On n - Regular Rings

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ABSTRACT

As a generalization of *regular rings*, that is a ring is called *n-regular* if $a \in aRa$ for all $a \in N(R)$. In this paper, we first give various properties of *n-regular* rings. Also, we study the relation between such rings and *reduced rings* by adding some types of rings, such as *NCI rings*, and other types of rings.

Keywords: *n-regular rings, N flat modules, reduced rings, nil-injective*

حول الحلقات المنتظمة من النمط – n

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الملخص

كتعميم للحلقات المنتظمة ، تلك الحلقات التي تسمى الحلقات المنتظمة من النمط n وهي لكل $a \in N(R)$ فأن $a \in aRa$. فقد تم في هذا البحث أعطاء خواص متنوعة للحلقات المنتظمة من النمط n وكذلك درسنا العلاقة بين تلك الحلقات والحلقات المختزلة بإضافة بعض أنواع الحلقات ومنها مثلاً الحلقات من النمط *NCI* وأنواع أخرى من الحلقات.

الكلمات المفتاحية: منتظمة من النمط n, مسطح من النمط n , منتظمة , غامرة من النمط n i l

1. Introduction

Throughout this paper *R* is associative ring with identity and all modules are unitary. For a subset *X* of R, the left(right) annihilator of *X* in *R* is denoted by *l(X)(r(X))*. If *X={a}*, we usually abbreviate it to *l(a)(r(a))*. We write *J(R),Z(R)(Y(R)),N(R)*, for the Jacobson radical, the left (right) singular ideal, the set of nilpotent elements respectively.

A ring *R* is called zero commutative (briefly *ZC*) if for $a, b \in R$, $ab = 0$ implies $ba = 0$ [5]. A ring *R* is called *ZI* [5], if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. Every *ZC* ring is *ZI* [5]. A ring R is called *reduced* if $N(R) = 0$ [7], or equivalently, $a^2=0$ implies $a=0$ in R for all $a \in R$.

In [2], we see the three following condition.

C1: Every non zero right ideal is essential in a direct summand.

C3: If $eR \cap fR=0$ where *e* and *f* are idempotent in *R* then $eR \oplus fR$ is a direct summand of *R*.

A ring *R* is called a right *CS-ring* if it satisfies *C1* and *R* is called *Quasi-Continuous* if it satisfies *C1* and *C3* [2].

2. n - Regular Ring

This section is devoted to give the definition of n-regular rings with some of its characterizations and basic properties.

A ring R is called (Von Neumann) *regular* if for any $a \in R$ there exists $b \in R$ such that $a=aba$. The concept of regular rings has been studied extensively.

As a generalization of this concept, Wei and Chen in [8] introduced n-regular rings, a ring R is called *n-regular* if for every $a \in N(R)$, $a \in aRa$.

Examples:

1- Every regular ring is n-regular.
2- Every reduced ring is n-regular.
3- The ring Z_6 of integers modulo 6, is n-regular.
4- Let $R=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;/ a,b,c,d \in Z_2\right\}$ is a ring with identity

   $N(R)=\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$, R is *n-regular* which is not reduced.
5- The ring Z of integer number is *n-regular* ring but not Von Neumann regular.

According to Wei and Chen [9], a right R-module M is called ***N flat*** if for any $a \in N(R)$, the mapping $1_M \otimes i: M \otimes_R Ra \rightarrow M \otimes_R R$ is monic, where $i : Ra \rightarrow R$ is the inclusion mapping .

Clearly, flat modules are *N flat*. By definition, we know that every module over any reduced ring is *N flat*. Since there exists a reduced ring R which is not von Neumann regular, there exists a module over R which is not flat. So there exists a *N flat* module which is not *flat*. [9]

Lemma 2.1 [9]

The following conditions are equivalent for a ring R.

1- R is *n-regular*.
2- Every right R-module is *N flat*.
3- Every cyclic right R-module is *N flat.*

Lemma 2.2 [7]

If L_i ($i \in I$) are right *R-module* and M is a left *R-module*, then there is a natural isomorphism

$$(\bigoplus_I L_i) \otimes_R M \cong \bigoplus_I (L_i \otimes_R M).$$

The following results give characterizes *n-regular* rings.

Proposition 2.3

R is *n-regular* ring if and only if for all $a \in N(R)$, Ra is a direct summand in R.

Proof:

Assume that Ra is a direct summand in R for all $a \in N(R)$. Let M_R be a right R-module. Then there exists a left ideal K such that R$a \oplus K=R$

$1_M \otimes i : M \otimes_R Ra \rightarrow M \otimes_R R$

$1_M \otimes i : M \otimes_R Ra \rightarrow M \otimes_R (Ra \oplus K)$

$1_M \otimes i : M \otimes_R Ra \rightarrow (M \otimes_R Ra) \oplus (M \otimes_R K)$ (Lemma 2.2) since $1_M \otimes i(m \otimes ra) = m \otimes ra$, then clearly that $1_M \otimes i$ is a monic , then M_R is a N flat right R-module. From Lemma 2.1, we get that R is n-regular.

