Hosoya Polynomials of The Width Distance of Some Cog-Special Graphs

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ABSTRACT

In this paper, we give some definitions and properties of the width distance and find the Hosoya polynomials, Wiener indices, and the average distances of some special cog-graphs with respect to the width distance.

Keyword: Width distance, Hosoya polynomial, Wiener index, average distance, cog-graphs.

متعددات حدود هوسويا للمسافة العرضية لبعض البيانات المسننة الخاصة

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الملخص

في هذا البحث تم إعطاء بعض التعاريف والخواص للمسافة العرضية مع إيجاد متعددة حدود هوسويا ودليل وينر ومعدل المسافة للمسافة العرضية لبعض البيانات المسننة الخاصة.

الكلمات المفتاحية: المسافة العرضية ، متعددة حدود هوسويا ، دليل وينر ، معدل المسافة، بيانات مسننة.

1. Introduction:

Because of the development of parallel architectures for interconnection computer networks, there has recently been interest in generalizations of the distance concept, for examples, the n-distance [1], Steiner distance [5], and width distance [2].

In [2] A. S. Aziz obtained Hosoya polynomials of the width distance of some special graphs and some compound graphs. It would be interesting to compute this polynomial for various graphs and to study its properties. It would also be interesting to see if this object yields any useful information in chemistry , group theory , or computer science. For more information about these concepts and their relation to networks , see the article of Hsu [6].

Let u and v be any two distinct vertices in a connected graph G, define the container C(u,v) as a set of vertex-disjoint paths between vertices u and v, i.e., any two paths in C(u,v) have only u and v in common. The container width $w = w(C(u,v))$, is the number of paths in the container, i.e., $w(C(u,v)) = | C(u,v) |$. The length of a container $l = l(C(u,v))$ is the length of the longest path in C(u,v). For a fixed positive integer w, define the **width distance** (w-distance) between u and v ([2],[7]) as:

$$d_w^*(u,v|G) = \min_{C(u,v)} l(C(u,v)),$$

where the minimum is taken over all containers C(u,v) of width w. We may denote the width distance between any two distinct vertices u and v in G by $d_w^*(u,v)$ if there is no confusion.

Note that when $w = 1$, then $d_w^*(u, v)$ reduces to the usual distance between vertices u and v which denoted by $d(u, v)$ ([4],[7]). Therefore, in this paper, we take $2 \le w \le k_0$, in which k_0 is the connectivity of G [3].

To define the diameter and radius of any connected graph G with respect to the width distance , we first define the eccentricity.

The **w-eccentricity** $e_w^*(v)$ of the vertex v in a graph G is the greatest possible w-distance from v to all other vertices of G, that is

$$e_w^*(v) = \max_{u \in V(G)} \{d_w^*(u, v)\}.$$

The **w-diameter** $\delta_w^* = \delta_w^*(G)$ of G is defined by

$$\delta_w^*(G) = \text{diam}_w^* G = \max_{v \in V(G)} e_w^*(v) = \max_{v, u \in V(G)} \{d_w^*(u, v)\}.$$

The **minimum w-distance** $m_w = m_w(G)$ of G is defined by

$$m_w(G) = \min_{u, v \in V(G)} \{d_w^*(u, v)\}.$$

The **w-radius** $r_w^* = r_w^*(G)$ of G is defined by

$$r_w^*(G) = \text{rad}_w^* G = \min_{v \in V(G)} e_w^*(v).$$

A vertex v in G is said to be **w-central vertex** if $e_w^*(v) = \text{rad}_w^* G$, a graph G may contain more than one w-centeral vertex ,we define the **w-center of a graph G**, $Z_w^*(G)$, as the set of all w-central vertices of G.

It is clear that the w-distance in connected graphs does not satisfy the metric axioms, because for $w \ge 2$, $d_w^*(u, v) \ge 2$, $d_w^*(u, v) = d_w^*(v, u)$, but $d_w^*(u, y) + d_w^*(y, v) \ge d_w^*(u, v)$ is not satisfactory for all vertices u and v of G. To show that, we take the next example .

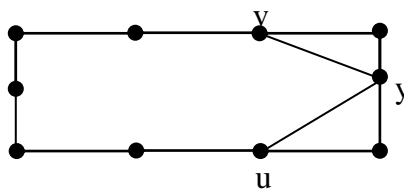**Example**: Let G be a graph of order 10 and size 12, as shown in Fig.1.1.

Fig. 1.1.

We notice that $d_2^*(v, y) = 2 = d_2^*(u, y)$, but $d_2^*(u, v) = 6$, so $d_2^*(u, y) + d_2^*(y, v) < d_w^*(u, v)$.

Moreover, it is clear that for all connected graphs G

- $\delta_w^*(G) \ge \delta(G)$, where $\delta(G)$ is diameter of G.

- $r_w^*(G) \ge r(G)$, where $r(G)$ is radius of G.

Equality holds if $w = 1$.

Concerning the width minimum distance $m_2(G)$, we have

- $m_2(G) = 2$ iff G contains a triangle or a cycle of order 4.

