem gives characterizations of strongly Alappnq compactly packed modules.

Theorem 4.10

Let Q be an R-module. Then the following statements are equivalent:

1. Q is strongly Alappnq compactly packed module.
2. For each $F \subset Q$, there exists $q \in F$ such that $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$.
3. For each $F \subset Q$, if $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, then there exists $\beta \in \Lambda$ such that $F \subseteq Alappnqrad_Q(F_\beta)$.
4. For each $F \subset Q$, if $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of Alappnq-prime radical submodule of Q, with $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, then there exists $\beta \in \Lambda$ such that $F \subseteq F_\beta$.

Proof

(1) $\Longrightarrow$(2) Let $F \subset Q$. Suppose that $Alappnqrad_Q(F) \neq Alappnqrad_Q(Rq)$ for all $q \in F$. implies that for all $q \in F$ there exists an Alappnq-prime submodule K_q containing Rq and $F \not\subset K_q$. But $F = \bigcup_{q\in F} Rq \subseteq \bigcup_{q\in F} K_q$ and since Q is strongly Alappnq compactly packed, then there exists $q \in F$ such that $F \subseteq K_q$ which is a contradiction. Hence there exists $q \in F$ such that $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$.

(2) $\Longrightarrow$(3) Let $F \subset Q$, and $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$. Hence, by hypothesis there exists $q \in F$ such that $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$. Since $q \in F$, then $q \in \bigcup_{\alpha\in\Lambda} F_\alpha$ implies that $q \in F_\beta$ for some $\beta \in \Lambda$. Hence, $Rq \subseteq F_\beta$ and $F \subseteq Alappnqrad_Q(F) = Alappnqrad_Q(Rq) \subseteq Alappnqrad_Q(F_\beta)$. That is $F \subseteq Alappnqrad_Q(F_\alpha)$.

(3) $\Longrightarrow$(4) Let $F \subset Q$, and $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of Alappnq-prime radical submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, then by hypothesis there exists $\beta \in \Lambda$ such that $F \subseteq Alappnqrad_Q(F_\beta) = F_\beta$. Since $F_\beta$is Alappnq-prime radical submodules of Q.

(4) $\Longrightarrow$(1) Let $F \subset Q$, and suppose $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of Alappnq-prime submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$. Since F_α is Alappnq-prime submodules for each $\alpha \in \Lambda$ then $F_\alpha = Alappnqrad_Q(F_\alpha)$. Thus $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda} Alappnqrad_Q(F_\alpha)$. Then by hypothesis, there exists $\beta \in \Lambda$ such that $F \subseteq Alappnqrad_Q(F_\alpha) = F_\alpha$. Thus Q is strongly Alappnq compactly packed.

Proposition 4.11

Let R be Alappnq compactly packed ring. Let Q be a faithful cyclic R-module such that for every submodule F_1, F_2 of Q with $F_1 \subseteq F_2$, whenever $[F_1 :_R Q] \subseteq [F_2 :_R Q]$. Then Q is strongly Alappnq compactly packed.

Proof

Let F be a proper submodule of Q, and $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of Alappnq-prime submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$. Since Q is cyclic, then $[\bigcup_{\alpha\in\Lambda} F_\alpha :_R Q] = \bigcup_{\alpha\in\Lambda}[F_\alpha :_R Q]$ implies that $[F:_R Q] \subseteq [\bigcup_{\alpha\in\Lambda} F_\alpha :_R Q] \subseteq \bigcup_{\alpha\in\Lambda}[F_\alpha :_R Q]$. Since Q is cyclic then Q is multiplication [12], then by Proposition 2.14 $[F_\alpha :_R Q]$ is Alappnq-prime ideal of R. But R is Alappnq compactly packed ring then there exists $\alpha_j \in \Lambda$ such that $[F:_R Q] \subseteq [F_{\alpha_j} :_R Q]$ and by hypothesis, hence $F \subseteq F_{\alpha_j}$. Thus Q is strongly Alappnq compactly packed.

The following proposition gives a necessary and sufficient condition for Z-regular module to be strongly Alappnq compactly packed.

Proposition 4.12

Let Q be a Z-regular R-module, then Q is strongly Alappnq compactly packed if and only if every proper submodule of Q is cyclic.

Proof

($\Longrightarrow$) Suppose that Q is a strongly Alappnq compactly packed R-module and let $F \subset Q$. Since Q is a strongly Alappnq compactly packed, then by Theorem 4.10, there exists $q \in F$ such that $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$. But Q is Z-regular module, then by Proposition 4.4, we have $F = Rq$, thus F is cyclic.