Conversely, Since R is n-regular, then there exists $b \in R$ such that $a=aba$. Put $e=ba$ so $e^2=baba=ba=e$, so $a=ae$. Let $x \in Ra$, then there exists $r \in R$ such that $x=ra=rae \in Re$, so $Ra \subseteq Re$. Let $y \in Re$ then there exists $s \in R$ such that $y=se$. Since $e=ba$, so $y=se=sba \in Ra$, $Re \subseteq Ra$ then $Ra=Re$, $Re \oplus R(1-e)=R$, $Ra \oplus R(1-e)=R$. Therefore Ra is a direct summand. ■

Lemma 2.4 [10]

For a ring R, if $Y(R) \neq 0$. Then there exists $0 \neq y \in Y(R)$ such that $y^2=0$.

Theorem 2.5

Let R be *n-regular* ring. Then R is non singular.

Proof:

Let R be *n-regular* ring and $Y(R) \neq 0$. By (Lemma 2.4), there exist a non zero element $y \in Y(R)$ such that $y^2=0$ implies that $y \in N(R)$. Since R is *n-regular* then there exists $0 \neq x \in R$ such that $y=yxy$. Let $r(y) \cap xyR=0$. If not, there exist $0 \neq z \in r(y) \cap xyR$, then $yz=0$ and $z=xyr$ for some $r \in R$, implies $yz=yxyr=yr=0$ $(y=yxy)$, but $z=xyr=x0=0$ then $z=0$. So $r(y) \cap xyR=0$. But $r(y)$ is essential. Therefore $xyR=0$ implies $xy=0$. Since $y=yxy$, so $y=0$. Therefore R a right nonsingular. Similarly prove that R is left non singular, so we get that R is non singulars. ■

3. The Connection Between N-Regular and Other Rings

Proposition 3.1

The Center of any n-regular ring is reduced.

Proof:

Let $a \in$ Cent(R) such that $a^2=0$, since R is n-regular then there exists $b \in R$ such that a=aba, since $a \in$ Cent(R), $a=a^2b=0b=0$. Therefore Center R is reduced. ■

Corollary 3.2

Let R be commutative ring. Then R is reduced if and only if R is n-regular.

In [8] the following result is proved.

Lemma 3.3

If R is *n-regular* ring then $N(R) \cap J(R)=0$.

Lemma 3.4 [4]

Every one sided or two sided nil ideal of R is contained in $J(R)$.

Theorem 3.5

Let $N(R)$ be an ideal of R. Then R is *strongly regular* ring if and only if R is *n-regular* and $R/N(R)$ is *regular*.

Proof:

Let R be a strongly regular ring. Then R is reduced and regular. Therefore $R/N(R)$ is regular and R is n-regular.

Conversely, assume that $R/N(R)$ is regular and R is n-regular. Since $N(R)$ is an ideal then by Lemma 3.4, $N(R)\subseteq J(R)$. But R is n-regular then $N(R)\cap J(R)=0$, (Lemma 3.3). So $N(R)\cap J(R)= N(R)=0$, since R/N(R) is regular ring, R/N(R)$\cong$R/{0}=R. Therefore R is regular, since R is reduced, hence R is strongly regular ring. ■

In general n-regular ring is not reduced. The following result gives the relation between n-regular and reduced ring.

Following [3], a ring R is called *NCI* provided that $N(R)$ contains a non zero ideal of R whenever $N(R)\neq 0$.

Theorem 3.6

Let R be *NCI ring*. Then R is *n-regular* if and only if R is *reduced*.

Proof:

Let R be *n-regular* ring and assume $N(R) \neq 0$. Since R is *NCI*, then R contains a non zero nil ideal, say I. Take $0\neq a\in I$. Since R is *n-regular*, there exists $b\in R$ such that $a=aba$. Since I is a right nil ideal and $ab\in I$ then there exists appositive integer n, such that $(ab)^n=0$. Then $a=aba=ababa=\ldots$ consequently we have $0\neq a=aba=\ldots=(ab)^n a=0$, which is a contradiction. Therefore R is reduced.

Conversely, it is clear. ■

Following [6], a ring R is called *Weakly Reversible* if and only if for all $a,b,r\in R$ such that $ab=0$, $Rbra$ is a nil left ideal of R (equivalently $braR$ is nil right ideal of R). Clearly *ZI* ring are weakly reversible [6].

Theorem 3.7

Let R be a *n-regular* ring. If R satisfies one if the following conditions, then R is *reduced*.

1- R is *weakly reversible*.
2- aR is an ideal for all $a\in N(R)$.

Proof (1):

Let $a\in R$, such that $a^2=0$. Since R is n-regular then there exists $b\in R$, such that $a=aba$, since R is weakly reversible, then $Rara$ is nil for all $r\in R$, so $Raba$ is nil left ideal, implies that $Raba \subseteq J(R)$, by Lemma 3.4, so $a=aba \in Raba\subseteq J(R)$, $a\in J(R)\cap N(R)=0$, by Lemma 3.3, so $a=0$. Therefore R is reduced.

