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Cartesian Product of Intuitionistic Fuzzy Modular Spaces

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Abstract:

In this paper, we define the concepts of intuitionistic fuzzy modular space and cartesian product in intuitionistic fuzzy modular space. Also, some properties of them are considered.

Keywords : modular space, cartesian product, fuzzy modular space, intuitionistic fuzzy modular space.

1. Introduction:

The concept of fuzzy sets was introduced by Zadeh [8]in1965 and study the it properties .1986, Atanassov [1] defined the notion of intuitionistic fuzzy set. The concept of modular space was introduced by Nakano [4] in 1950. Soon after, Musielak and Orlicz [3] redefined and generalized the notion of modular space in 1959. The concept of fuzzy modular space was introduced by Young Shen and Wei Chen [7] in 2013.The definition of cartesian product of two fuzzy modular spaces was introduced by Noor F. Al-Mayahi and Al-ham S. Nief [5]in 2019 and prove some results related with it .In this paper, we define the concepts of intuitionistic fuzzy modular space and cartesian product in intuitionistic fuzzy modular space . Also, some properties will be considered.

2. Preliminaries:

Definition (2.1)[8] :

Let X be a non-empty set and Let $I = [0,1]$ be the closed interval of real numbers . A fuzzy set μ in X (or a fuzzy subset form X) is a function from X to I=[0,1].
If μ is a fuzzy set in X then μ is described as characteristic function which connects every $x \in X$ to real number $\mu(x)$ in the interval I. $\mu(x)$ is the grade of membership function to x in μ. μ can be described completely as:

μ $=\{\langle x, \mu(x)\rangle : x \in X, 0 \leq \mu(x) \leq 1\} or \ \mu = \{\frac{\mu(x)}{x} : x \epsilon X\}$ where $\mu(x)$ is called the membership function for the fuzzy set μ . The family of all fuzzy sets in X is denoted by I^X.

Definition (2.2)[1]:

Let X be a non-empty set . An intuitionistic fuzzy set A is given by :
$A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \epsilon X \}$,where the functions $\mu_A : X \to I$ and

$\nu_A: X \to I$ denote the degree of membership and the degree of non-membership to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for each $x \in X$.The set of all intuitionistic fuzzy sets in X denoted by $IFS(X)$.

Definition (2.3)[4]:

Let X be a vector space over a field F.

(1) A function $\rho: X \to [0, \infty]$ is called modular if

(a) $\rho(x) = 0$ if and only if $x = 0$;

(b) $\rho(\alpha x) = \rho(x)$ for $\alpha \in F$ with $|\alpha| = 1$, for all $x \in X$;

(c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ iff $\alpha, \beta \geq 0$ whenever $\alpha + \beta = 1$,

for all $x, y \in X$. If (c) is replaced by

(c') $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ iff $\alpha, \beta \geq 0, \alpha + \beta = 1$ for all $x, y \in X$, then the modular ρ is called convex modular .

(2) A modular ρ defines a corresponding modular space , i.e., the space X_ρ given by

$$X_\rho = \{x \in X: \rho(\alpha x) \to 0 \ as \ \alpha \to 0\}.$$

Definition (2.4)[6]:

Let $*$ be a binary operation on the set $I = [0,1]$, i.e $* : [0,1] \times [0,1] \to [0,1]$ is a function, then $*$ is said to be t-norm (triangular-norm) on the set I if $*$ satisfies the following axioms:

(1) $*$ is commutative and associative.

(2) $a * 1 = a$ for all $a \in [0,1]$.

(3) If $b, c \in I$ such that $b \leq c$, then $a * b \leq a * c$ for all $a \in I$.

In addition, if $*$ is continuous then $*$ is called a continuous t-norm.

Theorem (2.5)[2]:

Let $*$ be a continuous t-norm on the set $I = [0,1]$, then:

(1) $1 * 1 = 1$

(2) $0 * 1 = 0$

(3) $0 * 0 = 0$

(4) $a * a \leq a, \forall a \in I$

(5) If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$ for all $a, b, c, d \in I$.