2. Hosoya Polynomial of The Width Distance

Let G be a connected graph of order p, size q, and let w be a fixed positive integer such that $2 \leq w \leq k_0$, and w-diameter is δ_w^*. The Wiener index of the w-distance (w-Wiener index) is the sum of all w-distances in G, that is [2]

$$W_w^*(G) = \sum_{u,v \in V(G)} d_w^*(u,v), \ u \neq v.$$

The Hosoya polynomial with respect to the width distance function d_w^* (called w-Hosoya polynomial) is defined by [2]:

$$H_w^*(G;x) = \sum_{u,v \in V(G)} x^{d_w^*(u,v)}.$$

If $C_w^*(G,k)$ is the number of unordered pairs of distinct vertices that has k w-distance apart, $m_w \leq k \leq \delta_w^*$, then

$$H_w^*(G;x) = \sum_{k=m_w}^{\delta_w^*} C_w^*(G,k)x^k,$$

If $w \geq 2$, then w-distance is not less than 2, ($d_w^*(u,v) \geq 2$, $w \geq 2$).
It is clear that

$$W_w^*(G) = \frac{d}{dx} H_w^*(G;x)\Big|_{x=1} = \sum_{k=m_w}^{\delta_w^*} k C_w^*(G,k),$$

The average w-distance, $\mu_w^*(G)$, of G is defined as :

$$\mu_w^*(G) = W_w^*(G) \Big/ \binom{p}{2},$$

We define the w-Hosoya polynomial of a vertex v in G as :

$$H_w^*(v,G;x) = \sum_{k=m_w}^{\delta_w^*} C_w^*(v,G,k)x^k \ , \text{ for } 2 \leq w \leq k_0.$$

where $C_w^*(v,G,k)$ is the number of vertices in V(G)-{v} that are at w-distance k from vertex v. We note that

$$\sum_{v \in V(G)} C_w^*(v,G,k) = 2C_w^*(G,k), \text{ for } \ m_w \leq k \leq \delta_w^*, \text{ and } 2 \leq w \leq k_0.$$

From the previous discussion, we note that

$$\sum_{v \in V(G)} H_w^*(v,G;x) = 2H_w^*(G;x).$$

Let S be a nonempty subset of V(G), then define

$$H_w^*(S;x) = \sum_{v \in S} H_w^*(v;G;x),$$

Moreover, we define $C_w^*(\tilde{S},G,k)$ as the number of pairs {u,v} of vertices of S such that

$d_w^*(u,v|G) = k$, therefore we define $H_w^*(\tilde{S},G;x) = \sum_{k=2} C_w^*(\tilde{S},G,k)x^k$.

Finally, if $T_1, T_2, \ldots, T_r$ is a partition of V(G), then

$$2H_w^*(G;x) = \sum_{i=1}^{r} H_w^*(T_i;G;x).$$

3. On The Coefficients of The w-Hosoya Polynomial

If $H_w^*(G;x) = \sum_{k=m_w}^{\delta_w^*(G)} C_w^*(G,k)x^k$, $2 \le w \le k_0$ is the w-Hosoya polynomial of a connected graph G for a fixed w, then the sequence of a graph G invariant $\{C_w^*(G,k)\}_{k=m_w}^{\delta_w^*(G)}$, for a fixed w, $2 \le w \le k_0$, often possesses some distinguished properties.

1. The polynomial $H_w^*(G;x)$ is called **unimodal** if, for some index h,

$$C_w^*(G,m_w) \le C_w^*(G,m_w+1) \le C_w^*(G,m_w+2) \le \dots \le C_w^*(G,h) \ge C_w^*(G,h+1) \ge \dots \ge C_w^*(G,\delta_w^*(G)), \quad (3.1)$$

and it is **strong-unimodal** if (3.1) holds without equalities.
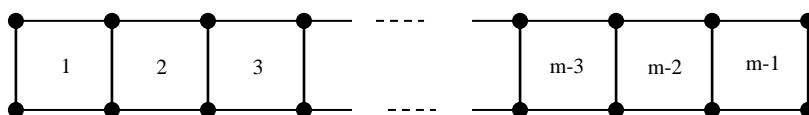For example, [2] : If $G = k_2 \times P_m$ is the graph as shown in Fig. 3.1, then

Fig. 3.1. G

$$H_2^*(G;x) = 2(m-1)x^2 + (5m-6)x^3 + 2\sum_{i=2}^{m-1}[2m-2i-1]x^{\lceil(2i+3)/2\rceil},$$

is strong-unimodal at index h=3.

2. The polynomial $H_w^*(G;x)$ is called **palindromic** if

$$C_w^*(G,k) = C_w^*(G,\delta_w^*(G)-k), \text{ for } m_w \le k \le \delta_w^*(G), \qquad \dots(3.2)$$

and it is **j-semi-palindromic** if (4.2) holds for $k = m_w + j, m_w + j+1, \dots, \delta_w^*(G) - j$, for some $j \ge 1$.
For example, [2] : If C_p is a cycle of order p, $p \ge 3$, then

$$H_2^*(C_p;x) = \begin{cases} p(x^{(p+1)/2} + x^{(p+3)/2} + \dots + x^{p-2} + x^{p-1}), & \text{if } p \text{ is odd}, \\ p(\dfrac{x^{p/2}}{2} + x^{p/2+1} + \dots + x^{p-2} + x^{p-1}), & \text{if } p \text{ is even}. \end{cases} \qquad \dots(3.3)$$

is palindromic if p is odd, and it is 1-semi-palindromic if p is even.

3. The polynomial $H_w^*(G;x)$ is called **monotonically increasing** (or **monotonically decreasing**) if $C_w^*(G,k) \le C_w^*(G,k+1)$ or $C_w^*(G,k \ge C_w^*(G,k+1)$, respectively, for all $m_w \le k \le \delta_w^*(G)$.

For example, [2] : If P_m^2 is the square of a path P_m of order $m \ge 5$, then

$$H_2^*(P_m^2;x) = 3(m-2)x^2 + \sum_{i=3}^{\lceil m/2 \rceil}[2m-4k+3]x^k,$$

is monotonically decreasing.

