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ɷ −Semi-p Open Set

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Abstract

Csaszar introduced the concept of generalized topological space and a new open set in a generalized topological space called ɷ-preopen in 2002 and 2005, respectively. Definitions of ɷ-preinterior and ɷ-preclosuer were given. Successively, several studies have appeared to give many generalizations for an open set. The object of our paper is to give a new type of generalization of an open set in a generalized topological space called ɷ-semi-p-open set. We present the definition of this set with its equivalent. We give definition of ɷ-semi-p-interior and ɷ-semi-p-closure of a set and discuss their properties. Also the properties of ɷ-preinterior and ɷ-preclosuer are discussed. In addition, we give a new type of continuous function in a generalized topological space as $(ɷ_1, ɷ_2)$-semi-p-continuous function and $(ɷ_1, ɷ_2)$-semi-p-irresolute function. The relationship between them is showed. We prove that every ɷ-open (ɷ-preopen) set is an ɷ-semi-p-open set, but not conversely. Every $(ɷ_1, ɷ_2)$-semi-p-irresolute function is an $(ɷ_1, ɷ_2)$-semi-p-continuous function, but not conversely. Also we show that the union of any family of ɷ-semi-p-open sets is an ɷ-semi-p-open set, but the intersection of two ɷ-semi-p-open sets need not to be an ɷ-semi-p-open set.

Keywords: ɷ-semi-p-open , ɷ-semi-p-interior , ɷ-semi-p-closure, $(ɷ_1, ɷ_2)$-semi-p-irresolute and $(ɷ_1, ɷ_2)$-semi-p-continuous.

1.Introduction and Preliminaries

In this paper, we denote a topological space by $(Ƶ, Ϫ)$ and the closure (interior) of a subset Ħ of Ƶ by cl(Ħ)(int(Ħ)), respectively.

1. The interior of Ħ is the set $\text{int}(Ħ) = \cup\{Ш: Ш \in Ϫ \textit{ and } Ш \subseteq Ħ\}$.

2. The closure of Ƕ is the set $cl(Ƕ) = \bigcap\{Ƒ : Ƒ \in Ӽ' \textit{ and } Ƕ \subseteq Ƒ\}$ [1], where Ӽ' symbolizes the family of closed subsets of Z.

The term "preopen" was introduced for the first time in 1984 [2]. A subset A of a topological space $(Z, Ӽ)$ is called a preopen set if $A \subseteq Int(clA)$. The complement of a preopen set is called a preclosed set. The family of all preopen sets of Z is denoted by $PO(Z)$. The family of all preclosed sets of Z is denoted by $PC(Z)$. In 2000, Navalagi used "preopen" term to define a "Semi-p-open set" [3]. A subset A of a topological space $(Z, Ӽ)$ is said to be semi-p-open set if there exists a preopen set U in Z such that $U \subseteq A \subseteq$ pre-cl U. The family of all semi-p-open sets of Z is denoted by S-PO(Z). The complement of a semi-p-open set is called semi-p-closed set. The family of all semi-p-closed sets of Z is denoted by S-PC(Z). A function $f: (Z_1, Ӽ_1) \to (Z_2, Ӽ_2)$ is said to be a continuous function if the inverse image of any open set in Z_2 is an open set in Z_1 [4]. Navalagi used the term "preopen" to introduce new types of a continuous function "pre-irresolute function" and "pre-continuous function". A function $f: (Z_1, Ӽ_1) \to (Z_2, Ӽ_2)$ is called pre-irresolute(pre-continuous) function if the inverse image of any pre-open set in Z_2 is a pre-open set inZ_1 (the inverse image of any open set in Z_2 is a pre-open set Z_1). In [5], Al-Khazraji used the term of "Semi-p-open set" to define new types of continuous functions "semi-p-irresolute" and "semi-p-continuous" function. A function $f: (Z_1, Ӽ_1) \to (Z_2, Ӽ_2)$ is called a semi-p-irresolute (semi-p-continuous) function if the inverse image of any semi-p-open set in Z_2 is a semi-p-open set in Z_1(the inverse image of any open set in Z_2 is a semi-p- open set in Z_1). Let Z be a nonempty set, a collection ɷ of subsets of Z is called a generalized topology (in brief, *GT*) on Z if $\emptyset$ belongs to ɷ and the arbitrary unions of elements of ɷ is an element in ɷ, (Z, ɷ) is called generalized topological space (in brief, *GTS*) [6]. Every set in ɷ is called ɷ-open, while the complement of ɷ-open is called ɷ-closed; the family of all ɷ-closed sets is denoted by ɷ'. The union of all ɷ-open set contained in a set Ƕ is called the ɷ- interior of Ƕ and is denoted by $int_{ɷ}(Ƕ)$, whereas the intersection of all ɷ-closed set containing Ƕ is called the ɷ-closure of Ƕ and is denoted by $cl_{ɷ}(Ƕ)$[7].