Definition(2.6)[7]:

The 3- tuple $(X, \mu, *)$is said to be a fuzzy modular space (shortly,F-modular space) if X is a vector space, $*$ is a continuous t-norm and μ is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions, for all $x, y \in X, t, s > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$:

$(\boldsymbol{FM.1})$ $\mu(x, t) > 0$,

$(\boldsymbol{FM.2})$ $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = 0$,

$(\boldsymbol{FM.3})$ $\mu(x, t) = \mu(-x, t)$,

$(\boldsymbol{FM.4})$ $\mu(\alpha x + \beta y, t + s) \geq \mu(x, t) * \mu(y, s)$,

$(\boldsymbol{FM.5})$ $\mu(x, .): (0, \infty) \to (0,1]$ is continuous.

Generally ,if $(X, \mu, *)$ is fuzzy modular space,we say that $(\mu, *)$ is a fuzzy modular on X.

Definition(2.7)[6]:
Let ◊ be a binary operation on the set $I = [0,1]$, then ◊ is said to be t-conorm (triangular-conorm) on the set I if ◊ satisfies the following axioms:
(1) ◊ is commutative and associative,
(2) $a \lozenge 0 = a$ for all $a \in [0,1]$,
(3) If $b, c \in I$ such that $b \leq c$, then $a \lozenge b \leq a \lozenge c$ for all $a \in I$.
In addition, If ◊ is continuous then ◊ is called a continuous t-conorm.
Theorem (2.8)[2]:
Let ◊ be a continuous t-conorm on the set $I = [0,1]$, then :
(1) $0 \lozenge 0 = 0$
(2) $1 \lozenge 0 = 1$
(3) $1 \lozenge 1 = 1$
(4) $a \lozenge a \geq a, \forall a \in I$

(5) If $a \leq c$ and $b \leq d$, then $a \lozenge b \leq c \lozenge d$ for all $a, b, c, d \in I$

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3. Main Results:
Definition (3.1):
The 5-tuple $(X, \mu, v, *, \lozenge)$ is said to be an intuitionistic fuzzy modular space (shortly,IF-modular space) if X is a vector space,$*$ is a continuous t-norm,◊ is a continuous t-conorm and μ, v are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions: for all $x, y \in X, t, s > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,
$(IFM.1)$ $\mu(x,t) + v(x,t) \leq 1$,
$(IFM.2)$ $\mu(x,t) > 0$,
$(IFM.3)$ $\mu(x,t) = 1$ if and only if $x = 0$,
$(IFM.4)$ $\mu(x,t) = \mu(-x,t)$,
$(IFM.5)$ $\mu(\alpha x + \beta y, t + s) \geq \mu(x,t) * \mu(y,s)$,
$(IFM.6)$ $\mu(x,.): (0,\infty) \to (0,1]$ is continuous,
$(IFM.7)$ $v(x,t) < 1$,
$(IFM.8)$ $v(x,t) = 0$ if and only if $x = 0$,
$(IFM.9)$ $v(x,t) = v(-x,t)$,
$(IFM.10)$ $v(\alpha x + \beta y, t + s) \leq v(x,t) \lozenge v(y,s)$,
$(IFM.11)$ $v(x,.): (0,\infty) \to (0,1]$ is continuous.

Definition(3.2):
Let $(X, \mu, v, *, \lozenge)$ be an intuitionistic fuzzy modular space,Then
1) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, if for every $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in Z^+$ such that
$\mu(x_n - x, t) > 1 - \epsilon$ and $v(x_n - x, t) < \epsilon$ for all $n \geq n_0$. (or equivalently $\lim_{n\to\infty} \mu(x_n - x, t) = 1$ and $\lim_{n\to\infty} v(x_n - x, t) = 0$).

2) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon \in (0,1)$ and $t > 0$,there exists $n_0 \in Z^+$ such that $\mu(x_n - x_m, t) > 1 - \epsilon$ and $v(x_n - x_m, t) < \epsilon$ for all $n, m \geq n_0$. (or equivalently $\lim_{n,m\to\infty} \mu(x_n - x_m, t) = 1$ and $\lim_{n,m\to\infty} v(x_n - x_m, t) = 0$).

3) An intuitionistic fuzzy modular space $(X, \mu, v, *, \lozenge)$ is said to be Complete if every Cauchy sequence is convergent.

Definition(3.3):

Let $(X, \mu, v, *, \lozenge)$ be an intuitionistic fuzzy modular space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with center $x \in X$ and radius $0 < r < 1, t > 0$ are defined as follows:

$B(x, r, t) = \{y \in X: \mu(x - y, t) > 1 - r$ and $v(x - y, t) < r\}$,

$B[x, r, t] = \{y \in X: \mu(x - y, t) \geq 1 - r$ and $v(x - y, t) \leq r\}$.

Definition(3.4):

Let $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ be two intuitionistic fuzzy modular spaces. the cartesian product of $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ is the product space $(X \times Y, \mu, v, *, \lozenge)$ where $X \times Y$ is the cartesian product of the sets X and Y and μ, v are functions

$\mu, v: (X \times Y \times (0, \infty)) \to [0,1]$ is given by:

$\mu\big((w, z), t\big) = \mu_1(w, t) * \mu_2(z, t)$,

$v\big((w, z), t\big) = v_1(w, t) \lozenge v_2(z, t)$ for all $(w, z) \in X \times Y$ and $t, > 0$.

Theorem(3.5):

Let $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ be two intuitionistic fuzzy modular Spaces .Then $(X \times Y, \mu, v, *, \lozenge)$ is an intuitionistic fuzzy modular space.

Proof:

Let $(w, z) \in X \times Y$,we have

1) since $\mu_1(w, t) > 0$, $\mu_2(z, t) > 0\ \forall t > 0$,then

$\mu\big((w, z), t\big) = \mu_1 (w, t) * \mu_2(z, t) > 0$ and

since $v_1(w, t) < 1$, $v_2(z, t) < 1\ \forall t > 0$,then

$v\big((w, z), t\big) = v_1 (w, t) \lozenge v_2(z, t) < 1$.

2) $\mu_1(w, t) = 1 \Leftrightarrow w = 0$, also $\mu_2(z, t) = 1 \Leftrightarrow z = 0$. Then

$\mu_1(w, t) * \mu_2(z, t) = 1 \Leftrightarrow (w, z) = 0$ .Hence $\mu\big((w, z), t\big) = 1 \Leftrightarrow (w, z) = 0\ \forall t > 0$ and $v_1(w, t) = 0 \Leftrightarrow w = 0$, also $v_2(z, t) = 0 \Leftrightarrow z = 0$.

Then $v_1(w, t) \lozenge v_2(z, t) = 0 \Leftrightarrow (w, z) = 0$. Hence $v((w, z), t) = 0 \Leftrightarrow (w, z) = 0\ \forall t > 0$.

3) since $\mu_1(w, t) = \mu_1(-w, t), v_1(w, t) = v_1(-w, t)\ \forall t > 0$ and $\mu_2(z, t) = \mu_2(-z, t), v_2(z, t) = v_2(-z, t)\ \forall t > 0$, then

$$\mu\big((w, z), t\big) = \mu_1 (w, t) * \mu_2 (z, t) = \mu_1(-w, t) * \mu_2(-z, t) = \mu(-(w, z), t)$$

and

$$v\big((w, z), t\big) = v_1 (w, t) \lozenge v_2 (z, t) = v_1(-w, t) \lozenge v_2(-z, t)$$

