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Bifurcation Theory by Melnikov method in fast slow system

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Abstract:

The Melmkov method for smooth dynamical systems is extended to be applicable to the non smooth one for nonlinear impact systems. This paper deals with studying a new subject of a singularity perturbed ordinary differential equations system. It is studied the ways to deal with the perturbation parameter > 0. Then the bifurcation theory is applied on the last system according to singularity perturbed ODEs. In addition, sufficient conditions for the occurrence of some types of bifurcation in the solution are given, such as (Fold, Pitchfork and Transcritical Bifurcation). Depending on the proof of theories to reduce the singular perturbation ODEs. For this purpose, proof of bifurcation that occurs in singular perturbation Teorem in this kind of situations is depended on the nature and behavior of the solution at the level of each state of bifurcation.

INTRODUCTION:

In fast-slow system we study the case when $\epsilon > 0$, we have two basic scales to

formulate singularity perturbed ODEs, slow time τ and fast time *t*. [10-15]

$$\frac{dx}{dt} = \epsilon f(x, y, \epsilon) \quad , \quad (2.1)$$

$$\frac{dy}{dt} = g(x, y, \epsilon), \quad (2.2)$$

where $(x,y) \in R^m \times R^n$, $0 < \epsilon \ll 1$ is a small positive parameter known as the singular perturbation parameter, with functions $f : R^m \times R^n \times R \longrightarrow R^m$ and $g : R^m \times R^n \times R \longrightarrow R^n$ are assumed to be suciently smooth. Variables (x) are called fast variables and variables (y) are called slow variables. For fast time scale$t = \epsilon\tau$we
have:

$$(2.3) \quad \frac{dx}{d\tau} = f(x, y, \epsilon),$$

$$\frac{dy}{d\tau} = \epsilon g(x, y, \epsilon) \quad , \quad (2.4)$$

we will study the singularity perturbed ODEs (2.3,2.4) with the case when$\epsilon > 0$. The

set
$C = \{f(x, y) \in R^m \times R^{n:} f(x, y, 0) = 0$
called a critical set. If C is a submanifold of $R^m \times R^n$, then C is called a critical manifold. Homoclinic chaos is a very common phenomenon in two degree of freedom Hamiltonian systems. For those systems that are close to being integrable, its presence can be Melnikov method, This Method is completely general, and more importantly, very simple to use. As opposed to the usual Melnikov theory [12-2], which gives the Melnikov function as an integral, an idea of Feng [1990] makes it possible to write down the Melnikov function at the resonance in closed form, and even to make certain that italways has zeros. The first developed for this particular situtation in Holmesand Marsden [1982] and Robinson [1988], and then systematically extended to n-degree of freedom systems in the book by Wiggins [1988],

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MELNIKOV, S METHOD:

The Melnikov method gives an analytic tool to predict the occurrence of chaotic orbits in non-autonomous smooth nonlinear systems under periodic perturbation [12]. According to the method is possible to construct a function called" Melnikov function", and hence to predict either regular or chaotic behavior of a studied dynamical system of the form D. Zheng and Z. Weinian [2005]:

$$\dot{x} = f(x, y) + \epsilon g(x, y, t, \epsilon), \qquad (3.1)$$

we consider systems (2.3,2.4) of the form where

$$f = \begin{bmatrix} f_1(x, y, \epsilon) \\ f_2(x, y, \epsilon) \end{bmatrix},$$

$$g = \begin{bmatrix} g_1(x, y, \epsilon, t) \\ g_2(x, y, \epsilon, t) \end{bmatrix},$$

with f is a Hamiltonian vector field on R^3, g is a periodic of period Tin t which need not be Hamiltonian itself and$\epsilon \in R$. For $= 0$ the system (2.3,2.4)a Homoclinic orbit
$x = \omega^0_{(t)}, -\infty < t < \infty$. Finally, we define the melnikov function

Z

$$M(t_0) = \quad f(\omega^0(t - t_0) \wedge g(\omega^0(t - t_0)))\, dt.$$

We assume that the unpeturped system is Hamiltonian with $f_1 = \frac{\partial H}{\partial y}$ and $f_2 = -\frac{\partial H}{\partial x.}$ The research area of bifurcation theory is the syllabus that concerns the study of changes of dynamical properties, as the rules defining the dynamical system changes. Accordingly, these bifurcation points are critical points for dynamic stability analysis of nonlinear systems, which deal with local properties such as the dynamic stability of equilibrium points under small variations of parameter p. Stationary bifurcations involve changes in the mechanism of equilibrium point when created and destroyed as parameter varies [16-13]. Periodic bifurcations involve changes in the periodic solutions or limit cycle. Consider the differential equation

$x^0 = f(\tau, x)$,
where $\tau, x \in R$,
$f(\tau, x) = p - x^2$.