2. ɷ-Pre-Open Set

Definition 2.1 [8]

In a *GTS* (Z, ɷ) by an ɷ-pre-open (in brief, $ɷ - p - o$) set, we mean a subset Ƕ of Z with $Ƕ \subseteq int_{ɷ}cl_{ɷ} Ƕ$. An ɷ-pre-closed (in brief, $ɷ - p - c$) set is the complement of an ɷ-pre-open set. The collection of all $ɷ - p - o$ ($ɷ - p - c$) subsets of Z will be denoted by ɷ-PO(Z) (ɷ-PC(Z), respectively).

Proposition 2.2

For a subset Ƕ of a (Z, ɷ) , we have $\bigcup_{\alpha\in\Lambda} int_{ɷ} cl_{ɷ} Ƕ_\alpha \subseteq int_{ɷ}cl_{ɷ} \bigcup_{\alpha\in\Lambda} Ƕ_\alpha$.

Proof:

$Ƕ_\alpha \subseteq \bigcup_{\alpha\in\Lambda} Ƕ_\alpha$, for every $\alpha \in \Lambda$, so $cl_{ɷ}Ƕ_\alpha \subseteq cl_{ɷ} \bigcup_{\alpha\in\Lambda} Ƕ_\alpha$ for every $\alpha \in \Lambda$, it follows that $int_{ɷ}cl_{ɷ}Ƕ_\alpha \subseteq int_{ɷ}cl_{ɷ} \bigcup_{\alpha\in\Lambda} Ƕ_\alpha$ $\forall \alpha \in \Lambda$.

Hence $\bigcup_{\alpha\in\Lambda} int_{ɷ} cl_{ɷ} Ƕ_\alpha \subseteq int_{ɷ}cl_{ɷ} \bigcup_{\alpha\in\Lambda} Ƕ_\alpha$.

Proposition 2.3

The union of any collection of $\omega - p - o$ sets is an $\omega - p - o$ set.

Proof:

Let $\{ H_{\alpha} : \alpha \in \Lambda \}$ be a family of $\omega - p - o$ sets, so $H_{\alpha} \subseteq int_{\omega}\, cl_{\omega}\, H_{\alpha}, \forall \alpha \in \Lambda$. Which means $\bigcup_{\alpha \in \Lambda} H_{\alpha} \subseteq \bigcup_{\alpha \in \Lambda} int_{\omega}\, cl_{\omega}\, H_{\alpha}$, but $\bigcup_{\alpha \in \Lambda} int_{\omega}\, cl_{\omega}\, H_{\alpha} \subseteq int_{\omega} cl_{\omega} \bigcup_{\alpha \in \Lambda} H_{\alpha}$ (by Proposition 2.2), therefore, we obtain $\bigcup_{\alpha \in \Lambda} H_{\alpha} \subseteq int_{\omega}\, cl_{\omega} \bigcup_{\alpha \in \Lambda} H_{\alpha}$, hence $\bigcup_{\alpha \in \Lambda} H_{\alpha}$ is an $\omega - p - o$ set .

Corollary 2.4

The intersection of any collection of $\omega - p - c$ sets is an $\omega - p - c$ set.

Definition 2.5: [6]

Let (Z, ω) be a GTS, and H be a subset of Z

1. The union of all $\omega - p - o$ sets contained in H is called the ω-preinterior of H and denoted by pre-$int_{\omega}H$.
2. The intersection of all $\omega - p - c$ sets containing H is called the ω-preclosuer of H and denoted by pre-$cl_{\omega}H$.

Theorem 2.6

Let H and T be subsets of (Z, ω). Then, the following properties are true:

1. $H \subseteq$ pre-$cl_{\omega}H$.
2. pre-$int_{\omega}H \subseteq H$.
3. If $H \subseteq T$, then pre-$int_{\omega}H \subseteq$ pre-$int_{\omega}T$.
4. If $H \subseteq T$, then pre-$cl_{\omega}H \subseteq$ pre-$cl_{\omega}T$.

Proof:

1. From Definition of pre-$cl_{\omega}H$.
2. From Definition of pre-$int_{\omega}H$.
3. Let $H \subseteq T$, we have from 2, pre-$int_{\omega}H \subseteq H$, so $pre - int_{\omega}H \subseteq T$, but $pre - int_{\omega}T$ is the largest $\omega - p - o$ set contained in T . So $pre - int_{\omega}H \subseteq pre - int_{\omega}T$.
4. Let $H \subseteq T$, we have from 1, $T \subseteq$ pre-$cl_{\omega}T$, so $H \subseteq$ pre-$cl_{\omega}T$, but pre-$cl_{\omega}H$ is the smallest $\omega - p - c$ set containing H . So pre-$cl_{\omega}H \subseteq$ pre-$cl_{\omega}T$.

Proposition 2.7

Let (Z, ω) be a GTS let H be a subset of Z. Then:

1. H is an $\omega - p - c$ set, if and only if $H =$ pre-$cl_{\omega}H$.
2. H is an $\omega - p -$ oset, if and only if $H =$ pre-$int_{\omega}H$.