($\Longleftarrow$) Suppose that every proper submodule of Q is cyclic. Let F be a proper submodule of Q then, F is cyclic, thus there exists $q \in F$ such that $F = Rq$, so we have $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$. Hence by Theorem 4.10, Q is strongly Alappnq compactly packed.

The following proposition gives condition under which strongly Alappnq compactly packed module satisfy ascending chain condition on Alappnq-prime radical submodules.

Proposition 4.13

Let Q be strongly Alappnq compactly packed R-module which has at least one maximal submodule, then Q satisfies the ascending chain condition on Alappnq-prime radical submodules.

Proof

Let $F_1 \subseteq F_2 \subseteq \cdots$ be an ascending chain condition for Alappnq-prime radical submodules of Q, let $L = \bigcup_i F_i$ then L is a submodule of Q. We claim that $L \subset Q$. In fact, if $L = Q$ and H is a maximal submodule of Q, then $H \subsetneq \bigcup_i F_i$. Since Q is strongly Alappnq compactly packed then by Theorem 4.10 $H \subseteq F_j$ for some j. But H is maximal submodule then $H = F_j$ and this implies $\bigcup_i F_i \subseteq F_j$ that is $Q \subseteq F_j$ which is a contradiction. So $L \subset Q$ and by Theorem 4.10 there exists j such that $L \subseteq F_j$, so $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_j$ that is Q satisfies the ascending chain condition on Alappnq-prime radical submodules.

The following corollaries are direct consequence of Proposition 4.13.

Corollary 4.14

Let Q be strongly Alappnq compactly packed R-module such that $J(Q) \neq Q$. Then Q satisfies the ascending chain condition on Alappnq-prime radical submodules.

Corollary 4.15

If Q is finitely generated strongly Alappnq compactly packed R-module, then Q satisfies the ascending chain condition on Alappnq-prime radical submodules.

Corollary 4.16

If Q is multiplication strongly Alappnq compactly packed R-module, then Q satisfies the ascending chain condition on Alappnq-prime radical submodules.

In the following proposition we give a condition under which the convers of Proposition 4.13 is hold.

Proposition 4.17

Let Q be a Bezout R-module. If Q satisfies the ascending chain condition for Alappnq-prime radical submodules, then Q is strongly Alappnq compactly packed module.

Proof

Let F be a proper submodule of Q, then by Proposition 4.5 there exists a finitely generated submodule L of F such that $F = Alappnqrad_Q(L)$ and hence by Proposition 4.3 $Alappnqrad_Q(F) = Alappnqrad_Q\left(Alappnqrad_Q(L)\right) = Alappnqrad_Q(L)$. But Q is Bezout module then L is cyclic submodule, then there exists $q \in L$ such that $L = Rq$, thus implies that $q \in F$ and $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$. Therefore, by Theorem 4.10 Q is a strongly Alappnq compactly packed module.

Proposition 4.18

Let $f: Q \to Q'$ be an R-epimorphism, and $\ker f$ is a small submodule of Q such that $\ker f \subseteq P$ for each Alappnq-prime submodule P of Q. Then Q is strongly Alappnq compactly packed if and only if Q' is strongly Alappnq compactly packed.

Proof

($\Longrightarrow$) Suppose that Q is strongly Alappnq compactly packed R-module, and $F' \subseteq \bigcup_{\alpha\in\Lambda} P'_\alpha$, where F' is proper submodule of Q' and P' is an Alappnq-prime submodule of Q' for all $\alpha \in \Lambda$. Since f is an epimorphism, then $f^{-1}(F') \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} P'_\alpha)$. Thus $f^{-1}(F') \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(P'_\alpha)$. But P'_α is an Alappnq-prime submodule of Q', then by Proposition 2.5 we have $f^{-1}(P'_\alpha)$ is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. Since Q is strongly Alappnq compactly packed then $f^{-1}(F') \subseteq f^{-1}(P'_\beta)$ for some $\beta \in \Lambda$. Therefore $F' \subseteq P'_\beta$ for some $\beta \in \Lambda$. Hence F' is strongly Alappnq compactly packed submodule of Q'. Thus Q' is strongly Alappnq compactly packed.