4. w-Hosoya Polynomial of Some Cog- Special Graphs:

4.1. Cog-complete Graph K_m^c :

Definition: A cog-complete graph K_m^c is the graph constructed from a complete graph $K_m, m \geq 3$, [3], of vertex set $V = \{v_1, v_2, \ldots, v_m\}$ with m additional vertex set $U = \{u_1, u_2, \ldots, u_m\}$, and 2m edges $\{u_i v_i, u_i v_{i+1} : i = 1, 2, \ldots, m\}$, $(v_{m+1} \equiv v_1)$, as shown in Fig. 4.1.
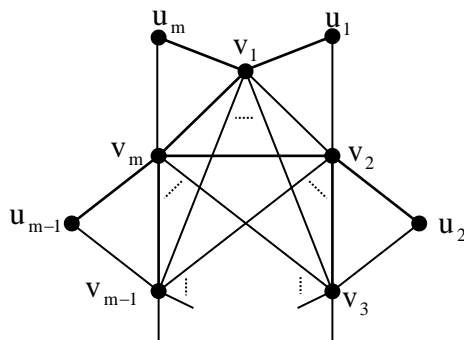

Fig. 4.1. K_m^c

It is clear that $p(K_m^*) = 2m, q(K_m^*) = \dfrac{1}{2}m(m+3)$, and $k_0 = 2$. Thus, w must be 1 or 2 only.

Therefore, we take $w = 2$.

We notice that $m_2(K_m^c) = 2$, and $\delta_2^*(K_m^c) = 3$.

Proposition 4.1.1: 2-Hosoya polynomial of $K_m^c, m \geq 4$ is

$$H_2^*(K_m^c; x) = (m(3m-1)/2) x^2 + (m(m-1)/2) x^3.$$

Proof: Let u and v be any two distinct vertices of , then we consider the following cases:

To find 2-Hosoya polynomial of $K_m^c, m \geq 4$, there are three cases:

Case (1): If $u, v \in U$, then

(i). If u and v are adjacent to a common vertex of V, then there are two internally disjoint paths between the vertices u and v of lengths 2 and 3; therefore $d_2^*(u, v) = 3$. Hence , the number of such pairs of vertices is m.

(ii). If u and v are not adjacent to a common vertex of V, then there are two internally disjoint paths between the vertices u and v, each has length 3, therefore $d_2^*(u, v) = 3$. Hence, the number of such pairs of vertices is $\dfrac{1}{2}m(m-3)$.

(i) and (ii) produce the polynomial $H_2^*(\tilde{U}, K_m^c; x) = \dfrac{1}{2}m(m-1)x^3$.

Case (2): If $u, v \in V$, then there is always a container $C(u, v)$ of length 2. Thus, $H_2^*(\tilde{V}, K_m^c; x) = \dfrac{1}{2}m(m-1)x^2$.

Case (3): If $u \in U$, and $v \in V$, then

(i). There are two internally disjoint paths between the vertices u and v of lengths 1 and 2, when $uv \in E(K_m^c)$, then $d_2^*(u, v) = 2$. Hence, the number of such pairs of vertices is 2m.

(ii). There are two internally disjoint paths between the vertices u and v each of length 2, when $uv \notin E(K_m^c)$, then $d_2^*(u, v) = 2$. Hence, the number of such pairs of vertices is $m(m-2)$.

(i) and (ii) produce the polynomial $H_2^*(M, K_m^c; x) = m^2 x^2$, where $M = \{(u, v) : u \in U, v \in V\}$.

From three cases, we obtain

$$H_2^*(K_m^c; x) = (m(3m-1)/2) x^2 + (m(m-1)/2) x^3.$$
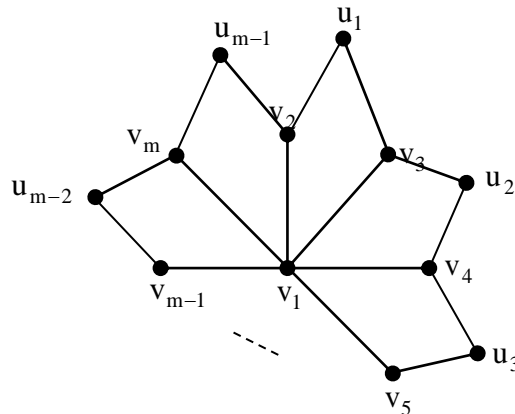
Corollary 4.1.2: For $m \geq 4$,

$$W_2^*(K_m^c) = m(9m-5)/2, \text{ and } \mu_2^*(K_m^c) = \frac{9m-5}{4m-2}; \text{ and } \mu_2^*(K_m^c) \leq 2.25.$$

Remark:

- $H_2^*(K_3^c; x) = 12x^2 + 3x^3$, $W_2^*(K_3^c) = 33$, and $\mu_2^*(K_3^c) = 11/5$.

4.2. Cog-star Graphs:

Definition: [8] A cog-star graph S_m^c is the graph constructed from a star $S_m, m \geq 4$, of vertex set $V = \{v_1, v_2, \ldots, v_{m-1}, v_m\}$ with (m-1) additional vertex set $U = \{u_1, u_2, \ldots, u_{m-2}, u_{m-1}\}$, and edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1, 2, \ldots, m-1\}, (v_{m+1} \equiv v_2)$, as shown in Fig. 4.2.

Fig. 4.2. S_m^c

It is clear that $p(S_m^*) = 2m-1$, $q(S_m^*) = 3(m-1)$, we notice that w = 2, $\delta_2^*(S_m^c) = m-1$, and $m_2(S_m^c) = 2$, $m \geq 5$.