Equilibria occurrence if $f(\tau, x) = 0$ and lie on the curve $p - x^2$.
Thus there are no equilibria if $p < 0$, one equilibrium at 0 if $p = 0$ and two equilibria

4.3

$\mp \sqrt{p}$ if $p > 0$. If $p = 0$, then $Df_0(0) = 0$. If $p > 0$, then $Df p(\sqrt{-p}) = -2\sqrt{-p} < 0$ if $p < 0$

4.4 √ √ √

and $Df_p(-p) = 2$ $p > 0$. This means that p is a stable equilibrium (sink), and $-p$ is unstable (source). Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits. It has two types Local bifurcation and global bifurcation
, now we explain the first type [6-7-8]:

Local bifurcations [4]:

which can be analyzed through changes in the local stability properties of equilibria or periodic orbits as parameters cross through thres holds. Now we give definition for some kinds of Bifurcation used in this paper. First three kinds depending on the branch of equilibrium point.

Definition 3.1.

Fold bifurcation [3]: a unique curve equilibrium points passed through the origin point. Moreover, the curve lies entirely on one side of $p = 0$ *in the* $p - x$ *plane.*

Definition 3.2.

Trainscritical bifurcation [3]: two curves of critical points intersected at the origin in the $p - x$ *plane. Both curves existed on either side of* $p = 0$*. However, the stability of equilibrium for* $p < p_0$ *becomes unstable for* $p > p_0$ *and vice versa.*

Definition 3.3.

Pitchfork bifurcation [3]: two curves of equilibrium points intersected at the origin in the $p - x$ *plane. Only one curve* $x = 0$ *existed on both sides of* $p = 0$*.*

FOLD BIFURCATION:

In this section, we study fold bifurcation of the singularity perturbed system by applying implicit function theorem according to the conditions of Melnikov method, with a bifurcation parameter = 0 and the non-hyperbolic equilibrium points at $x = 0$,

$y = 0,\ \epsilon = 0$. Consider the fast system:

$$\dot{x} = \frac{dx}{d\tau} = f(x, y, \epsilon)$$

$$\dot{y} = \frac{dy}{d\tau} = f(x, y, \epsilon) + \epsilon g(x, y, \epsilon)$$

where $x \in R^m$ and $y \in R^n$, x is a slow variable and y is a fast variable,$0 < \epsilon \ll 1$. **Example 4.1.** *Consider the fast slow system*

$$x^{\cdot} = f(x, y, 0), \qquad (4.1)$$
$$\epsilon\dot{y} = x^2 - p$$

where $(x, y) \in R \times R$*. The critical manifold is a parabola*

$$S = \{(x, y) \in R^{2:}\ p = x^2\},$$

now we situation the origin point $(x,p) = (0,0)$ *in the Jacobian matrix then we get* $DF(0,0,0) = 0$ *since the matrix* $\frac{\partial f}{\partial x}(0,0,0) = 0$ *its mean that* $(0,0,0)$ *not regular and normally hyperbolic. but the matrix*

$$\frac{\partial^2 f}{\partial x^2}(0,0,0) = -1 \neq 0$$

which is non degeneracy condition. The slow subsystem

$$\dot{x} = f(x,p,0),$$
$$0 = x^2 - p,$$

this case, $p \in R$ *is a parameter and*

$$\dot{x} = f(x,p,0),$$
$$0 = x^2 - p,$$

is the normal form for a fold bifurcation at $\epsilon = 0$*, Alternative terms for a fold bifurcation are saddle-node bifurcation, turning point, and limit point.*

When > 0 we have the simpler form for Melnikov finction:

$$M(t_0) = \int f(\omega^0(t - t_0) \wedge g(\omega^0(t - t_0)))\, dt.$$

With the non-hyperbolic conditions

$$\frac{\partial M}{\partial x}(0,0,0,0) = 0,$$

and

$$\frac{\partial M}{\partial y}(0,0,0,0) = 0$$

Now suppose K be a set of all equilibrium points $(x_0, y_0, \epsilon_0, t_0)$ defines as follows:

$$K = \{f((x_0, y_0, \epsilon_0, t_0) \in R^m \times R^n \times R \times R : M(x_0, y_0, \epsilon_0, t_0) = 0,)\}$$

The following theorem to prove the origin point is fold bifurcation by applied Melnikov method on singular perturbation systems.

