On Almost Tℓ-m- continuous Multifunctions

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Abstract

We introduce and study the concept of almost Tℓ-m- continuous multifunctionsby using the concept of T- open set and minimal spaces, which is stronger than the concept of almost $T\ell$- continuous multifunctions. Several properties and characterizations of this new concept are proved .

1. Introduction:

The notions ofalmost continuous multifunctions, m- continuous multifunctionsand their properties are studied by ValeriuPopa, and Takashi Noiri[7],[9],[10], Whileat 2006,A. Kanibir and I.L. Reilly [2]investigate the conceptof Almost ℓ-continuous multifunctions. After thatHadiJaber Mustafa and Muayad G. Mohsen [6] introduced a stronger concept than almost ℓ-continuous multifunctions, between topological spaces namely almost $T\ell$-continuous multifunctions (briefly, a.$T\ell$-c.mf.).

In this paper we shall focus on a new class of functions, lies between almost continuous multifunctionsand Almost Tℓ-continuous multifunctions. By using the concepts of T-opensets[4],m_X_open sets[10] and T-Lindelöf[5] we shall introduce the concept of almost Tℓ-m continuous multifunction (briefly, a.Tℓ-m-c.mf.) which is stronger than the concept of almost Tℓ-continuous multifunctions (a. ℓ-c.m.f.). Throughout this paper , the closure (resp. interior) of a subset B in a topological space (Y,τ) is denoted by $\mathrm{cl}B$ (resp. $\mathrm{int}B$), while B is called regular open if $B = \mathrm{int}(\mathrm{cl}B)$.

2. Preliminiries:

In this section, we introduce and recall the basic definition and facts needed in this work.

2.1 Definition:

Let (X,τ,T) be an operator topological space [3], and let $A \subseteq X$. Then,

1) A is called$T$- open if given $x \in A$,then there exist $V \in \tau$ such that $x \in V \subseteq T(V) \subseteq A$. The complement of a T-open set is called T-closed[4].
2) A is calledT-regular open[6] (briefly TRO) if and only if A is regular open and T- open. The set of all T-regular open is denoted by TRO (X,τ,T). The complement of T- regular open set is T-regular closed (briefly TRC).
3) X is called T-Lindelöf if every T-open cover of X has a countable subcover .

2.2 Remark :

The family of all T- open subsets of X does not form in general a topology on X [4] .

2.3Definition:[10]

Let X be a non-empty set and let $m_X \subseteq P(X)$,where $P(X)$ denoted to power set of X. Then m_X is called an m-structure (or a minimal structure) on X, if $\emptyset$ and X belong to m_X.

2.4Remark:

1) The members of the minimal structure m_X are called$m_X - open$ sets,and the pair (X, m_X) is called an $m - space$.
2) The complement of $m_X - open$ sets is said to be$m_X - closed$ sets.
3) It's clear that if (X,τ) is a space then the topology τ on X is a minimal space but the convers is not true in general.

2.5Definition:[8]

By a multifunction $F:(X,\sigma)\longrightarrow(Y,\tau)$, we mean a point –to-set correspondence from$(X,\sigma)$to(Y,τ), and we always assume that $F(x)\neq\emptyset,\forall\, x\in X$.

2.6Definition:[8]

Let $F:(X,\sigma)\longrightarrow(Y,\tau)$ be a topological multifunction, $A\subseteq X$, and $B\subseteq Y$. Then,

i) $F^{+}(B)=\{x\in X: F(x)\subseteq B\}$ is called the upper inverse of B.

ii) $F^{-}(B)=\{x\in X: F(x)\cap B\neq\emptyset\}$ is called the lower inverse of the set B.

iii) $F(A)=\bigcup_{x\in A}F(x)$ is called the image of the set A.