Theorem 4.2.1: For $m \geq 6$,

$$H_2^*(S_m^c; x) = (m-1)\left[3x^3 + x^4 + x^{m-1} + 2\sum_{i=2}^{m-2} x^i\right]. \quad \ldots(4.2.1)$$

Proof: When w = 2, there are four cases:

Case (1): If $u, u' \in U, u \neq u'$, then there are two internally disjoint paths between u and u' having lengths 2 and 4 when u and u' are adjacent to a common vertex of V, then $d_2^*(u, u'|S_m^c) = 4$.

The number of such pairs of vertices $\{u, u'\}$ such that $d_2^*(u, u'|S_m^c) = 4$ is (m-1).

Also, there are two internally disjoint paths between u and u' having length 4 and $d(u, u'|S_m^c - \{v_1\})$ when u and u' are not adjacent to a common vertex of V.

It is clear that then $d(u, u'|S_m^c - \{v_1\}) = 4, 6, 8, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$, then

$$d_2^*(u, u'|S_m^c) = d(u, u'|S_m^c - \{v_1\}).$$

Let $d(u, u'|S_m^c - \{v_1\}) = k$, for $k = 4, 6, 8, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$.

Hence, the number of pairs $\{u, u'\}$ when u and u' are not adjacent to a common vertex of V such that $d_2^*(u, u'|S_m^c) = k$, for $k = 4, 6, 8, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$ is m-1, except when m is odd, and $d_2^*(u, u'|S_m^c) = m - 1$, then the number of such pairs $\{u, u'\}$ is $\dfrac{m-1}{2}$.

Thus, from this case, we get

$$H_2^*(\tilde{U}, S_m^c; x) = (m-1) \begin{cases} x^4 + \displaystyle\sum_{i=2}^{\frac{m-3}{2}} x^{2i} + \dfrac{1}{2} x^{m-1}, & \text{if m is odd}, \\[4mm] x^4 + \displaystyle\sum_{i=2}^{\frac{m}{2}-1} x^{2i}, & \text{if m is even}. \end{cases}$$

Case (2): If $v, v' \in R = V - \{v_1\}, v \neq v'$, then there are three internally disjoint paths between v and v' having lengths 2, $d(v, v'|S_m^c - \{v_1\})$ and $2(m-1) - d(v, v'|S_m^c - \{v_1\})$. Thus,

$$d_2^*(v, v'|S_m^c) = \min\big[\max\{2, d(v, v'|S_m^c - \{v_1\})\}, \max\{2, 2(m-1) - d(v, v'|S_m^c - \{v_1\})\},$$
$$\max\{d(v, v'|S_m^c - \{v_1\})\}, 2(m-1) - d(v, v'|S_m^c - \{v_1\})\}\big].$$

Since $d(v, v'|S_m^c - \{v_1\}) = 2, 4, 6, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$, then $d_2^*(v, v'|S_m^c) = d(v, v'|S_m^c - \{v_1\})$.

Let $d(v, v'|S_m^c - \{v_1\}) = k$, for $k = 2, 4, 6, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$.

Hence, the number of pairs $\{v, v'\}$ such that $d_2^*(v, v'|S_m^c) = k$, for $k = 2, 4, 6, \ldots, 2\left\lfloor \dfrac{m-1}{2} \right\rfloor$ is m-1, except when m is odd, and $d_2^*(v, v'|S_m^c) = m - 1$, then the number of such pairs $\{v, v'\}$ is $\dfrac{m-1}{2}$.

Thus, from this case, we get

$$H_2^*(\tilde{R}, S_m^c; x) = (m-1) \begin{cases} \sum_{i=1}^{\frac{m-3}{2}} x^{2i} + \frac{1}{2} x^{m-1}, & \text{if m is odd}, \\ \sum_{i=1}^{\frac{m}{2}-1} x^{2i}, & \text{if m is even.} \end{cases}$$

Case (3): If $u \in U$, and $v \in V - \{v_1\}$, then

either there are two internally disjoint paths between the vertices u and v of length 1 and 3 when $uv \in E(S_m^c)$, then $d_2^*(u, v | S_m^c) = 3$.

It is clear that the number of pairs of vertices $\{u, v\}$ with $uv \in E(S_m^c)$ is 2(m-1).

 Or there are two internally disjoint paths between the vertices u and v of length 3 , and $d(u, v | S_m^c - \{v_1\})$ when $uv \notin E(S_m^c)$.

It is clear that $d(u, v | S_m^c - \{v_1\}) = 3, 5, 7, ..., 2\left\lfloor \frac{m}{2} \right\rfloor - 1$, then $d_2^*(u, v | S_m^c) = d(u, v | S_m^c - \{v_1\})$.

Let $d(u, v | S_m^c - \{v_1\}) = k$, for $k = 3, 5, 7, ..., 2\left\lfloor \frac{m}{2} \right\rfloor - 1$.

Hence, the number of pairs of $\{u,v\}$, $uv \notin E(S_m^c)$, such that $d_2^*(u, v | S_m^c) = k$, for $k = 3, 5, 7, ..., 2\left\lfloor \frac{m}{2} \right\rfloor - 1$, is 2(m-1), except when m is even and $d_2^*(u, v | S_m^c) = m - 1$, then the number of pairs $\{u,v\}$ is (m-1).

Thus, in this case, we get

$$H_2^*(M, S_m^c; x) = 2(m-1)x^3 + 2(m-1) \begin{cases} \sum_{i=1}^{\frac{m-3}{2}} x^{2i+1}, & \text{if m is odd}, \\ \sum_{i=1}^{\frac{m}{2}-2} x^{2i+1} + \frac{1}{2} x^{m-1}, & \text{if m is even}, \end{cases}$$

where $M = \{(u, v) : u \in U, v \in V - \{v_1\}\}$.