Proposition 2.8

$$\bigcup_{\alpha \in \Lambda} pre - cl_{\omega}H_{\alpha} \subseteq pre - cl_{\omega} \bigcup_{\alpha \in \Lambda} H_{\alpha}$$

Proof:

$H_\alpha \subseteq \bigcup_{\alpha\in\Lambda} H_\alpha$, for every $\alpha \in \Lambda$, so pre-cl$_{\omega}H_\alpha \subseteq$ pre-cl$_{\omega}$ $\bigcup_{\alpha\in\Lambda} H_\alpha$ for every $\alpha \in \Lambda$, therefore, $\bigcup_{\alpha\in\Lambda} \text{pre}-\text{cl}_{\omega} H_\alpha \subseteq \text{pre}-\text{cl}_{\omega} \bigcup_{\alpha\in\Lambda} H_\alpha$.

Remark 2.9

The reverse of Proposition 2.8 is not correct in general, as we show in the following example:

For example

$Z = \{a, b, c\}$, $\omega = \{Z, \emptyset, \{a, b\}\}$, and $\omega' = \{Z, \emptyset, \{c\}\}$, then:

ω-PO(Z) = $\{Z, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

ω-PC(Z) = $\{\emptyset, Z, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$,

let $H = \{b\}$ and $T = \{a\}$, so pre-cl$_{\omega}$ $H = \{b\}$ and pre-cl$_{\omega}$ $T = \{a\}$, note that $H \cup T = \{a, b\}$, and pre-cl$_{\omega}$ $(H \cup T) = Z$, while pre-cl$_{\omega}H \cup$pre-cl$_{\omega}T = \{a, b\}$.

Hence, pre-cl$_{\omega}(H \cup T) \nsubseteq$ pre-cl$_{\omega}H \cup$ pre-cl$_{\omega}T$.

Proposition: 2.10

If H is any subset of a topological space $(Z, \mathfrak{X})$, then:

1. $[\text{pre}-\text{int}\,(H)]^c = \text{pre}-\text{cl}\,(H^c)$.
2. $\text{pre}-\text{int}(H^c) = [\text{pre}-\text{cl}\,(H)]^c$.

3. ω-Semi-P-Open Set

Definition 3.1

A subset G of a GTS (Z, ω) is said to be ω-semi-p-open(in brief, $\omega-\text{sp}-\text{o}$) set if there exists an $\omega-\text{p}-\text{o}$ set H in Z such that $H \subseteq G \subseteq$ pre-cl$_{\omega}H$. Any subset of Z is called ω-semi-p-closed(in brief, $\omega-\text{sp}-\text{c}$) set if its complement is ω-semi-p-open set .The collection of all $\omega-\text{sp}-\text{o}$ subsets of Z will be denoted by ω-SPO(Z). The collection of all $\omega-\text{sp}-\text{c}$ subsets of Z will be denoted by ω-SPC(Z).

Theorem 3.2

Let (Z, ω) be a GTS and $G \subseteq Z$. Then G is an $\omega-\text{sp}-\text{oset} \Leftrightarrow G \subseteq$ pre-cl$_{\omega}$pre-int$_{\omega}G$.

Proof:

The "if" part

Assume that G is an $\omega-\text{sp}-\text{oset}$, then there exists a $\omega-\text{p}-\text{o}$ subset H of Z such that $H \subseteq G \subseteq$ pre-cl$_{\omega}H$, it follows by Theorem 2.6 (4) that pre-int$_{\omega}H \subseteq$pre-int$_{\omega}G$, but pre-int$_{\omega}H = H$, therefore $H \subseteq$ pre-int$_{\omega}G$. It follows by Theorem 2.6 (3) that pre-cl$_{\omega}H \subseteq$ pre-cl$_{\omega}$pre-int$_{\omega}G$. Now, we get $G \subseteq$pre-cl$_{\omega}H \subseteq$ pre-cl$_{\omega}$pre-int$_{\omega}G$. Thus $G \subseteq$ pre-cl$_{\omega}$pre-int$_{\omega}G$.

The "only if" part

Assume that $G \subseteq$ pre-cl$_{\omega}$pre-int$_{\omega}G$, we have to show that G is a $\omega-\text{sp}-\text{oset}$. Take pre-int$_{\omega}G = H$, then H is a $\omega-\text{p}-\text{o}$ set and $H \subseteq G \subseteq$ pre-cl$_{\omega}H$. Hence G is an $\omega-\text{sp}-\text{oset}$.

Corollary 3.3

Let (Z, ω) be a *GTS* and $F \subseteq Z$. Then Ḩ is $\omega - sp -$ cif and only if pre- int_{ω}(pre-cl_{ω}H) $\subseteq$ H.

Proof:

The "if" part

Let F be an $\omega - sp - c$ subset of Z, then pre-cl_{ω} H $=$ H (by Proposition 2.9(1)) which implies pre-int_{ω}(pre-cl_{ω}H) $\subseteq$ H, since pre-int_{ω}Ḩ $\subseteq$ Ḩ (by Theorem 2.3(2).

