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The q-Exponential Operator and the Bivariate Carlitz Polynomial with Numerical Applications

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Abstract:

In this paper, we introduce two operator roles for the q-exponential operator $E(b\theta)$, we represent the bivariate Carlitz polynomial $g_n(x,y;q)$ by the operator $E(b\theta)$ to derive the generating function, symmetry property, Mehler's formula, Rogers formula, linearization formula, the inverse linearization formula, another Rogers-type formula. Also we give an extended generating function, extended Mehler's formula extended Rogers formula, and another extended identities for the bivariate Carlitz polynomial by using the roles of the q-exponential operator $E(b\theta)$. Finally, we test the convergence conditions of all identities given in this paper and their effect numerically.

Keywords: The q-exponential operator, generating function, symmetry property, Mehler's formula, Rogers formula, linearization formula, extended generating function,

1. Introduction and Notation:

Using of operators approach to some basic hypergeometric series given in the work of Goldman and Rota [20,21], Andrews [4] and Roman [22]. In 1998 Chen and Liu [13] developed a method of deriving hypergeometric identities by parameter augmentation, this method has more realizations as in [1, 2, 3, 8, 12,14, 15, 16, 17, 23, 24, 26, 27].

In this paper, we derive some new identities of the polynomials $g_n(x,y;q)$ and give an operator proof for these identities. Let us review some common notation and terminology for basic hypergeometric series in [18]. Throughout this paper, we assume that $0<q<1$, the q −shifted factorial is defined for any real or complex variable a by:

$$(a;q)_0=1,\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\qquad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad n\in Z^+.$$

The following notation refers to the multiple q −shifted factorials:

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n\,(a_2;q)_n\ \dots\ (a_m;q)_n\,,$$
$$(a_1,a_2,\dots,a_m;q)_\infty=(a_1;q)_\infty\,(a_2;q)_\infty\ \dots\ (a_m;q)_\infty\,.$$

The q −binomial coefficients, or the Gaussian polynomials, are given by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

The basic hypergeometric series ${}_{r+1}\phi_r$ are defined by:

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, b_2, \dots, b_r; q)_n} x^n,$$

where a_i, b_j, q and x may be real or complex [25].

The Cauchy identity is defined as:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \qquad \dots (1.1)$$

Putting $a = 0$, (1.1) becomes Euler's identity:

$$\sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \qquad \dots (1.2)$$

and its inverse relation:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty, \qquad \dots (1.3)$$

The q −difference operator D_q and the q −shift operator η are given by:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a} \quad and \quad \eta\{f(a)\} = f(aq).$$

In 1998, Chen and Liu [13] constructed the operator $\theta = \eta^{-1} D_q$.
with the Leibniz formula for θ:

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\} \theta^{n-k}\{g(aq^{-k})\}.$$

Where:

$$\theta^k\{x^n\} = \frac{(q;q)_n}{(q;q)_{n-k}} q^{-nk+\binom{k}{2}+k} x^{n-k}. \qquad \dots (1.4)$$

Also, they introduced the following exponential operator:

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{\binom{k}{2}}}{(q;q)_k}, \qquad \dots (1.5)$$

and gave the following operator identities [13], where they assume that the operator acts on parameter a :

Proposition 1.1.

$$E(b\theta)\{(at;q)_\infty\} = (at, bt; q)_\infty \qquad \dots (1.6)$$

$$E(b\theta)\{(as, at; q)_\infty\} = \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty} \qquad \dots (1.7)$$

where $|abst/q| < 1$.

In 2005, Zhang and Wang [26] proved the following operator identity:

Proposition 1.2.

$$E(b\theta)\left\{\frac{(at,as,aw;q)_\infty}{(av;q)_\infty}\right\} = \frac{(at,as,aw,bs,bw;q)_\infty}{(av,absw/q\,;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} t/v,\ q/as\ ,q/aw \\ q/av,\ \ q^2/absw \end{matrix}\ ;q,q\right), \quad \dots(1.8)$$

where $\max\{|av|,|absw/q|\} < 1$.