Theorem 4.1. *Consider the system*

$$\frac{dx}{dt} = \epsilon f(x, y, \epsilon)$$

$$\frac{dy}{dt} = g(x, y, \epsilon),$$

with the set of equilibrium points C. If the following condition hold:

1 $H_\epsilon \neq 0,$

2 $H_{xx} \neq 0,$

where H is Hamiltonian system then $(0,0,0,0)$ *is a saddle node bifurcation.*

Proof. By condition 1 above

$$M_\epsilon \neq 0,$$

then by implicit function theorem can be written as:

$$M(x, y, \epsilon(x, y)) = 0 \tag{4.2}$$

Differentiate (4.2) w, r, t.*x* to get:

$$M_x + (M_y \acute{y} + M_\epsilon(\epsilon_x + \epsilon_y \acute{y}) = 0.$$

Evaluating at the origin we obtain:

$$M_x(0,0,0,0) + M_\epsilon(0,0,0,0)\epsilon_x(0,0,0,0) = 0.$$

Apply the non-hyperbolic condition we
find:

$$\epsilon_x(0,0,0,0) = -H_x(0,0,0,0)H_\epsilon(0,0,0,0)^{-1} = 0$$

Now take the second derivative w.r.t. *x:*

$$M_{xx} + M_{xy}\acute{y} + M_{x\epsilon}(\epsilon_x + \epsilon_y \acute{y}) + M_y \acute{\acute{y}}[M_{yx} + M_{yy}\acute{y} + M_{yp}(\epsilon_x + \epsilon_y \acute{y}) + M_\epsilon$$
$$[\epsilon_{xx} + \epsilon_{xy}\acute{y} + \epsilon_y \acute{\acute{y}} + (\epsilon_{yx} + p_{yy}\acute{y})\acute{y}] + [p_x + p_y \acute{y}][M_{\epsilon x} + M_{py}\acute{y} + M_{\epsilon\epsilon}(\epsilon_x + \epsilon_y \acute{y})] = 0.$$

evaluating at (0,0,0,0) and substituting ϵ_x^0= 0 and implies:

$$M_{xx}(0,0,0,0) + M_y(0,0,0,0)\acute{\acute{y}} + M_\epsilon \epsilon_{xx}(0,0,0,0) + M_\epsilon \epsilon_y(0,0,0,0)\acute{\acute{y}} = 0,$$

$$H_{xx}(0,0,0,0) + (H_y(0,0,0,0) + H_\epsilon \epsilon_y(0,0,0,0))\acute{\acute{y}} H_\epsilon \epsilon_{xx}(0,0,0,0) = 0,$$

then by condition one above we can write:

$$\epsilon_{xx}(0,0,0,0) = -[H_{xx}(0,0,0,0) + (H_y(0,0,0,0) + H_\epsilon \epsilon_y(0,0,0,0))\acute{\acute{y}}$$
$$+((H_y(0,0,0,0)](H_\epsilon(0,0,0,0))^{-1}.$$

Lastly,with both condition 1 and 2 ,it is certainly $\epsilon_{xx}(0,0,0,0) \neq 0$, which ends the proof.□

TRANSCRITICAL BIFURCATION IN DAEs:

In this section applied the conditions of this kind of bifurcation on Melnikov Method. we write the function *M* given as following where the functions *U* and *V* are defined
as following:

$$h_x(x,y,\epsilon) = xU(x,y,\epsilon),$$
$$h_y(x,y,\epsilon) = xV(x,y,\epsilon),$$

Since

$$M(x,y,\epsilon) = \begin{bmatrix} h_x(x,y,\epsilon) \\ h_y(x,y,\epsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Then

, (5.1)

and

So *x* = 0 is a surface of critical points passing through the origin from side to side of *p* = 0. Hence, all what we need to do is finding another surface, in *U* = 0 or *V* = 0, passing through the origin from side to side of $p^0 = 0$, (we need to prove $\epsilon_x^0 = 0$.)