$= \nu(-(w,z),t).$

4) $\mu(\alpha(w_1,z_1)+\beta(w_2,z_2),t) \geq \mu((\alpha w_1+\beta w_2,\alpha z_1+\beta z_2),t)$
$\geq \mu_1(\alpha w_1+\beta w_2,t) * \mu_2(\alpha z_1+\beta z_2,t)$
$\geq \mu_1(w_1,t) * \mu_1(w_2,t) * \mu_2(z_1,t) * \mu_2(z_2,t)$
$\geq \mu_1(w_1,t) * \mu_2(z_1,t) * \mu_1(w_2,t) * \mu_2(z_2,t)$
$\geq \mu\big((w_1,z_1),t\big) * \mu((w_2,z_2),t)$ and
$\nu(\alpha(w_1,z_1)+\beta(w_2,z_2),t) \leq \nu((\alpha w_1+\beta w_2,\alpha z_1+\beta z_2),t)$
$\leq \nu_1(\alpha w_1+\beta w_2,t) \lozenge \nu_2(\alpha z_1+\beta z_2,t)$
$\leq \nu_1(w_1,t) \lozenge \nu_1(w_2,t) \lozenge \nu_2(z_1,t) \lozenge \nu_2(z_2,t)$
$\leq \nu_1(w_1,t) \lozenge \nu_2(z_1,t) \lozenge \nu_1(w_2,t) \lozenge \nu_2(z_2,t)$
$\leq \nu((w_1,z_1),t) \lozenge \nu((w_2,z_2),t)$

5) since $\mu_1(w,t):(0,\infty) \to (0,1]$ is continuous , $\mu_2(z,t):(0,\infty) \to (0,1]$ is continuous and since $\nu_1(w,t):(0,\infty) \to (0,1]$ is continuous , $\nu_2(z,t):(0,\infty) \to (0,1]$ is continuous then $\mu\big((w,z),t\big):(0,\infty) \to (0,1]$ is continuous and $\nu\big((w,z),t\big):(0,\infty) \to (0,1]$ is continuous.

Theorem(3.6) :

Let $\{w_n\}$ be a sequence in intuitionistic fuzzy modular space $(X,\mu_1,\nu_1,*,\lozenge)$ which converges to w in X and $\{z_n\}$ is a sequence in the intuitionistic fuzzy modular space $(Y,\mu_2,\nu_2,*,\lozenge)$ which converges to z in Y. Then $\{(w_n,z_n)\}$ is a sequence in intuitionistic fuzzy modular space$(X \times Y,\mu,\nu,*,\lozenge)$ converges to (w,z) in $X \times Y$.

Proof:

To prove that sequence $\{(w_n,z_n)\}$ in $X \times Y$ converges to (w,z) we show that $\lim_{n\to\infty} \mu\big((w_n,z_n)-(w,z),t\big) = 1$ and
$\lim_{n\to\infty} \nu\big((w_n,z_n)-(w,z),t\big) = 0$
by theorem(3.5) $(X \times Y,\mu,\nu,*,\lozenge)$ is an intuitionistic fuzzy modular space
since $\{w_n\}$ be a sequence in $(X,\mu_1,\nu_1,*,\lozenge)$ convergence to w
then $\lim_{n\to\infty} \mu_1(w_n-w,t) = 1$ and $\lim_{n\to\infty} \nu_1(w_n-w,t) = 0$
since $\{z_n\}$ be a sequence in $(Y,\mu_2,\nu_2,*,\lozenge)$ convergence to z
then $\lim_{n\to\infty} \mu_2(z_n-z,t) = 1$ and $\lim_{n\to\infty} \nu_2(z_n-z\,,t) = 0$
then that $\lim_{n\to\infty} \mu\big((w_n,z_n)-(w,z),t\big) = \lim_{n\to\infty} \mu_1(w_n-w,t)$
$* \lim_{n\to\infty} \mu_2(z_n-z,t) = 1 * 1 = 1$ and
$\lim_{n\to\infty} \nu\big((w_n,z_n)-(w,z),t\big) = \lim_{n\to\infty} \nu_1(w_n-w,t) \lozenge \lim_{n\to\infty} \nu_2(z_n-z,t) = 0 \lozenge 0 = 0.$
Hence $\{(w_n,z_n)\}$ converges to (w,z).