Also in 2006, Zhang and Liu [27] derived two operator identities as:

Proposition 1.3.

$$E(b\theta)\{a^n(as;q)_\infty\} = a^n(as,bs;q)_\infty\ {}_2\phi_1\left(\begin{matrix} q^{-n},\ \ q/as \\ 0 \end{matrix}\ ;q,bs\right), \quad \dots(1.9)$$

where $|bs| < 1$.

$$E(b\theta)\{a^n(as,at;q)_\infty\} = a^n\frac{(as,at,bs,bt;q)_\infty}{(abst/q)_\infty}\ {}_3\phi_2\left(\begin{matrix} q^{-n}, q/as\ ,q/at \\ 0, q^2/abst \end{matrix}\ ;q,q\right), \quad \dots(1.10)$$

where $|abst/q| < 1$.

In this paper, we introduce two operator roles for the q-exponential operator $E(b\theta)$ and represent the bivariate Carlitz polynomial $g_n(x,y;q)$ by this operator to derive their basic and extended identities, where in Section 2, we introduce two operator roles for the q-exponential operator and derive the generating function, symmetry property, Mehler's formula, Rogers formula, linearization formula, the inverse linearization formula and another Rogers-type formula for $g_n(x,y;q)$ polynomial. In Section 3, we give five extended identities such that: extended generating function, extended Mehler's formula and extended Rogers formula. Then, in Section 4, we test numerically the convergence conditions and their effects for all given formulas.

2. New Operator Roles and the Basic Identities for $g_n(x,b;q)$.

In this section, we introduce two operator roles for the q-exponential operator and derive the generating function, symmetry property, Mehler's formula, Rogers formula, linearization formula, the inverse linearization formula and another Rogers-type formula for $g_n(x,y;q)$ polynomial.

<u>Theorem 2.1:</u> We have:

$$E(b\theta)\left\{\frac{(at,aw;q)_\infty}{(av;q)_\infty}\right\} = \frac{(at,aw,bw;q)_\infty}{(av;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} t/v,\ q/aw \\ q/av \end{matrix}\ ;q,qbw\right), \quad \dots(2.1)$$

where $\max\{\,|av|,|qbw|\} < 1$.

Proof: From identity (1.8)

$$E(b\theta)\left\{\frac{(at,as,aw;q)_\infty}{(av;q)_\infty}\right\} = \frac{(at,as,aw,bs,bw;q)_\infty}{(av,absw/q\,;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} t/v,\ q/as\ ,q/aw \\ q/av,\ \ q^2/absw \end{matrix}\ ;q,q\right)$$

$$= \frac{(at, as, aw, bs, bw; q)_\infty}{(av, absw/q\,; q)_\infty} \sum_{n=0}^{\infty} \frac{(t/v, q/as, q/aw; q)_n\, q^n}{(q/av,\ q^2/absw,\ q\,; q)_n} \qquad \ldots (2.2)$$

Since:

$$\frac{(q/as; q)_n}{(q^2/absw; q)_n} = \frac{(1-q/as)(1-q^2/as)\ldots(1-q^n/as)}{(1-q^2/absw)(1-q^3/absw)\ldots(1-q^{n+1}/absw)}$$

$$= \frac{1/(as)^n}{1/(absw)^n} \frac{(as-q)(as-q^2)\ldots(as-q^n)}{(absw-q^2)(absw-q^3)\ldots(absw-q^{n+1})}$$

$$= \frac{(bw)^n\ \ (as-q)(as-q^2)\ldots(as-q^n)}{(absw-q^2)(absw-q^3)\ldots(absw-q^{n+1})} \qquad \ldots (2.3)$$

Substitute (2.3) in (2.2), we get:

$$E(b\theta)\left\{\frac{(at, as, aw; q)_\infty}{(av; q)_\infty}\right\}$$

$$= \frac{(at, as, aw, bs, bw; q)_\infty}{(av, absw/q\,; q)_\infty} \sum_{n=0}^{\infty} \frac{(t/v, q/aw; q)_n}{(q, q/av,; q)_n} \frac{(as-q)(as-q^2)\ldots(as-q^n)\,(qbw)^n}{(absw-q^2)(absw-q^3)\ldots(absw-q^{n+1})}$$