Theorem 5.1. *Consider the system*

$$\frac{dx}{dt} = \epsilon f(x,y,\epsilon)$$

$$\frac{dy}{dt} = g(x,y,\epsilon),$$

with the surface of equilibrium point C. If

1. $V_p = 0$,
2. V_{xp} (0,0,0) 6= 0, *and*
3. V_{xx} (0,0,0) 6= 0, *then* (0,0,0) *is a Transcritical point.*

Proof. Consider the function$V(x, y, \epsilon)$in(5.1) .Since M_{xp} 6= 0 when which by definition of the function V means

$$V_\epsilon(x, y, \epsilon) \neq 0.$$

Then by implicit function theorem

$$V(x, y, \epsilon) = 0,$$

which can have differentiated w, r, t, *x*.

$$V_x(x, y, \epsilon) + V_y(x, y, \epsilon)\acute{y} + V_\epsilon(x, y, \epsilon)(\epsilon_x + \epsilon_y(x, y)\acute{y}) = 0.$$

If we evaluate at the origin, and because of assumption two and three we get:

$$\epsilon^0_x(0, 0, 0) \neq 0$$

which ends the proof.

PITCHFORK BIFURCATION:

To get a pitchfork bifurcation at the origin, we need to find two surfaces of critical points passing through the origin. One from side to side of p = 0 and the other should be placed entirely in one side of*p* = 0, i.e, px (0,0,0,0) = 0 and *pxx* (0,0,0,0) = 0. The following Theorem formulate the conditions of Pitchfork bifurcation to occur.

Theorem 6.1. *Consider the system*

$$\frac{dx}{dt} = \epsilon f(x, y, \epsilon)$$

$$\frac{dy}{dt} = g(x, y, \epsilon),$$

with the surface of equilibrium point C. If

1 $V_\epsilon(0, 0, 0) = 0,$
2 V_{xp} (0,0,0) 6= 0,
3 V_{xx} (0,0,0) = 0
4 V_{xxx} (0,0,0) 6= 0, *then* (0,0,0,0) *is a pitchfork point.*

Proof. consider the function $V(x, y, \epsilon)$ (5.1) . Since V_{xp} (0,0,0,0) 6= 0 which by definition of the function *V* means

$$V_\epsilon(x, y, \epsilon) \neq 0.$$

Then by implicit function Theorem

$$V(x, y, \epsilon) = 0, \quad (6.1)$$

which can have differentiated w, r, t, *x*.

$$V_x(x, y, \epsilon) + V_y(x, y, \epsilon)\acute{y} + V_\epsilon(x, y, \epsilon)(\epsilon_x(x, y) + \epsilon_y(x, y)\acute{y}) = 0.$$

If we evaluate at the origin, we get:

$$M_{xx}(0, 0, 0) + M_{x\epsilon}(0, 0, 0)\epsilon_x(0, 0, 0) = 0.$$

By condition 2 and 3 we get:

$$\epsilon_{xx}(0, 0, 0) \neq 0$$

It remains to establish that

, to do so we differentiate (6.1) two times w, r, t, *x*.:

Vxx + *Vxyy´*+ *Vxp* (*px* + *pyx´* + *Vyy´´*+ [*Vyx* + *Vyyy´*+ *Vyp* (*px* + *pyy´*+

$$V_\epsilon[\epsilon_{xx} + \epsilon_{xy}\acute{y} + \epsilon_y\acute{\acute{y}} + (y\epsilon_x + \epsilon_{yy}\acute{y})\acute{y}] + [V_{\epsilon x} + V_{\epsilon y}\acute{y} + V_{\epsilon\epsilon}(\epsilon_x + \epsilon_y\acute{y})][\epsilon_x + \epsilon_y\acute{y}] = 0.$$

If we evaluate at the origin, then:

$$\epsilon_{xx}(0,0,0) = -[M_{xxx}(0,0,0) + (M_{xy}(0,0,0) + M_{x\epsilon}\epsilon_y(0,0,0))\acute{y}](M_{xp})(0,0,0)^{-1}.$$

Hence by condition 2 and 4 above we get:

$$\epsilon_{xx}(0,0,0) \neq 0,$$

which ends the proof.

Example 6.1. *Consider DAE system*

$$x^{\cdot} = f(x, y, p),$$
$$0 = yx - y^3 + py.$$

From this yield two curve of equilibrium point the first curve is (0,0) *exists on one side of* $p = 0$ *and the second curve is* $(0, \mp \sqrt{p})$ *exists on only one side.* $(0, \mp \sqrt{p})$ *should satisfies* $p_x(0) = 0$ *and* $p_{xx}(0) 6= 0$.

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