Case (4): There are two internally disjoint paths between v_1 and u_i, $1 \leq i \leq m - 1$, that have length 2 , then $d_2^*(v_1, u_i | S_m^c) = 2$, $1 \leq i \leq m - 1$, and there are three internally disjoint paths between v_1 and v_i, $2 \leq i \leq m$, that have length 1, 3, and 3, then $d_2^*(v_1, v_i | S_m^c) = \min[\max\{1,3\}, \max\{1,3)\}, \max\{3,3)\}] = 3$, $2 \leq i \leq m$.

Hence , $H_2^*(v_1, S_m^c; x) = (m-1)x^2 + (m-1)x^3$

From $H_2^*(S_m^c; x) = H_2^*(\tilde{U}, S_m^c; x) + H_2^*(\tilde{R}, S_m^c; x) + H_2^*(M, S_m^c; x) + H_2^*(v_1, S_m^c; x)$, we get (4.2.1).

We notice from (4.2.1), that the 2-Hosoya polynomial of S_m^c, $m \geq 9$, is 3-semi-palindromic. ∎
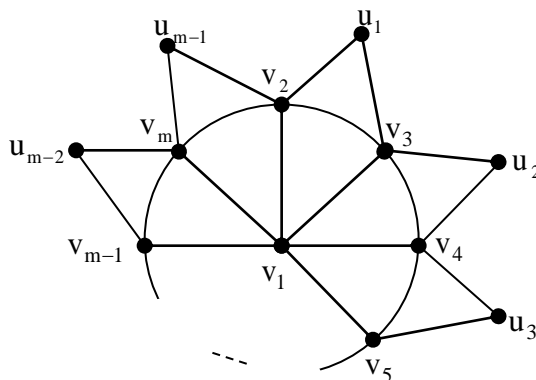
Remark:

- $H_2^*(S_4^c; x) = 6x^2 + 12x^3 + 3x^4$.
- $H_2^*(S_5^c; x) = 8x^2 + 20x^3 + 8x^4$.

Corollary 4.2.2: For $m \geq 4$,

(a) $W_2^*(S_m^c) = (m-1)(m^2 - 2m + 12)$.

(b) $\mu_2^*(S_m^c) = (m^2 - 2m + 12)/(2m - 1)$.

(c) $\mu_2^*(S_m^c) \leq \dfrac{m+1}{2}$, for all $m \geq 5$. ∎

4.3. Cog-wheel Graphs:

Definition: [8] A cog-wheel graph W_m^c is the graph constructed from a wheel W_m, $m \geq 4$, of order m, with vertex set $V = \{v_1, v_2, \dots, v_{m-1}, v_m\}$, and with (m-1) additional vertex set $U = \{u_1, u_2, \dots, u_{m-2}, u_{m-1}\}$, and edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1, 2, \dots, m-1\}$, $(v_{m+1} \equiv v_2)$, as shown in Fig. 4.3.

Fig. 4.3 W_m^c

It is clear that $p(W_m^*) = 2m - 1$, $q(W_m^*) = 4(m-1)$, and $w = 2$.

$\delta_2^*(W_m^c) = \left\lfloor \dfrac{m+1}{2} \right\rfloor$, $m \geq 7$, and $m_w(S_m^c) = 2$, $m \geq 4$.

Theorem 4.3.1: For $m \geq 8$, then

$$H_2^*(W_m^c; x) = 2(m-1)x^2 + (m-1)x^3 + 2(m-1)x^4$$

$$+ 4(m-1) \begin{cases} \displaystyle\sum_{i=2}^{\frac{m-3}{2}} x^i + \frac{7}{8} x^{\frac{m-1}{2}} + \frac{1}{8} x^{\frac{m+1}{2}}, & \text{if } m \text{ is odd}, \\[3mm] \displaystyle\sum_{i=2}^{\frac{m}{2}-1} x^i + \frac{1}{2} x^{\frac{m}{2}}, & \text{if } m \text{ is even}, \end{cases} \quad \dots(4.3.1)$$

Proof: To prove (4.3.1), we consider three cases

Case (1): There are two internally disjoint paths between u_i and u_{i+1}, $1 \leq i \leq m-1$, $(u_m \equiv u_1)$ having length 2 and 4, then $d_2^*(u_i, u_{i+1} | W_m^c) = 4$, $1 \leq i \leq m-1$, $(u_m \equiv u_1)$, and

the number of pairs of vertices $\{u_i, u_{i+1}\}$, $1 \le i \le m-1$, $(u_m \equiv u_1)$ is (m-1). In addition , there are two internally disjoint paths between u_i and u_{i+2}, $1 \le i \le m-1$, $(u_m \equiv u_1$ and $u_{m+1} \equiv u_2)$ having length 3 and 4, then $d_2^*(u_i, u_{i+2}|W_m^c) = 4$, $1 \le i \le m-1$, $(u_m \equiv u_1$ and $u_{m+1} \equiv u_2)$, and the number of pairs of vertices $\{u_i, u_{i+2}\}$, $1 \le i \le m-1$, $(u_m \equiv u_1$ and $u_{m+1} \equiv u_2)$ is (m-1).