Set $s = 0$, then:

$$E(b\theta)\left\{\frac{(at, aw; q)_\infty}{(av; q)_\infty}\right\} = \frac{(at, aw, bw; q)_\infty}{(av; q)_\infty} \sum_{n=0}^{\infty} \frac{(t/v, q/aw; q)_n}{(q, q/av; q)_n} \frac{(-1)^n q^{\binom{n}{2}}}{(-1)^n q^{\binom{n}{2}}}\ (qbw)^n$$

$$= \frac{(at, aw, bw; q)_\infty}{(av; q)_\infty} \sum_{n=0}^{\infty} \frac{(t/v, q/aw; q)_n}{(q, q/av; q)_n}\ (qbw)^n$$

$$= \frac{(at, aw, bw; q)_\infty}{(av; q)_\infty}\ {}_2\phi_1\left(\begin{matrix} t/v,\ q/aw \\ q/av \end{matrix}\ ; q, qbw\right).$$

□

Theorem 2.2: **We have:**

$$E(b\theta)\left\{\frac{(aw; q)_\infty}{(av; q)_\infty}\right\}$$

$$= \frac{(aw, bw; q)_\infty}{(av; q)_\infty}\ {}_2\phi_1\left(\begin{matrix} 0,\ q/aw \\ q/av \end{matrix}\ ; q, qbw\right), \qquad \ldots (2.4)$$

$where \max\{|av|, |qbw|\} < 1$.

Proof: By setting $t = 0$ in **Theorem (2.1)**.

□

Definition 2.3 The bivariate Carlitz polynomials is defined in [9] as follows:

$$g_n(x, y; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k\, y^{n-k}.$$

Which can be represented by the q-exponential operator based on the definition of the operator (1.5) and identity (1.4) as follows:

Theorem 2.4 Suppose that the operator $E(b\theta)$ acts on the variable x, then we have:

$$E(y\theta)\{x^n\} = g_n(x,y;q) \qquad \dots (2.5)$$

Proof:

$$E(y\theta)\{x^n\} = \sum_{k=0}^{\infty} \frac{y^k q^{\binom{k}{2}}}{(q;q)_k} \theta^k\{x^n\} = \sum_{k=0}^{\infty} \frac{y^k q^{\binom{k}{2}}}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} q^{-nk+\binom{k}{2}+k} x^{n-k}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}-nk+\binom{k}{2}+k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^{n-k} y^k .$$

Which is equal the required identity after setting $k \to n-k$.

□

According to **Theorem (2.4)** we derive the generating function for the bivariate Carlitz polynomials by using identity (1.6) of q-exponential operator to give the same results of [9], as follows:

Theorem 2.5 (The Generating Function for $g_n(x,y;q)$) We have:

$$\sum_{n=0}^{\infty} g_n(x,y;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n} = (yt,xt;q)_\infty . \qquad \dots (2.6)$$

$where\ |yt| < 1.$

Proof:

$$\sum_{n=0}^{\infty} g_n(x,y;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E(y\theta)\{x^n\} \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= E(y\theta)\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n} \right\}$$

$$= E(y\theta)\{(xt;q)_\infty\}$$

$$= (yt,xt;q)_\infty .$$

□

It is clear, the polynomials $g_n(x,y;q)$ is symmetric with x and y when we set $k \to n-k$ in **definition 2.3**, also we can prove this symmetry property by using the generating function (2.6) as follows:

Theorem 2.6 (Symmetry Property for $g_n(x,y;q)$) We have:

$$g_n(x,y;q) = g_n(y,x;q). \qquad \dots (2.7)$$

Proof: Since

$$\sum_{n=0}^{\infty} g_n(x,y;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n} = (yt,xt;q)_\infty = (xt,yt;q)_\infty$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (xt)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (yt)^k}{(q;q)_k}$$