2.7Definition:[11]

The multifunction $\mathrm{F}:(\mathrm{X},\sigma)\longrightarrow(\mathrm{Y},\tau)$ is called upper semicontinuous briefly u.s.c.(resp. lower semicontinuous briefly l.s.c.) if $\mathrm{F}^{+}(\mathrm{B})$ (resp.$\mathrm{F}^{-}(\mathrm{B})$ is open in(X,σ) for every V open set of (Y,τ).

2.8Definition[6]

i) The almost co T- Lindelöf topology τ on Y is denotedby $\mathrm{q}(\tau,\mathrm{T})$ and it has abase $\mathrm{q}'(\tau,\mathrm{T})=\{\mathrm{U}\in \mathrm{TRO}(\mathrm{Y},\tau,\mathrm{T}):\mathrm{U}^{c}\text{is T}-\text{Lindelöf}\}$

3. AlmostT$\boldsymbol{\ell}$-m-continuous multifunctions(simplya.T$\boldsymbol{\ell}$-m-c.mf.):

We now introduce a new class of multifunctionsthat related between two topological spaces with the following definition .

3.1 Definition :

A multifunction $F:(X,m_X)\longrightarrow(Y,\tau,T)$

is defined to be

i) Upper Almost $T\ell$-m-continuous or$u.a.T\ell$-m-c. at a point $x\in X$, if for each T-regular open subset V(briefly TRO) of Y with $F(x)\subseteq V$ and having T-Lindelöf complement, there exist an m_X −open neighbourhood U of x such that $F(U)\subseteq V$.

ii) Lower Almost $T\ell$-m-continuous or $l.a.T\ell$-m-c. at a point $x\in X$, if for each T-regular open subset V of Y with $F(x)\cap V\neq\emptyset$ and having T-Lindelöf complement , thereexist an m_X −open neighbourhood U of x such that $F(z)\cap V\neq\emptyset$ for every point $z\in U$.

iii) Almost $T\ell$-m-continuous, at a point $x\in X$, if it is both$u.a.T\ell$-m-c. and$l.a.T\ell$-m-c. at $x\in X$.

iv) Almost $T\ell$-m-continuous ($resp.$ $u.a.T\ell$-m-c. , $l.a.T\ell$-m-c.) if it is Almost $T\ell$-m-continuous ($resp.$ $u.a.T\ell$-m-c. , $l.a.T\ell$-m-c.) at each point of X.

3.2Example:

Consider$(\mathrm{X},\mathrm{m_X})$ be minimal space s.t.$\mathrm{X}=\{\mathrm{a},\mathrm{b},\mathrm{c}\}$and $\mathrm{m_X}=\{\emptyset,\mathrm{X},\{\mathrm{a}\},\{\mathrm{a},\mathrm{b}\},\{\mathrm{a},\mathrm{c}\}\}$, and let $(\mathrm{Y},\tau,\mathrm{T})$ be operator topological space s.t.$\mathrm{Y}=\{1,2,3,4\}$with the topology $\tau=\{\emptyset,\mathrm{Y},\{1\},\{2\},\{1,2\}\}$, and the operator $\mathrm{T}:\mathrm{P}(\mathrm{Y})\longrightarrow\mathrm{P}(\mathrm{Y})$ defined as $\mathrm{T}(\mathrm{A})=\mathrm{int}(\mathrm{cl}(\mathrm{A}))$, $\mathrm{A}\subseteq\mathrm{Y}$.

Define $\mathrm{F}:(\mathrm{X},\mathrm{m_X})\longrightarrow(\mathrm{Y},\tau,\mathrm{T})$ as follows:

$\mathrm{F}(\mathrm{a})=\mathrm{F}(\mathrm{b})=\{1\}$, $\mathrm{F}(\mathrm{c})=\{1,2\}$.

Then F is continuous, almost continuous and almost Tℓ-m-continuous multifunction.