($\Longleftarrow$) Suppose that Q' is strongly Alappnq compactly packed R-module and $\ker f \subseteq P$ for each Alappnq-prime submodule P of Q. Let F be a proper submodule of Q such that $F \subseteq \bigcup_{\alpha\in\Lambda} P_\alpha$, where P_α is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. Then $f(F) \subseteq f(\bigcup_{\alpha\in\Lambda} P_\alpha)$ implies that $f(F) \subseteq \bigcup_{\alpha\in\Lambda} f(P_\alpha)$. But $\ker f \subseteq P_\alpha$ for each α. Then by Proposition 2.6 $f(P_\alpha)$ is an Alappnq-prime submodule of Q' for all $\alpha \in \Lambda$. Since Q' is strongly Alappnq compactly packed R-module, then $f(F) \subseteq f(P_\beta)$ for some $\beta \in \Lambda$. Thus, for every $x \in F$, $f(x) \in f(F) \subseteq f(P_\beta)$, then $f(x) \in f(P_\beta)$. Therefore, there exists $b \in P_\beta$ such that $f(x) = f(b)$, then $f(x) - f(b) = 0$, and $f(x - b) = 0$ so $x - b \in \ker f \subseteq P_\beta$. That is $x \in P_\beta$. Therefore $F \subseteq P_\beta$ for some $\beta \in \Lambda$ and hence F is strongly Alappnq compactly packed submodule. Hence Q is strongly Alappnq compactly packed R-module.

Proposition 4.19

Let Q be an R-module, and S a multiplicatively closed set in R. If Q is strongly Alappnq compactly packed module, then Q_S is strongly Alappnq compactly packed module.

Proof

Assume Q is strongly Alappnq compactly packed. Let F be proper submodule of Q_S with $F \subseteq \bigcup_{\alpha\in\Lambda} P_\alpha$, where P_α is an Alappnq-prime submodule of Q_S for all $\alpha \in \Lambda$. Define $f: Q \to Q_S$ by $f(q) = \frac{q}{1}$ for every $q \in Q$. Thus f is an epimorphism. Therefore $f^{-1}(F) \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} P_\alpha)$, implies that $f^{-1}(F) \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(P_\alpha)$. Since P_α is an Alappnq-prime submodule of Q_S for all $\alpha \in \Lambda$ and f is an epimorphism, then by Proposition 2.5 we have $f^{-1}(P_\alpha)$ is an Alappnq-prime submodule of Q for all $\alpha \in \Lambda$. But Q is strongly Alappnq compactly packed, then $f^{-1}(F) \subseteq f^{-1}(P_\beta)$ for some $\beta \in \Lambda$. Therefore $(f^{-1}(F))_S \subseteq \left(f^{-1}(P_\beta)\right)_S$. We need to show that $(f^{-1}(F))_S = F$ for any submodule F of Q_S. Let $\frac{x}{s} \in (f^{-1}(F))_S$, where $x \in f^{-1}(F)$ and $s \in S$. Then $f(x) \in F$, therefore

$\frac{x}{1} \in F$, hence $\frac{x}{s} = \frac{1}{s}\frac{x}{1} \in F$. Thus $(f^{-1}(F))_S \subseteq F$. Now let $\frac{x}{s} \in F$, then $\frac{1}{s}\frac{x}{1} \in F$ and hence $\frac{x}{1} \in F$, implies that $f(x) \in F$, therefore $x \in f^{-1}(F)$ and $\frac{x}{s} \in (f^{-1}(F))_S$. Thus $F \subseteq (f^{-1}(F))_S$. Therefore $F = (f^{-1}(F))_S$ for any submodule F of Q_S. Now since $(f^{-1}(F))_S \subseteq \left(f^{-1}(P_\beta)\right)_S$ for some $\beta \in \Lambda$ we have $F \subseteq P_\beta$ for some $\beta \in \Lambda$. Thus F is strongly Alappnq compactly packed submodule. Therefore Q_S is strongly Alappnq compactly packed module.

5. Conclusion

The main results of this paper are:

- Let Q be Alappnq compactly packed module with $J(Q) \neq Q$, then, Q satisfies the ascending chain condition for Alappnq-prime submodules.
- If Q is an Alappnq compactly packed module, then, Q_S is an Alappnq compactly packed module, for each multiplicatively closed set S of R.
- An R-module Q is strongly Alappnq compactly packed if and only if every proper submodule of Q is Alappnq-prime radical of a cyclic submodule of it.
- Let Q be an R-module. Then the following statements are equivalent:
  1. Q is strongly Alappnq compactly packed module.
  2. For each $F \subset Q$, there exists $q \in F$ such that $Alappnqrad_Q(F) = Alappnqrad_Q(Rq)$.
  3. For each $F \subset Q$, if $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of submodules of Q, such that $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, then there exists $\beta \in \Lambda$ such that $F \subseteq Alappnqrad_Q(F_\beta)$.
  4. For each $F \subset Q$, if $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of Alappnq-prime radical submodule of Q, with $F \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, then there exists $\beta \in \Lambda$ such that $F \subseteq F_\beta$.

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