Set $n \to n-k$, get:

$$= \sum_{n=k}^{\infty} \frac{(-1)^{n-k} q^{\binom{n-k}{2}} (xt)^{n-k}}{(q;q)_{n-k}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (yt)^k}{(q;q)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n q^{\binom{n-k}{2}+\binom{k}{2}}\, x^{n-k}\, y^k}{(q;q)_{n-k}\,(q;q)_k}\, t^n.$$

By comparing the coefficients of t^n in both sides, get:

$$g_n(x,y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^{n-k}\, y^k = g_n(y,x;q).$$

□

Now we derive Mehler's formula by using identity (1.7) of the q-exponential operator, where we can represent the polynomials $g_n(x,y;q)$ by this operator as $E_x(y\theta)\{x^n\}$ or the polynomials $g_n(z,w;q)$ as $E_z(w\theta)\{z^n\}$ or using the two representations together to get the same following result. Notice that Mehler's formula was given in another way in [9].

Theorem 2.7 (**Mehler's Formula for $g_n(x,y;q)$**) We have:

$$\sum_{n=0}^{\infty} g_n(x,y;q)\, g_n(z,w;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \frac{(xwt, xzt, ywt, yzt;q)_\infty}{(xywzt^2/q\,;q)_\infty}, \qquad \ldots (2.8)$$

where $|xywzt^2/q| < 1$.

Proof:

$$\sum_{n=0}^{\infty} g_n(x,y;q)\, g_n(z,w;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E_x(y\theta)\{x^n\}\, g_n(z,w;q) \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= E_x(y\theta)\left\{\sum_{n=0}^{\infty} g_n(z,w;q) \frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= E_x(y\theta)\{(xwt, xzt;q)_\infty\}$$

$$= \frac{(xwt, xzt, ywt, yzt;q)_\infty}{(xywzt^2/q\,;q)_\infty}.$$

□

Here, we introduce two forms of the Roger's formula depending on identity (1.7) of the q-exponential operator $E(b\theta)$.

Theorem 2.8 (**The Rogers Formula for $g_n(x,y;q)$**) We have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{n+m}(x,y;q) \frac{(-1)^n t^n}{(q;q)_n} \frac{(-1)^m s^m}{(q;q)_m}\, q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \frac{(yt, ys, xt, xs;q)_\infty}{(xyts/q;q)_\infty}, \qquad \ldots (2.9)$$

where $|xyts/q| < 1$.

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m}(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(y\theta)\{x^{n+m}\}\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$=E(y\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m (xs)^m}{(q;q)_m} q^{\binom{m}{2}}\right\}$$
$$=E(y\theta)\{(xt,xs;q)_\infty\}$$
$$=\frac{(yt,ys,xt,xs;q)_\infty}{(xyts/q;q)_\infty}.$$

In the R.H.S. of (2.9), the terms $(yt,xt;q)_\infty$ and $(ys,xs;q)_\infty$ can be written as product of generating functions for the polynomials $g_n(x,y;q)$ and $g_m(x,y;q)$ to get the following identity:

$$\frac{(yt,ys,xt,xs;q)_\infty}{(xyts/q;q)_\infty}$$
$$=\frac{1}{(xyts/q;q)_\infty}\sum_{n=0}^{\infty} g_n(x,y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{n=0}^{\infty} g_m(x,y;q)\frac{(-1)^m s^m q^{\binom{m}{2}}}{(q;q)_m}$$
$$=\frac{1}{(xyts/q;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_n(x,y;q) g_m(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$

Therefore, we get another Roger's formula as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m}(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$=\frac{1}{(xyts/q;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_n(x,y;q)\, g_m(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}} \quad \text{... (2.10)}$$

□

Now we derive the linearization formula as an applications of the Roger's formula (2.10) as follows:

Corollary 2.8.1 *For* $n,m \in N$, we have:

$$g_n(x,y;q) g_m(x,y;q)$$
$$=\sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(q;q)_k (-xy)^k q^{k(3k-2m-2n-1)/2} g_{n+m-2k}(x,y;q). \quad \text{... (2.11)}$$