The "only if" part

Assume that pre- int_{ω}pre-cl_{ω}H $\subseteq$ H. We have to show Ḩ is an $\omega - sp -$ cset. Since pre-int_{ω}pre-cl_{ω}H $\subseteq$ H, then $H^c \subseteq$ [pre- int_{ω}(pre-cl_{ω} H)]^c, so we obtain from Proposition 2.10 $H^c \subseteq$ pre-cl_{ω}(pre-cl_{ω}H)^c and $H^c \subseteq$ pre- cl_{ω}pre-$int_{\omega}H^c$. Hence H^c is an $\omega - sp - o$ set by Theorem (2.2.2) which means H is an $\omega - sp - c$.

Proposition 3.4

The union of any collection of $\omega - sp - o$ sets is an $\omega - sp - o$ set.

Proof:

Let $\{G_\alpha, \alpha \in \Lambda\}$ be any family of $\omega - sp - o$ sets. Then there exists an $\omega - p - o$ set H_α for each G_α , $\alpha \in \Lambda$ such that $H_\alpha \subseteq G_\alpha \subseteq$ pre-cl_{ω} H_α, so $\bigcup_{\alpha\in\Lambda} H_\alpha \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha \subseteq \bigcup_{\alpha\in\Lambda} pre - cl_{\omega} H_\alpha$, but $\bigcup_{\alpha\in\Lambda} H_\alpha$ is an $\omega - p - o$ set by Theorem 2.3, and $\bigcup_{\alpha\in\Lambda} pre - cl_{\omega} H_\alpha \subseteq$ pre-cl_{ω} $\bigcup_{\alpha\in\Lambda} H_\alpha$ by Proposition 2.8. Now we get $\bigcup_{\alpha\in\Lambda} H_\alpha \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha \subseteq$ pre-cl_{ω} $\bigcup_{\alpha\in\Lambda} H_\alpha$. Hence $\bigcup_{\alpha\in\Lambda} G_\alpha$ is an $\omega - sp - o$ set.

Corollary 3.5

The intersection of any collection of $\omega - sp - c$ sets is an $\omega - sp - c$ set.

Proof:

Let $\{ F_\alpha : \alpha \in \Lambda \}$ be any family of $\omega - sp - c$ subsets of Z. we have to show that $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - c$ set, we know that $Z - \bigcap_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda}(Z - F_\alpha)$ (De Morgan's laws). But $\bigcup_{\alpha\in\Lambda}(Z - F_\alpha)$ is an $\omega - sp - c$ set, so $Z - \bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - o$ set. Hence $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an $\omega - sp - c$.

Remark 3.6

The intersection of two $\omega - sp - o$ sets need not to be an $\omega - sp - o$ set, as we show in the following example:

Example

Let $Z = \{a, b, c, d\}$, $\omega = \{Z, \emptyset, \{a\}, \{d\}, \{a, d\}\}$,

ω-PO(Z) $= \{Z, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$, and

ɰSPO(Z)=
{Z, Ø, {a}, {d}, {a, b}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}}. Let Ƕ = {a, b, c} and Ƭ = {b, d} , Ƕ and Ƭ are ɰ − sp − o sets, but Ƕ ∩ Ƭ = {b} which is not an ɰ − sp − o set because there is not ɰ − p − o set $V_{ɰ}$, therefore V ⊆ {b} ⊆ pre- $cl_{ɰ}V$.

Remark 3.7

If Ƕ and Ƭ are two ɰ − sp − c sets, then Ƕ ∪ Ƭ need not be ɰ − sp − c as we show in the following example:

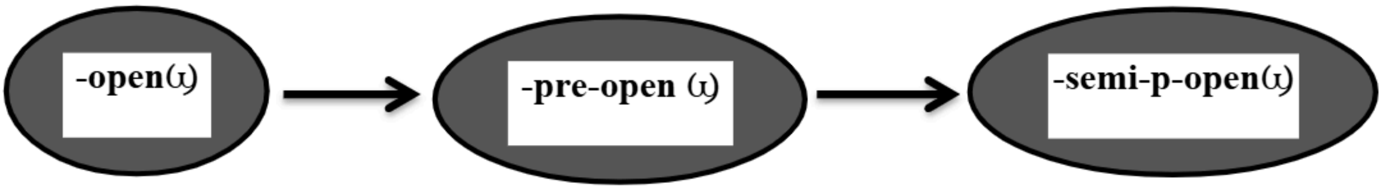
Example

From the example of Remark 3.7 Let Ƕ = {d} and Ƭ = {a, c}.

Ƕ and Ƭ are ɰ − sp − c set, but Ƕ ∪ Ƭ = {a, c, d} is not an ɰ − sp − c set because

Z - {a, c, d} = {b} is not an ɰ − sp − o set.

The following diagram illustrates the relation among ɰ-open, ɰ-pre-open, and ɰ-semi-p-open set



Definition 3.8

1. The union of all ɰ − sp − o sets contained in Ƕ is called the ɰ-semi-p-interior of Ƕ, denoted by s-p-$int_{ɰ}$(Ƕ).
2. The intersection of all ɰ − sp − c sets containing Ƕ is called the ɰ-semi-p-closure of Ƕ, denoted by s-p-$cl_{ɰ}$(Ƕ).