Theorem(3.7) :

Let $\{w_n\}$ be a Cauchy sequence in intuitionistic fuzzy modular space $(X,\mu_1,\nu_1,*,\lozenge)$ and $\{z_n\}$ is a Cauchy sequence in intuitionistic fuzzy modular space $(Y,\mu_2,v_2,*,\lozenge)$ then $\{(w_n,z_n)\}$ is a Cauchy sequence in intuitionistic fuzzy modular space $(X\times Y,\mu,v,*,\lozenge)$.

Proof:

By theorem (3.5) $(X\times Y,\mu,v,*,\lozenge)$ is intuitionistic fuzzy modular space since $\{w_n\}$ be a Cauchy sequence in intuitionistic fuzzy modular space $(X,\mu_1,v_1,*,\lozenge)$

then $\lim_{n,m\to\infty}\mu_1(w_n-w_m,t)=1$ and $\lim_{n,m\to\infty}v_1(w_n-w_m,t)=0$

since $\{z_n\}$ be a Cauchy sequence in intuitionistic fuzzy modular space $(Y,\mu_2,v_2,*,\lozenge)$

then $\lim_{n,m\to\infty}\mu_2(z_n-z_m,t)=1$ and $\lim_{n,m\to\infty}v_2(z_n-z_m,t)=0$

then $\lim_{n,m\to\infty}\mu\big((w_n,z_n)-(w_m,z_m),t\big)=\lim_{n,m\to\infty}\mu_1(w_n-w_m,t)$ $*\lim_{n,m\to\infty}\mu_2(z_n-z_m,t)=1*1=1$ and

$\lim_{n,m\to\infty}v\big((w_n,z_n)-(w_m,z_m),t\big)=\lim_{n,m\to\infty}v_1(w_n-w_m,t)$ $\lozenge\lim_{n,m\to\infty}v_2(z_n-z_m,t)=0\lozenge 0=0$.

Hence $\{(w_n,z_n)\}$ is a Cauchy sequence in $(X\times Y,\mu,v,*,\lozenge)$.

Theorem(3.8):

If $(X\times Y,\mu,v,*,\lozenge)$ is an intuitionistic fuzzy modular space, then? $(X,\mu_1,v_1,*,\lozenge)$ and $(Y,\mu_2,v_2,*,\lozenge)$ are intuitionistic fuzzy modular spaces by defining

$\mu_1(w,t)=\mu\big((w,0),t\big),v_1(w,t)=v((w,0),t)$ and

$\mu_2(z,t)=\mu\big((0,z),t\big),v_2(z,t)=v((0,z),t)$ for all $w\in X,z\in Y$ and $t>0$.

Proof:

1) $\mu_1(w,t)=\mu\big((w,0),t\big)>0,v_1(w,t)=v\big((w,0),t\big)<1\ \forall w\in X$

2) For all $t>0,1=\mu_1(w,t)=\mu\big((w,0),t\big)\Leftrightarrow w=0$ and
$0=v_1(w,t)=v\big((w,0),t\big)\Leftrightarrow w=0$.

3) For all $t>0,\mu_1(w,t)=\mu_1(-w,t)=\mu(-(w,0),t)$ and
$v_1(w,t)=v_1(-w,t)=v(-(w,0),t)$.

4) $\mu_1(\alpha w+\beta w_1,t)=\mu\big((\alpha w+\beta w_1,0),t\big)$
$\geq\mu\big((w,0),t\big)*\mu\big((w_1,0),t\big)\geq\mu_1(w,t)*\mu_1(w_1,t)$ and
$v_1(\alpha w+\beta w_1,t)=v\big((\alpha w+\beta w_1,0),t\big)$
$\leq v((w,0),t)\lozenge v((w_1,0),t)\leq v_1(w,t)\lozenge v_1(w_1,t)$.