Now, there are two internally disjoint paths between the vertices u_i and u_j, $1 \le i \le m-4$, $i+3 \le j \le m-1$,(except for $i=1, j \ne m-2, m-1$; and for $i=2$, $j \ne m-1$) having lengths 4 , and $d(u_i, u_j|W_m^c - \{v_1\})$. Since, $4 \le d(u_i, u_j|W_m^c - \{v_1\}) \le \left\lfloor \dfrac{m+1}{2} \right\rfloor$, then

$$d_2^*(u_i, u_j|W_m^c) = d(u_i, u_j|W_m^c - \{v_1\}).$$

Let $d(u_i, u_j|W_m^c - \{v_1\}) = k$, for all $4 \le k \le \left\lfloor \dfrac{m+1}{2} \right\rfloor$.

The number of pairs $\{u_i, u_j\}$ such that $d_2^*(u_i, u_j|W_m^c) = k$, for all $4 \le k \le \left\lfloor \dfrac{m+1}{2} \right\rfloor$, is m-1, except if m is odd , then the number of $\{u_i, u_j\}$ such that $d_2^*(u_i, u_j|W_m^c) = \dfrac{m+1}{2}$,

is $\dfrac{m-1}{2}$, $1 \le i \le m-4$, $i+3 \le j \le m-1$, (except for $i=1$, $j \ne m-2, m-1$; and for $i=2$, $j \ne m-1$).

Hence,

$$H_2^*(\tilde{U}, W_m^c; x) = 2(m-1)x^4 + (m-1)\begin{cases} \displaystyle\sum_{i=4}^{\frac{m-1}{2}} x^i + \frac{1}{2} x^{\frac{m+1}{2}}, & \text{if m is odd}, \\[4mm] \displaystyle\sum_{i=4}^{\frac{m}{2}} x^i, & \text{if m is even}. \end{cases}$$

Case (2): If $v, v' \in V$, $v \ne v'$, then we consider 3 subcases:

(I) If $v, v' \in V - \{v_1\}$, and $vv' \in E(W_m^c)$, then there are four internally disjoint paths between v and v' having lengths 1, 2, 2 , m-2, then $d_2^*(v, v'|W_m^c) = \min\{2, m-2\} = 2$, $m \ge 4$.

Hence, the number of pairs $\{v, v'\}$, such that $d_2^*(v, v'|W_m^c) = 2$, $vv' \in E(W_m^c)$ is (m-1).

(II) If $v = v_1, v' = v_i$, $2 \le i \le m$, then there are three internally disjoint paths between the vertices v_1 and v_i, $2 \le i \le m$ having lengths 1,2, and 2, then $d_2^*(v_1, v_i|W_m^c) = 2$, $2 \le i \le m$.

It is clear that the number of such pairs $\{v_1, v_i\}$, $2 \le i \le m$, is (m-1).

(III) If $v, v' \in V - \{v_1\}$, and $vv' \notin E(W_m^c)$, then there are three internally disjoint paths between the vertices v and v', having lengths 2, $d(v, v'|W_m^c - \{v_1\})$ and $(m-1) - d(v, v'|W_m^c - \{v_1\})$. Then

$$d_2^*(v, v'|W_m^c) = \min\{\max\{2, d(v, v'|W_m^c-\{v_1\})\}, \max\{2, (m-1)-d(v, v'|W_m^c-\{v_1\})$$
$$, \max\{d(v, v'|W_m^c-\{v_1\}), (m-1)-d(v, v'|W_m^c-\{v_1\})\}.$$

Since $2 \le d(v, v'|W_m^c-\{v_1\}) \le \left\lfloor \dfrac{m-1}{2} \right\rfloor$, then,

$$d_2^*(v, v'|W_m^c) = \min\{d(v, v'|W_m^c-\{v_1\}), (m-1)-d(v, v'|W_m^c-\{v_1\}) = d(v, v'|W_m^c-\{v_1\}).$$

Let $d(v, v'|W_m^c-\{v_1\}) = k$, for all $2 \le k \le \left\lfloor \dfrac{m-1}{2} \right\rfloor$, then the number of pairs $\{v, v'\}$,

$vv' \notin E(W_m^c)$ such that $d_2^*(v, v'|W_m^c) = k$, for all $2 \le k \le \left\lfloor \dfrac{m-1}{2} \right\rfloor$, is (m-1), except when

m is odd and $d_2^*(v, v'|W_m^c) = \dfrac{m-1}{2}$, then the number of pairs $\{v, v'\}$ is $\dfrac{m-1}{2}$.

Hence,

$$H_2^*(\tilde{V}, W_m^c; x) = 2(m-1)x^2 + (m-1)\begin{bmatrix} \displaystyle\sum_{i=2}^{\frac{m-3}{2}} x^i + \frac{1}{2}x^{\frac{m-1}{2}}, \text{ if m is odd}, \\[2em] \displaystyle\sum_{i=2}^{\frac{m}{2}-1} x^i, \text{ if m is even}. \end{bmatrix}$$

Case (3): If $u \in U$ and $v \in V$, then there are two internally disjoint paths between the vertices u and v for each of the following subcases :

(a) If $u = u_i$, $1 \le i \le m-1$, and $v = v_1$, then two paths each has length 2; therefore $d_2^*(v_1, u_i|W_m^c) = 2$, $1 \le i \le m-1$, and the number of these pairs is m-1.

(b) If $v \in V-\{v_1\}$ and $uv \in E(W_m^c)$, then the two paths have length 1, and 2, therefore $d_2^*(v, u|W_m^c) = 2$, and the number of such pairs $\{u, v\}$ is 2(m-1).

(c) If $v \in V-\{v_1\}$, and u and v are adjacent to a common vertex of $V-\{v_1\}$, then the two paths have lengths 2, and 3 , therefore $d_2^*(u, v|W_m^c) = 3$, and the number of such pairs $\{u, v\}$ is 2(m-1).