Proposition 3.9

Let Ƕ and Ƭ be two subsets of (Z, ɰ). Then, the following properties are true:

1. Ƕ ⊆ s − p − $cl_{ɰ}$ Ƕ.
2. If Ƕ ⊆ Ƭ, then s − p − $cl_{ɰ}$Ƕ ⊆ s − p − $cl_{ɰ}$Ƭ.
3. s − p − $cl_{ɰ}$Ƕ ∪ s − p − $cl_{ɰ}$Ƭ ⊆ s − p − $cl_{ɰ}$ (Ƕ ∪ Ƭ).
4. s − p − $cl_{ɰ}$ (Ƕ ∩ Ƭ) ⊆ s − p − $cl_{ɰ}$ Ƕ ∩ s − p − $cl_{ɰ}$ Ƭ .

Proof:

1. It is clear from Definition 3.14(2).

2. Let Ƕ ⊆ Ƭ , from (1) we have Ƭ ⊆ s − p − $cl_{ɰ}$ Ƭ, so Ƕ ⊆ s − p − $cl_{ɰ}$ Ƭ which is ɰ − sp − c set, but s − p − $cl_{ɰ}$Ƕ is the smallest ɰ − sp − c set containing Ƕ, thus s − p − $cl_{ɰ}$Ƕ ⊆ s − p − $cl_{ɰ}$Ƭ.

3. Since $Ⱨ \subseteq Ⱨ \cup Ⴀ$ and $Ⴀ \subseteq Ⱨ \cup Ⴀ$, it follows from (1) that $s-p-\mathrm{cl}_{(\omega)}Ⱨ \subseteq s-p-\mathrm{cl}_{(\omega)}(Ⱨ \cup Ⴀ)$ and $s-p-\mathrm{cl}_{(\omega)}Ⴀ \subseteq s-p-\mathrm{cl}_{(\omega)}(Ⱨ \cup Ⴀ)$, therefore $s-p-\mathrm{cl}_{(\omega)}Ⱨ \cup s-p-\mathrm{cl}_{(\omega)}Ⴀ \subseteq s-p-\mathrm{cl}_{(\omega)}(Ⱨ \cup Ⴀ)$.

4. Since $(Ⱨ \cap Ⴀ) \subseteq Ⱨ$ and $(Ⱨ \cap Ⴀ) \subseteq Ⴀ$, so semi-p-$\mathrm{cl}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq$ semi-$\mathrm{cl}_{(\omega)}Ⱨ$ and $s-p-\mathrm{cl}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{cl}_{(\omega)}Ⴀ$, thus $s-p-\mathrm{cl}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{cl}_{(\omega)}Ⱨ \cap s-p-\mathrm{cl}_{(\omega)}Ⴀ$.

Theorem 3.10

$Ⱨ$ is $(\omega)-\mathrm{sp}-\mathrm{c}$ set $\Leftrightarrow H = s-p-\mathrm{cl}_{(\omega)}H$.

Proof: Is clear.

Corollary 3.11

$s-p-\mathrm{cl}_{(\omega)}Z = Z$.

Theorem 3.12

Let $Ⱨ$ and $Ⴀ$ be two subsets of (Z, ω). Then the following properties are true:

1. $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq Ⱨ$.
2. If $Ⱨ \subseteq Ⴀ$, then $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq s-p-\mathrm{int}_{(\omega)}Ⴀ$.
3. $s-p-\mathrm{int}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{int}_{(\omega)}Ⱨ \cap s-p-\mathrm{int}_{(\omega)}Ⴀ$
4. $s-p-\mathrm{int}_{(\omega)}Ⱨ \cup s-p-\mathrm{int}_{(\omega)}Ⴀ \subseteq s-p-\mathrm{int}_{(\omega)}Ⱨ \cup Ⴀ)$.

Proof:

1. Clear.

2. Let $Ⱨ \subseteq Ⴀ$, from (1) we have $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq Ⱨ$, so $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq Ⴀ$ where $s-p-\mathrm{int}_{(\omega)}Ⱨ$ is $(\omega)-\mathrm{sp}-\mathrm{o}$ set, but $s-p-\mathrm{int}_{(\omega)}Ⴀ$ is the largest $(\omega)-\mathrm{sp}-\mathrm{o}$ set contained in $Ⴀ$, hence $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq s-p-\mathrm{int}_{(\omega)}Ⴀ$.

3. Since $(Ⱨ \cap Ⴀ) \subseteq Ⱨ$ and $(Ⱨ \cap Ⴀ) \subseteq Ⴀ$, so $s-p-\mathrm{int}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{int}_{(\omega)}Ⱨ$ and $s-p-\mathrm{int}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{int}_{(\omega)}Ⴀ$, so $s-p-\mathrm{int}_{(\omega)}(Ⱨ \cap Ⴀ) \subseteq s-p-\mathrm{int}_{(\omega)}Ⱨ \cap s-p-\mathrm{int}_{(\omega)}Ⴀ$.