3.3Example

let $\mathbb{R}$ be the set of real number with minimal structure $\mathrm{m_X}=\{\emptyset,\mathbb{R},\mathbb{Q}^{c}\}$, and let . $\mathrm{Y}=\{\mathrm{a},\mathrm{b}\}$ equipped with the topology $\tau=\{\emptyset,\mathrm{Y},\{\mathrm{a}\}\}$, and let $\mathrm{T}:\mathrm{P}(\mathrm{Y})\longrightarrow\mathrm{P}(\mathrm{Y})$ be the identity operator , then the function $\mathrm{F}:(\mathbb{R},\mathrm{m_X})\longrightarrow(\mathrm{Y},\tau,\mathrm{T})$ which is defined bellow is almost continuous and almost Tℓ-m-continuous but not continuous at $\mathrm{x}\in\mathbb{R}$ if it's rational.

$$F(x) = \begin{cases} \{a\}, \; if \;\; x \in \mathbb{Q}, \\ \{b\}, \;\; if \;\; x \notin \mathbb{Q} \end{cases}$$

3.4 Example

Let $X = \mathbb{R}$ with minimal space $m_X = \{\emptyset, \mathbb{R}, (-\infty, -r), (\infty, r), (-r, r)\}$

and let $Y = \mathbb{R}$ with usual topology . Define $F(x) = \begin{cases} \frac{1}{x-2} \; if \; x \neq 2 \\ \frac{1}{2} if \;\; x = 2 \end{cases}$

And define the operator $T: P(Y) \longrightarrow P(Y)$ as $T(A) = int(cl(A))$

Then the function $F: (X, \, m_X) \longrightarrow (Y, \tau, T)$ is almost $T\ell$-m-continuous at the point 2.

3.5Remark

It's clear from the definition that:

Almost$T\ell$-m-continuous multifunction⟹ almost $T\ell$-continuous

3.6 Theorem:

The following conditions are equivalent for a multifunctions$F: (X, m_X) \longrightarrow (Y, \tau, T)$.

a) $F$is upper almost $T\ell$-m-continuous.
b) $F^+(V)$ is m_X −open for any T-regular open set V having T-Lindelöf complement in Y.
c) $F^+(V)$ is m_X −open for any $V \in q'(\tau, T)$.
d) $F^-(V)$ is m_X −closed for any T-regular closed Lindelöf set $V \; of \; Y$.
e) For each $x \in X$, and each net(x_α) which converges to x in X, and for each T-regular open subset V with T-Lindelöf complement V^C, such that $x \in F^+(V)$, the net (x_α) is eventually in$F^+(V)$.

Proof:

(a) ⟹ (b)

Let $V \in$ TRO(Y, T, τ), having T- Lindelöf complement.Let $x \in F^+(V)$.Then there exist an m_X −open set U containing x, such that $F(U) \subseteq V$ hence $x \in U \subseteq F^+(V)$. This show that $F^+(V)$ is m_X −open.

(b) ⟹ (a) . Let $x \in X$ and V be any TRO subset of Y having T- Lindelöf complement with $F(x) \subseteq V$, then $x \in F^+(V)$ and $F^+(V)$ ism_X − open. Put $U = F^+(V)$. Hence U is an m_X −open nbh. of x and $F(U) \subseteq V$.

(b) ⟹ (c) . Let $V \in \; q'(\tau, T)$. hence V is TRO subset in Y, V^c is T- Lindelöf by (b) $F^+(V)$ is m_X −open.

(c) ⟹ (b) . let V is TRO subset of Y , hence T- Lindelöf in Y.
$\therefore V \in \; q'(\tau, T)$. $\therefore F^+(V)$ is m_X −open.

(b) ⟹ (d) . Let V be any TRC- Lindelöf subset of Y.consider V^c is TRO subset of Y having T- Lindelöf complement, by (b) we have $F^-(V^c)$ is m_X −open. Hence by the fact$F^+(V^c) = (F^-(V))^c$, we have $F^-(V)$ is m_X −closed .