5) $\mu_1(w,.)=\mu((w,0),.)$ and $v_1(w,.)=v((w,0),.)$ are continuous from $(0,\infty)$ to $(0,1]$ for all $w\in X$.Then $(X,\mu_1,v_1,*,\lozenge)$ is intuitionistic fuzzy modular space Similarly we can prove that $(Y,\mu_2,v_2,*,\lozenge)$.

Theorem(3.9):

Let $(X,\mu_1,v_1,*,\lozenge)$ and $(Y,\mu_2,v_2,*,\lozenge)$ be two intuitionistic fuzzy modular spaces, then the product $(X\times Y,\mu,v,*,\lozenge)$ is complete intuitionistic fuzzy modular space if and only if $(X,\mu_1,v_1,*,\lozenge)$ and $(Y,\mu_2,v_2,*,\lozenge)$ are complete intuitionistic fuzzy modular spaces.

Proof:

Suppose that $(X \times Y, \mu, v, *, \lozenge)$ is complete intuitionistic fuzzy modular space

Since $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ are intuitionistic fuzzy modular spaces

By theorem (3.8)

Let $\{w_n\}$ be a Cauchy sequence in $(X, \mu_1, v_1, *, \lozenge)$

Then $\{(w_n, 0)\}$ be a Cauchy sequence in $X \times Y$

Since $X \times Y$ is complete intuitionistic fuzzy modular space

Then there is $(w, 0)$ in $X \times Y$ such that $\{(w_n, 0)\}$ convergent to $(w, 0)$

Now , $\lim_{n\to\infty} \mu_1(w_n - w, t) = \lim_{n\to\infty} \mu\big((w_n - w, 0), t\big) = 1$ and

$\lim_{n\to\infty} v_1(w_n - w, t) = \lim_{n\to\infty} v\big((w_n - w, 0), t\big) = 0$

Then $(X, \mu_1, v_1, *, \lozenge)$ is complete intuitionistic fuzzy modular space

Similarly we can prove that $(Y, \mu_2, v_2, *, \lozenge)$ is complete intuitionistic fuzzy modular space.

Conversely, suppose that $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ are complete intuitionistic fuzzy modular spaces

Let $\{(w_n, z_n)\}$ be a Cauchy sequence in $X \times Y$

since $(X, \mu_1, v_1, *, \lozenge)$ and $(Y, \mu_2, v_2, *, \lozenge)$ are complete intuitionistic fuzzy modular spaces

then $\exists\, w$ in X and z in Y such that $\{w_n\}$ convergent to w and $\{z_n\}$ convergent to z .

So $\lim_{n,\to\infty} \mu_1(w_n - w, t) = 1$, $\lim_{n\to\infty} v_1(w_n - w, t) = 0$ and

$\lim_{n\to\infty} \mu_2(z_n - z, t) = 1$, $\lim_{n\to\infty} v_2(z_n - z, t) = 0$ then

$\lim_{n\to\infty} \mu\big((w_n, z_n) - (w, z), t\big) = \lim_{n\to\infty} \mu_1(w_n - w, t) * \lim_{n\to\infty} \mu_2(z_n - z, t)$

$= 1 * 1 = 1$ and

$\lim_{n\to\infty} v\big((w_n, z_n) - (w, z), t\big) = \lim_{n\to\infty} v_1(w_n - w, t) \lozenge \lim_{n\to\infty} v_2(z_n - z, t)$

$= 0 \lozenge 0 = 0$

Hence $\{(w_n, z_n)\}$ convergent to (w, z) in $X \times Y$.

Hence $(X \times Y, \mu, v, *, \lozenge)$ is complete intuitionistic fuzzy modular space.

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