4. Since $Ⱨ \subseteq Ⱨ \cup Ⴀ$ and $Ⴀ \subseteq Ⱨ \cup Ⴀ$, then $s-p-\mathrm{int}_{(\omega)}Ⱨ \subseteq s-p-\mathrm{int}_{(\omega)}(Ⱨ \cup Ⴀ)$ and $s-p-\mathrm{int}_{(\omega)}Ⴀ \subseteq s-p-\mathrm{int}_{(\omega)}(Ⱨ \cup Ⴀ)$. Thus $s-p-\mathrm{int}_{(\omega)}Ⱨ \cup s-p-\mathrm{int}_{(\omega)}Ⴀ \subseteq s-p-\mathrm{int}_{(\omega)}(Ⱨ \cup Ⴀ)$.

Theorem 3.13

H is an $(\omega)-\mathrm{sp}-\mathrm{o}$ set $\Leftrightarrow H = s-p-\mathrm{int}_{(\omega)}H$.

Proof: Is Clear.

Corollary 3.14

$s-p-\mathrm{int}_{(\omega)}\emptyset = \emptyset$

4. (ω_1, ω_2)-semi-p-continuous function

Definition 4.1:[8]
Let (Z, ω_1) and (Y, ω_2) be two GTS's. A function f: $Z \to Y$ is said to be (ω_1, ω_2)-continuous function if the inverse image of any ω_2-open subset of Y is an ω_1-open set in Z.

Definition 4.2:[9]
A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is called (ω_1, ω_2)-M- pre-open function if the direct image of any ω_1- pre-open set in Z is an ω_2- pre-open set in Y.

Definition 4.3:
A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is called (ω_1, ω_2)-M-semi-p-open ((ω_1, ω_2)-M-semi-p-closed) function if the direct image of any ω_1-semi-p-open (ω_1-semi-p-closed) set in Z is an ω_2-semi-p-open (ω_2-semi-p-closed) set in Y.

Definition 4.4

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is said to be (ω_1, ω_2)-semi-p-continuous function if the inverse image of any ω_2-open set in Y is an ω_1-semi-p-open set in Z.

Theorem 4.5

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is an (ω_1, ω_2)-semi-p-continuous function $\Leftrightarrow$ the inverse image of any ω_2-closed set in Y is an ω_1-semi-p-closed set in Z .

Proof:

The "if" part. Let F be any ω_2-closed set in Y, thus (Y− F) is an ω_2-open set in Y, then $f^{-1}(Y - F)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1, ω_2)-semi-p-continuous function), but $f^{-1}(Y - F) = Z - f^{-1}(F)$, then $f^{-1}(F)$ is an ω_1-semi-p-closed set.

The "only if" part. Let H be any ω_2-open set in Y, thus (Y− H) is an ω_2-closed set in Y, then $f^{-1}(Y - H)$ is an ω_1-semi-p-closed set in Z (by hypothesis) but $f^{-1}(Y - H) = Z - f^{-1}(H)$, then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z , therefore f is an (ω_1, ω_2)-semi-p-continuous function.

Definition 4.6

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is said to be (ω_1, ω_2)-semi-p-irresolute function if the inverse image of any ω_2-semi-p-open set in Y is an ω_1-semi-p-open set in Z

Theorem 4.7

A function $f: (Z, \omega_1) \to (Y, \omega_2)$ is an (ω_1, ω_2)-semi-p-irresolute function $\Leftrightarrow$the inverse image of each ω_2-semi-p-closed set in Y is an ω_1-semi-p-closed set in Z .

Proof:

The "if" part. Let F be any ω_2-semi-p-closed set in Y, thus (Y− F) is an ω_2-semi-p-open set in Y, then $f^{-1}(Y-F)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1,ω_2)-semi-p-irresolute function) , but $f^{-1}(Y-F) = Z - f^{-1}(F)$, therefore $f^{-1}(F)$ is an ω_1-semi-p-closed set.

The "only if" part . Let H be any ω_2-semi-p-open set in Y, thus (Y− H) is an ω_1-semi-p-closed set in Y then $f^{-1}(Y-H)$ is an ω_1-semi-p-closed set in Z (by hypothesis) , but $f^{-1}(Y-H) = Z - f^{-1}(H)$, then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z, therefore f is an (ω_1,ω_2)-semi-p-irresolute function.

Proposition 4.8

Every (ω_1,ω_2)-semi-p-irresolute function is an (ω_1,ω_2)-semi-p-continuous function.

Proof:

Let f be any (ω_1,ω_2)-semi-p-irresolute function from (Z,ω_1) into (Y,ω_2) . Let H by any ω_2-open in Y , thus H is an ω_2-semi-p- open set (Corollary 3.11), then $f^{-1}(H)$ is an ω_1-semi-p-open set in Z(since f is (ω_1,ω_2)-semi-p-irresolute function), therefore f is an (ω_1,ω_2)-semi-p-continuous function.