(d) ⟹ (b) . Let V be any TRO subset of Y having T- Lindelöf complement . consider V^c is TRC- Lindelöf , by(d) $F^-(V^c)$ is closed ,hence $(F^-(V^c))^c$ is m_X −open , hence $F^+(V)$ is m_X −open.

(a) ⟹ (e). Let $J = (x_\alpha)$ be a net which converge to $x \in X$ and let V be any TRO subset of Y having T- Lindelöf complement V^c such that $x \in F^+(V)$,then there exist an m_X −open set $U \subseteq X$ containing x such that $U \subseteq F^+(V)$

Since (x_α) converge to x it follows that there exist $\alpha_0 \in \; \Omega$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Therefore $x_\alpha \in F^+(V)$ for all $\alpha \geq \alpha_0$. Hence the net(x_α) is eventually in $F^+(V)$.

(d) ⟹ (a) Suppose that (a) is not true .then there exist $x \in X$ and a TRO subset V of Y having T- Lindelöf complement with $F(x) \subseteq V$ such that $F(U) \nsubseteq V$ for each m_X −open set $U \subseteq X$ containing x .Therefore the nbh net (x_U), $x_U \rightarrow x$, but (x_U) is not eventually in $F^+(V)$.This is a contradiction. ■

Similarly , we can obtain the following conditions for the lower almost continuous multifunctions .

3.7 Theorem :

The following conditions are equivalent for a multifunctions$F: (X, m_X) \longrightarrow (Y, \tau, T)$

a) F is lower almost $T\ell$-m-continuous.
b) $F^-(V)$ is m_X −open for any T-regular open set V having T-Lindelöf complement in Y.
c) $F^-(V)$ is m_X −open for any $V \in q'(\tau, T)$.
d) $F^+(V)$ is m_X −closed for any T-regular closed T-Lindelöf set V of Y.
e) For each $x \in X$, and each net(x_α) which converges to x in X, and for each T-regular open subset V with T-Lindelöf complement V^c, such that $x \in F^-(V)$, the net (x_α) is eventually in$F^-(V)$.

3.8Remark :

Let $F: (X, \sigma) \longrightarrow (Y, \tau, T)$be a multifunction then we have:

i)If F is upper semi continuous (brieflyu.s.c.) then F is u.a. $T\ell$-m-continuous.

ii)If F is lower semicontinuous (briefly l.s.c.) then F is l.a. $T\ell$-m-continuous.

These implications are not reversible in general as the following examples shows:

3.9 Example:

Consider the minimal space $(\mathbb{R}, m_{\mathbb{R}})$ such that:

$m_{\mathbb{R}}$=$\{\emptyset, \mathbb{R}, (a,b): a, b \in \mathbb{R}\}$, where $\mathbb{R}$ the set of real numbers.

and $(\mathbb{R}, \tau_{CC}, T)$ be operator topological spaces , τ_{CC} is the co countabletopology on$\mathbb{R}$ and, T is the identity operator on P($\mathbb{R}$),(the power set of $\mathbb{R}$) .

Define $F: (\mathbb{R}, m_{\mathcal{R}}) \longrightarrow (\mathbb{R}, \tau_{CC}, T)$as follow:

$$F(x) = \begin{cases} \{x\}, & \text{if x is irrational} \\ Q, & \text{if x is rational} \end{cases}$$

Then the multifunction F is u.a. $T\ell$-m-continuous , since $q'(\tau_{CC}, T)$= $\{\mathbb{R}, \emptyset\}$. In fact F is a. $T\ell$-m-continuous. However F is not u.s.c. or l.s.c. , since $V = Q^c$ is open in $(\mathbb{R}, \tau_{CC})$, but $F^+(V)$ and $F^-(V)$are not$m_{\mathbb{R}}$- open in $(\mathbb{R}, m_{\mathcal{R}})$.