(d) If $v \in V-\{v_1\}$, and u and v are neither adjacent nor adjacent to a common vertex of $V-\{v_1\}$, then the two paths have lengths 3, and $d(u, v|W_m^c-\{v_1\})$. Since, $3 \le d(u, v|W_m^c-\{v_1\}) \le \left\lfloor \dfrac{m}{2} \right\rfloor$, then $d_2^*(u, v|W_m^c) = d(u, v|W_m^c-\{v_1\})$.

Let $d(u, v|W_m^c-\{v_1\}) = k$, for all $3 \le k \le \left\lfloor \dfrac{m}{2} \right\rfloor$, then the number of pairs $\{u, v\}$ such that $d_2^*(v, u|W_m^c) = k$, for all $3 \le k \le \left\lfloor \dfrac{m}{2} \right\rfloor$ is 2(m-1), except if m is even, in which the number of $\{u,v\}$ such that $d_2^*(v, u|W_m^c) = \dfrac{m}{2}$, is (m-1).

Thus,

$$H_2^*(R, W_m^c; x) = (m-1)x^2 + 2(m-1)x^3 + 2(m-1)\begin{cases} \sum_{i=2}^{\frac{m-1}{2}} x^i \, , \text{ if } m \text{ is odd}, \\[4mm] \sum_{i=2}^{\frac{m}{2}-1} x^i + \frac{1}{2} x^{\frac{m}{2}} \, , \text{ if } m \text{ is even}, \end{cases}$$

where $R = \{(u,v) : u \in U, v \in V\}$, such that $d_2^*(u, v | W_m^c) = k$, for all $2 \le k \le \left\lfloor \dfrac{m}{2} \right\rfloor$.

From $H_2^*(W_m^c; x) = H_2^*(\tilde{U}, W_m^c; x) + H_2^*(\tilde{V}, W_m^c; x) + H_2^*(R, W_m^c; x)$, we get (4.3.1).　■

We notice, from (4.3.1), that $H_2^*(W_m^c; x)$ is 3-semi-palindromic for $m \ge 16$.

Remark:

- $H_2^*(W_4^c; x) = 18x^2 + 2x^3$.
- $H_2^*(W_5^c; x) = 22x^2 + 10x^3 + 4x^4$.
- $H_2^*(W_6^c; x) = 30x^2 + 15x^3 + 10x^4$.
- $H_2^*(W_7^c; x) = 36x^2 + 27x^3 + 15x^4$.

Corollary 4.3.2: For $m \ge 6$,
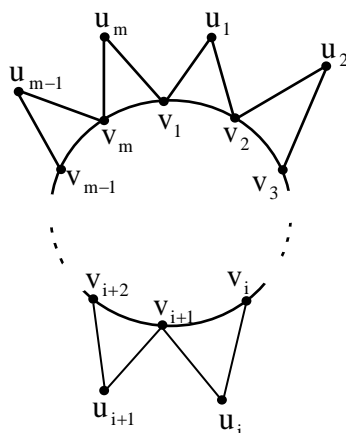
(a) $W_2^*(W_m^c) = (m-1)(m^2 + 22)/2$.

(b) $\mu_2^*(W_m^c) = (m^2 + 22)/(4m - 2)$. ■

Notice that $W_2^*(W_4^c) = 42$, $W_2^*(W_5^c) = 90$, $\mu_2^*(W_4^c) = 2$ and $\mu_2^*(W_5^c) = 2.5$.

Corollary 4.3.3: For $m \ge 4$, $\mu_2^*(W_m^c) \le \dfrac{m+5}{4}$, $m \ge 4$. ■

4.4. Cog Cycles C_m^c :

Let $C_m = v_1, v_2, \dots, v_m, v_1$ be a cycle of order m. The cog cycle C_m^c , $m \ge 3$, is obtained from C_m by adding m new vertices $u_1, u_2, \dots, u_m$, and edges $\{u_i v_i, u_i v_{i+1} : i = 1, 2, \dots, m\}$; $v_{m+1} = v_m$ [8], as shown in Fig.4.4.1.

Fig. 4.4.1. C_m^c

It is clear that $p(C_m^c) = 2m$, $q(C_m^c) = 3m$, $w = 2$, $\delta_2^*(C_m^c) = m$, and $m_w(C_m^c) = 2$.

Theorem 4.4.1: For $m \geq 7$, then

$$H_2^*(C_m^c; x) = 3mx^2 + 3mx^{m-1} + mx^m$$

$$+ 4m \begin{bmatrix} \sum_{i=2}^{\frac{m-3}{2}} x^{m-i} + \frac{1}{2}x^{\frac{m+1}{2}} \text{, if } m \text{ is odd ,} \\ \\ \sum_{i=2}^{\frac{m-4}{2}} x^{m-i} + \frac{7}{8}x^{\frac{m}{2}+1} + \frac{1}{8}x^{\frac{m}{2}} \text{, if } m \text{ is even .} \end{bmatrix} \qquad \dots(4.4.1)$$

Proof: Let $V = \{v_1, v_2, \dots, v_m\}$, and $U = \{u_1, u_2, \dots, u_m\}$, then there are three cases to prove (4.4.1).

Case (1): If $u, u' \in U$, $u \neq u'$, then there are two internally disjoint paths between u and u' having length $d(u, u'|C_m^c)$ and $(m+2) - d(u, u'|C_m^c)$.

Since $2 \leq d(u, u'|C_m^c) \leq \left\lfloor \dfrac{m+2}{2} \right\rfloor$, then $d_2^*(u, u'|C_m^c) = (m+2) - d(u, u'|C_m^c)$.