Remark 4.9

The reverse of Proposition 4.7 is not correct in general as we show in the following example:

Example

Let $Z = \{1,2,3,4\}$, $\omega_1 = \{Z,\emptyset,\{1\},\{4\},\{1,4\}\}$,

$\omega_1 - PO(Z) = \{Z,\emptyset,\{1\},\{4\},\{1,4\},\{1,2,4\},\{1,3,4\}\}$, and

$\omega_1 - SPO(Z) = \omega_1 - PO(Z) \cup \{\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},\}$.

Let $Y = \{a,b,c,d\}$, $\omega_2 = \{\emptyset,\{b,d\}\}$, $\omega_2 - PO(Y) = \{\emptyset,\{b,d\},\{b\},\{d\}\}$,

$\omega_2 - SPO(Y) = \mathbb{P}(Y)$ (The power set of Y).

Define $f : (Z,\omega_1) \rightarrow (Y,\omega_2)$ such that $f(1) = f(2) = \{d\}, f(3) = \{b\}$

f is an (ω_1,ω_2)-semi-p-continuous function. But not (ω_1,ω_2)-semi-p-irresolute function, since {b} is an ω_2-semi-p-open set in Y, but $f^{-1}(\{b\}) = \{3\}$ is not an ω_1-semi-p-open set in Z.

Proposition 4.10

Every (ω_1,ω_2)-continuous function is an (ω_1,ω_2)-semi-p-continuous function.

Proof:

Let f be any (ω_1,ω_2)- continuous function from (Z,ω_1) into (Y,ω_2) . Let H by any ω_2-open in Y , it follows from Definition 4.1 that $f^{-1}(H)$ is an ω_1- open set in Z, but every ω_1- open set is an ω_1-semi-p –open. Therefore f is an (ω_1,ω_2)-semi-p-continuous function.

Remark 4.11

The reverse of Remark 4.9 is not correct in general as we show in the following example:

Example

Let $Z = \{1,2,3\}$, $\omega_1 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and $\omega_1 - PO(Z) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$,

$\omega_1 - SPO(Z) = \mathbb{P}(Z)$ (The power set of Z).

Let $Y = \{a, b, c, d\}$, $\omega_2 = \{\emptyset, \{b, d\}\}$, $\omega_2 - PO(Y) = \{\emptyset, \{b, d\}, \{b\}, \{d\}\}$,

$\omega_2 - SPO(Y) = \mathbb{P}(Y)$ (The power set of Y).

Define $f : (Z, \omega_1) \rightarrow (Y, \omega_2)$ such that $f(1) = f(2) = \{a\}, f(3) = \{b\}$,

f is an (ω_1, ω_2)-semi-p-continuous function, but it is not an (ω_1, ω_2)-continuous function, since $\{b, d\}$ is an ω_2-open set in Y, but $f^{-1}(\{b, d\}) = \{3\}$ is not an ω_1-open set in Z .

Proposition 4.12

The composition of (ω_1, ω_2)-semi-p-irresolute function and (ω_2, ω_3)-semi-p-irresolute function is an (ω_1, ω_3)-semi-p- irresolute function.

Proof

Let $f: (Z, \omega_1) \rightarrow (Y, \omega_2)$ be (ω_1, ω_2)-semi-p-irresolute function and $g : (Y, \omega_2) \rightarrow (W, \omega_3)$ be (ω_2, ω_3)-semi-p-irresolute functions, we have to show that $g \circ f : (Z, \omega_1) \rightarrow (W, \omega_3)$ is an (ω_1, ω_3)-semi-p-irresolute function. Let Ḩ be any ω_3-semi-p-open set in W, then $(g \circ f)^{-1}(H) = f^{-1} \circ g^{-1}(H) = f^{-1}(g^{-1}(H))$, but $g^{-1}(H)$ is an ω_2-semi-p-open set in Y (since g is an (ω_2, ω_3)-semi-p-irresolute function), and $f^{-1}(g^{-1}(H))$ is an ω_1-semi-p-open set in Z (since f is an (ω_1, ω_2)-semi-p-irresolute functions) , therefore $g \circ f$ is an (ω_1, ω_3)-semi-p-irresolute functions .

Remark 4.13

The composition of (ω_1, ω_2)-semi-p-continuous function and (ω_2, ω_3)-semi-p-continuous function need not to be (ω_1, ω_3)-semi-p-continuous function as we show in the following example:

Example

Let $Z = \{1, 2, 3\}$, $\omega_1 = \{Z, \emptyset, \{1, 2\}\}$,

$Y = \{a, b, c\}$, $\omega_2 = \{Y, \emptyset, \{a, b\}\}$,

$W = \{i, j, k\}$, $\omega_3 = \{W, \emptyset, \{i, k\}\}$,

ω_1-PO(Z)= $\{Z, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = \omega_1$-SPO(Z)

ω_2-PO(Y)= $\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} = \omega_2$-SPO(Y), and

ω_3-PO(W)= $\{W, \emptyset, \{i\}, \{k\}, \{i, j\}, \{i, k\}, \{j, k\}\} = \omega_3$-SPO(W)

Define $f : (Z, \omega_1) \rightarrow (Y, \omega_2)$ by $f(1) = f(3) = \{b\}$, $f(2) = \{c\}$.