Proof (2):

Let $0\neq a\in R$, such that $a^2=0$. Since R is n-regular then $a=ara$, since aR is two sided, there exists $b\in R$ such that $ar=ba$, so $a=ara=ba^2=b0=0$, $a=0$. Therefore R is reduced. ■

Lemma 3.8 [5]

The following statements are equivalent:

1- R is ZI ring.
2- For each a$\in$R, l(a) (equivalently r(a)) is a two sided ideal of R.

Theorem 3.9

The following conditions are equivalent for a ring R.

1- R is *reduced*.
2- R is *n-regular* ring and *ZC*.

3- *R* is *n-regular* ring and *ZI*.
4- *R* is *n-regular* ring and *l(a)* is an ideal for all $a \in N(R)$.

Proof:

$1 \to 2 \to 3$ it is trivial. $3 \to 4$ by (Lemma 3.8)
$4 \to 1$

Let $a \in R$ satisfy $a^2=0$. since l(a) is an ideal then $l(a) \subseteq M$ where *M* is a maximal right ideal of *R*. Since *R* is n-regular, then there exists $b \in R$ such that $a=aba$ so $(1-ab)a=0$, $1-ab \in l(a) \subset M$, since $a \in l(a)$, then $ab \in M$, implies that $1 \in M$. Hence $a=0$ which is a contradiction Therefore *R* is reduced ring. ■

Definition 3.10 [8]

A right *R*-module *M* is said to be *nil*-injective, if for any $a \in N(R)$, any right *R*-homomorphism $f:aR \to M$ can be extended to $R \to M$, or equivalently $f=m.$, where $m \in M$.

The ring R is called right nil-injective if R_R is right nil-injective. Clearly a reduced ring is a right nil-injective and n-regular ring is a right nil-injective [8].

Proposition 3.11

Let *aR* be a *nil-injective* right *R-module*, for all $a \in N(R)$. Then *R* is *n-regular*.

Proof:

Let $a \in N(R)$ and $i: aR \to aR$ be the identity mapping, since aR is a nil-injective right R-module, then there exists $b \in aR$, such that i(ar)=bar for all $r \in R$, then i(a)=a and i(a)=ba. Since $b \in aR$, there exists $c \in R$, such that b=ac implies a=aca for all $a \in N(R)$. Therefore R is n-regular. ■

Lemma 3.12 [8]

The following conditions are equivalent for a ring R.
1- *R* is a *right nil-injective*.
2- $l(r(a))=Ra$ for every $a \in N(R)$.

Lemma 3.13 [8]

The following conditions are equivalent for a ring R.
1- *R* is a *n-regular* ring.
2- Every right R-module is *nil-injective*.

Lemma 3.14 [1]

Let *R* be a right *CS ring*, then $Y(R)=0$.

A ring *R* is called a right *Ikeda-Nakayama ring* (right *IN-ring*) if the left annihilator of the intersection of any two right ideals is the sum of the two left annihilators. [2]

Lemma 3.15 [2]

Every right *IN-ring* is right *quasi-continuous*.

It is clear that every reduced ring is nil-injective, the converse is not true. The following theorem gives a partial answer for the converse.

Theorem 3.16

Let R be a right *IN-ring.* Then R is *right nil-injective* ring if and only if R is *n-regular ring.*

Proof:

Let R be a *right nil-injective ring* and let Ra is a principal left ideal for a non zero $a\in N(R)$, since R is right nil-injective, (Lemma 3.12) $Ra=l(r(a))$. Since R is right *IN*, then $Y(R)=0$ (Lemma 3.15, 3.14). $r(a)$ is not an essential right ideal of R, hence $r(a)\oplus L$ is an essential right ideal for some non zero right ideal L of R. Now $l(r(a))+l(L)=l(r(a)\cap L)=l(0)=R$ while $l(r(a))\cap l(L)\subseteq l(r(a)\oplus L)=0$, so $l(r(a))\cap l(L)=0$ then $l(r(a))\oplus l(L)=R$. Since $Ra=l(r(a))$, then $Ra\oplus l(L)=R$. So Ra is a direct summand by (Proposition 2.4) R is n-regular ring.

Converse, Lemma 3.13. ■

Example:

The ring Z of integer number is *IN* and nil-injective, so it is n-regular.

Lemma 3.17 [8]

Let R be a *right nil-injective ring*. Then the following conditions are equivalent:

1- R is *a reduced*.
2- R is *right nonsingular* and *NI*.

Theorem 3.18

Let R be and *NI ring* then the following are equivalent:

1- R is *reduced*.
2- R is *n-regular*.
3- R is *right nil-injective* and *right nonsingular*.

Proof:

$1\rightarrow 2$ It is trivial.
$2\rightarrow 3$ by (Lemma 3.13) and (Theorem 2.5).
$3\rightarrow 1$ by (Lemma 3.17). ■

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