Let $d(u, u'|C_m^c) = k$, for all $2 \leq k \leq \left\lfloor \dfrac{m+2}{2} \right\rfloor$, then the number of pairs $\{u, u'\}$ such that $d_2^*(u, u'|C_m^c) = (m+2) - k$ is m, except if m is even, in which the number of $\{u, u'\}$ such that

$$d_2^*(u, u'|C_m^c) = \frac{m+2}{2}, \text{ is } \frac{m}{2}.$$

Hence,

$$H_2^*(\tilde{U}, C_m^c; x) = m \begin{bmatrix} \sum_{i=0}^{\frac{m-3}{2}} x^{m-i} \text{, if } m \text{ is odd ,} \\ \\ \sum_{i=0}^{\frac{m}{2}-2} x^{m-i} + \frac{1}{2}x^{\frac{m}{2}+1} \text{, if } m \text{ is even.} \end{bmatrix}$$

Case (2): If $v, v' \in V$, $v \neq v'$, then either $vv' \in E(C_m^c)$, or $vv' \notin E(C_m^c)$.

If $vv' \in E(C_m^c)$, then there are three internally disjoint paths between v and v', having lengths 1,2, and m-1, thus
$$d_2^*(v, v'|C_m^c) = \min[\max\{1,2\}, \max\{1, m-1\}, \max\{2, m-1\}] = 2.$$

The number of pairs $\{v, v'\}$, such that $d(v, v'|C_m^c) = 2$, $vv' \in E(C_m^c)$ is m.

Now, if $vv' \notin E(C_m^c)$, then there are two internally disjoint paths between v and v', having lengths $d(v, v'|C_m^c)$, and $m - d(v, v'|C_m^c)$.

Since $2 \leq d(u, v|C_m^c) \leq \left\lfloor \dfrac{m}{2} \right\rfloor$, then

$d_2^*(v, v' | C_m^c) = m - d(v, v' | C_m^c) = m - d(v, v' | C_m) = d_2^*(v, v' | C_m)$.

Hence , from this case and (3.3) , we have

$H_2^*(\tilde{V}, C_m^c; x) = H_2^*(C_m; x) - mx^{m-1} + mx^2$

$$= m \begin{bmatrix} \displaystyle\sum_{i=2}^{\frac{m-1}{2}} x^{m-i} + x^2 \text{, if m is odd ,} \\ \displaystyle\sum_{i=2}^{\frac{m}{2}-1} x^{m-i} + \frac{1}{2} x^{\frac{m}{2}} + x^2 \text{ , if m is even.} \end{bmatrix}$$

Case (3): If $u \in U$, and $v \in V$, then either $uv \in E(C_m^c)$, or $uv \notin E(C_m^c)$.

If $uv \in E(C_m^c)$, then there are two internally disjoint paths between the vertices u and v of length 1 and 2, then $d_2^*(u, v | C_m^c) = 2$.

It is clear that the number of pairs { u , v }, such that $d_2^*(u, v | C_m^c) = 2$, $uv \in E(C_m^c)$ is 2m.

Now, if $uv \notin E(C_m^c)$, then there are two internally disjoint paths between the vertices u and v having length $d(u, v | C_m^c)$, and $(m+1) - d(u, v | C_m^c)$.

Since $2 \leq d(u, v | C_m^c) \leq \left\lfloor \dfrac{m+1}{2} \right\rfloor$, then $d_2^*(u, v | C_m^c) = (m+1) - d(u, v | C_m^c)$.

Let $d(u, v | C_m^c) = k$, for all $2 \leq k \leq \left\lfloor \dfrac{m+1}{2} \right\rfloor$, then the number of pairs { u , v } such that $d_2^*(u, v | C_m^c) = (m+1) - k$ is 2m, except if m is odd, in which the number of { u , v } such that $d_2^*(u, v | C_m^c) = \dfrac{m+1}{2}$, is m.

Thus, from this case, we get

$$H_2^*(R, C_m^c; x) = 2m \begin{bmatrix} \displaystyle\sum_{i=1}^{\frac{m-3}{2}} x^{m-i} + \frac{1}{2} x^{\frac{m+1}{2}} + x^2 \text{, if m is odd ,} \\ \displaystyle\sum_{i=1}^{\frac{m}{2}-1} x^{m-i} + x^2 \text{, if m is even ,} \end{bmatrix}$$

where $R = \{(u, v) : u \in U, v \in V\}$.

From $H_2^*(C_m^c; x) = H_2^*(\tilde{U}, C_m^c; x) + H_2^*(\tilde{V}, C_m^c; x) + H_2^*(R, C_m^c; x)$, we get (5.7.1). ∎

Remark:

- $H_2^*(C_3^c; x) = 12x^2 + 3x^3$.
- $H_2^*(C_4^c; x) = 14x^2 + 10x^3 + 4x^4$.
- $H_2^*(C_5^c; x) = 15x^2 + 10x^3 + 15x^4 + 5x^5$
- $H_2^*(C_6^c; x) = 18x^2 + 3x^3 + 21x^4 + 18x^5 + 6x^6$.

Corollary 4.4.2: If C_m^c , $m \geq 3$ be a cog-cycle graph of order 2m, then

(a) $W_2^*(C_m^c) = m(3m^2 - 6m + 13)/2$.

(b) $\mu_2^*(C_m^c) = (3m^2 - 6m + 13)/(4m - 2)$.

(c) $\mu_2^*(C_m^c) \leq \dfrac{3m}{4}$.   ∎

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