And $g:(Y,\omega_2) \to (W,\omega_3)$ by $g(a) = g(c) = \{j\},\ g(b) = \{k\}$.

Then $g \circ f:(Z,\omega_1) \to (W,\omega_3)$ is defined by:

$g \circ f\,(1) = g(f(1)) = g(b) = \{k\}$,

$g \circ f\,(2) = g(f(2)) = g(c) = \{j\}$,

$g \circ f\,(3) = g(f(3)) = g(b) = \{k\}$,

f is an (ω_1,ω_2)-semi-p-continuous function and g is an (ω_2,ω_3)-semi-p-continuous function. But $g \circ f$ is not an (ω_1,ω_3)-semi-p-continuous function, since $\{i,k\}$ is an ω_3-semi-p-open set in W, but $f^{-1}(\{i,k\}) = \{3\}$ is not ω_1-semi-p-open set in Z.

Proposition 4.14

The composition of an (ω_1,ω_2)-semi-p-continuous function and (ω_2,ω_3)- continuous function is an (ω_1,ω_3)-semi-p-continuous function.

Proof:
Let $f: (Z,\omega_1) \to (Y,\omega_2)$ be any (ω_1,ω_2)-semi-p-continuous function and $g:(Y,\omega_2) \to (W,\omega_3)$ be any (ω_2,ω_3)- continuous function. We have to show that $g \circ f: (Z,\omega_1) \to (W,\omega_3)$ is an (ω_1,ω_3)-semi-p- continuous function. Let H be any ω_3- open set in W. Then, $g^{-1}(H)$ is an ω_2-open set in Y (since g is an (ω_2,ω_3)-continuous function), so $f^{-1}(g^{-1}(H)$ is an ω_1-semi-p-open set in Z (since f is an (ω_1,ω_2)-semi-continuous function), but $(g \circ f)^{-1}(H) = f^{-1} \circ g^{-1}(H) = f^{-1}(g^{-1}(H)$. Hence $g \circ f$ is an (ω_1,ω_3)-semi-p-continuous function.

Theorem 4.15

Let $f: (Z,\omega_1) \to (Y,\omega_2)$ be an onto function, then f is an (ω_1,ω_2)-M-semi-p-open function if and only if it is an (ω_1,ω_2)-M-semi-p-closed function.

Proof:

The "if" part. Let F be any ω_1-semi-p-closed set, so $(Z - F)$ is an ω_1-semi-p-open set, then $f(Z - F)$ is an ω_2-semi-p-open set (since f is an (ω_1,ω_2)-M-semi-p-open function), but $f(Z - F) = Y - f(F)$, therefore $f(F)$ is an ω_2-semi-p-closed. Hence f an (ω_1,ω_2)-M-semi-p-closed function.

The "only if" part. Let H be any ω_1-semi-p-open set , so $(Z - H)$ is an ω_1-semi-p-closed set, then $f(Z - H)$ is an ω_2-semi-p-closed set (since f is an (ω_1,ω_2)-M-semi-p-closed function), but $f(Z - H) = Y - f(H)$, therefore $f(H)$ is an ω_2-semi-p-open. Hence f an (ω_1,ω_2)-M-semi-p-closed function.

Theorem 4.16

Let $f: (Z,\omega_1) \to (Y,\omega_2)$ be a bijective function, then f is an (ω_1,ω_2)-M-semi-p-open function, $\Leftrightarrow f^{-1}: (Y,\omega_2) \to (Z,\omega_1)$ is an (ω_1,ω_2)-semi-p-irresolute function.

Proof

The "if" part. Suppose that f is an (ω_1,ω_2)-M-semi-p-open function, to show that f^{-1} is an (ω_1,ω_2)-semi-p-irresolute function. Let H be any ω_1-semi-p-open set in Z, then $(f^{-1})^{-1}(H) =$

$f(H)$ is an ω_2-semi-p-open set in Y (since f is an (ω_1, ω_2)-M-semi-p-open function), so $f^{-1}$is an (ω_1, ω_2)-semi-p- irresolute function.

The "only if" part. Suppose that f^{-1} is an (ω_1, ω_2)-semi-p-irresolute function, to show that f is an (ω_1, ω_2)-M-semi-p-open function. Let H be any ω_1-semi-p-open set in Z, then $(f^{-1})^{-1}(H) = f(H)$ is an ω_2-semi-p-open set in Y(since $f^{-1}$is an (ω_1, ω_2)-semi-p-irresolute function), so f is an (ω_1, ω_2)-M-semi-p-open function.

Definition 4.17

A bijection function $f : (Z, \omega_1) \rightarrow (Y, \omega_2)$ is called (ω_1, ω_2)-semi-p-homeomorphism function if f is both (ω_1, ω_2)-semi-p-irresolute function and (ω_1, ω_2)-M-semi-p-open function.

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