3.10 Theorem :

Let $F: (X, m_X) \longrightarrow (Y, \tau, T)$ be a multifunction then, it is l.a. $T\ell$-m-continuous iff$F_q: (X, m_X) \longrightarrow (Y, q(\tau, T), T)$is l.s.c.($where\ F_q = F$).

Proof⟹:

Assume F is l.a. $T\ell$-m-continuous. Let $V \in q(\tau, T)$ we can write $V \in \bigcup_{\alpha \in \Omega} V_\alpha$ where V_α is a TRO set having T-Lindelöf complement in Y for $\alpha \in \Omega$. Where $F_q^-(V) = F_q^-(\bigcup_{\alpha \in \Omega} V_\alpha) = \bigcup_{\alpha \in \Omega} F_q^-(V_\alpha)$= $\bigcup_{\alpha \in \Omega} F^-(V_\alpha)$ but $F^-(V_\alpha)$ is an ,m_X-open set for $\alpha \in \Omega$ by theorem (3.3), so $F_q^-(V)$ is an ,m_X-open set . Hence $F_q: (X, m_X) \longrightarrow (Y, q(\tau, T), T)$ is l.s.c.
⟸ abvious. l.s.c. ⟶ l.a. $T\ell$-m-continuous ■

The theorem (3.6) does not hold for upper almost continuous multifunctions as the following example shows .

3.11 Example :

Consider $X = N$ with the topology

$m_X = \{\emptyset, N, \{1\}, \{2,3,4, \dots\}\}$,and

$Y = \{1,2,3,4\}$with the topology $\tau = \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}\}$. let F be defined as

$F(1) = \{4\},\ F(2) = \{1,2\},\ F(\{3,4,\dots\}) = \{3\}$.

Let T= identity operator then $q'(\tau, T) = q'(\tau) = \{\emptyset, Y, \{1\}, \{2\}\}$.

The family $q'(\tau, T)$is abase consisting of TRO sets having T-Lindelöf complement in Y for $q(\tau, T) = \tau$. Then for any $V \in q'(\tau)$ we have $F^{+}(V) \in \sigma$, therefore $F: (X, m_X) \longrightarrow (Y, \tau, T)$ is u.a. $T\ell$-m-continuous . The topology $q(\tau)$ contains the set $\{1,2\}$ but $F_q^{+}(\{1,2\}) = \{2\} \notin m_X$. Hence $F_q: (X, m_X) \longrightarrow (Y, q(\tau, T), T)$ is not u.s.c. ■

In the next theorem we shall relate three different kinds of multifunction, m-continuous multifunction[10],almost $T\ell$continuous multifunctionand almost $T\ell$-m-continuous multifunction .

3.12 Theorem :

Let $F: (X, m_X) \longrightarrow (Y, \tau)$ and $G: (Y, \tau) \longrightarrow (Z, \sigma, T)$ be a multifunctions then, if F is l.m-continuous and G l.a. $T\ell$continuous then $G \circ F$ is l.a. $T\ell$-m-continuous.

Proof

Let V be a TRO open set having T-Lindelöf complement in Z ,since G l.a. $T\ell$continuous $G^{-}(V)$ is an open in Y. since $F$is l.m-continuous, then $F^{-}(G^{-}(V))= (G \circ F)^{-}(V)$ is an m_X −open in X ,therefore $G \circ F$ is l.a. $T\ell$-m-continuous. ■

In the next theorem we have the same result for upper almost $T\ell$-m continuous multifunction ,between minimal space (X, m_X), topological space (Y, τ), and operator topological space(Z, σ, T).

3.13 Theorem :

Let $F: (X, m_X) \longrightarrow (Y, \tau)$ and $G: (Y, \tau) \longrightarrow (Z, \sigma, T)$ be a multifunctions then, if F is u. m-continuous and Gu.a. $T\ell$continuous then $G \circ F$ is u.a. $T\ell$-m-continuous.

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