Proof: From (2.10) we have:

$$(xyts/q;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m}(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_n(x,y;q)\, g_m(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m} q^{\binom{n}{2}+\binom{m}{2}}.$$

Verify the L.H.S. by using Euler's identity (1.3):

$$\sum_{k=0}^{\infty}\frac{(-xyts/q)^k\, q^{\binom{k}{2}}}{(q;q)_k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m}(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-xy/q)^k\, q^{\binom{k}{2}}}{(q;q)_k} g_{n+m}(x,y;q)\frac{(-1)^n t^{n+k}}{(q;q)_n}\frac{(-1)^m s^{m+k}}{(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}.$$

Set $n\to n-k\,, m\to m-k$, the L.H.S. equals:

$$\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\sum_{k=0}^{\infty}\frac{(-xy/q)^k\, q^{\binom{k}{2}}}{(q;q)_k} g_{n+m-2k}(x,y;q)\frac{(-1)^{n-k} t^{n}}{(q;q)_{n-k}}\frac{(-1)^{m-k} s^{m}}{(q;q)_{m-k}}q^{\binom{n-k}{2}+\binom{m-k}{2}}$$

$$\sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\sum_{k=0}^{\min\{n,m\}}\frac{(-xy/q)^k\, q^{\binom{k}{2}}}{(q;q)_k} g_{n+m-2k}(x,y;q)\frac{(-1)^{n} t^{n}}{(q;q)_{n-k}}\frac{(-1)^{m} s^{m}}{(q;q)_{m-k}}q^{\binom{n-k}{2}+\binom{m-k}{2}}.$$

By comparing the coefficients of $t^n s^m$ with the R.H.S. get:

$$\frac{g_n(x,y;q)\, g_m(x,y;q)}{(q;q)_n(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}} = \sum_{k=0}^{\min\{n,m\}}\frac{(-xy/q)^k}{(q;q)_k} g_{n+m-2k}(x,y;q)\frac{q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}}}{(q;q)_{n-k}\,(q;q)_{m-k}}.$$

Then:

$$g_n(x,y;q)g_m(x,y;q) = \sum_{k=0}^{\min\{n,m\}}\frac{(q;q)_n(q;q)_m}{(q;q)_k(q;q)_{n-k}(q;q)_{m-k}}(-xy)^k\ g_{n+m-2k}(x,y;q)\ q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-k}$$

$$= \sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(-xy)^k\, g_{n+m-2k}(x,y;q)q^{\binom{n-k}{2}+\binom{m-k}{2}+\binom{k}{2}-\binom{n}{2}-\binom{m}{2}-k}.$$

$$= \sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(-xy)^k q^{k(3k-2m-2n-1)/2}\, g_{n+m-2k}(x,y;q).$$

□

Here we give the second application of the Roger's formula, it is the inverse relation of the linearization formula for $g_n(x,y;q)$ polynomial.

Corollary 2.8.2 *For* $n,m\in N$, we have:

$$g_{n+m}(x,y;q) = \sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(xy)^k q^{k(k-m-n)}\, g_{n-k}(x,y;q)\, g_{m-k}(x,y;q) \quad \dots(2.12)$$

where $|xyts/q|<1$.

Proof: In (2.10), expand $1/(xyts/q\,;q)_\infty$ by the Euler's identity (1.2), the R.H.S. can be rewritten as:

$$\sum_{k=0}^{\infty}\frac{(xyts/q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_n(x,y;q)g_m(x,y;q)\frac{(-1)^n t^n}{(q;q)_n}\frac{(-1)^m s^m}{(q;q)_m}\ q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(xy/q)^k}{(q;q)_k} g_n(x,y;q)g_m(x,y;q)\frac{(-1)^n t^{n+k}}{(q;q)_n}\frac{(-1)^m s^{m+k}}{(q;q)_m}\ q^{\binom{n}{2}+\binom{m}{2}}.$$

Set $n\to n-k\,,\ m\to m-k$, the R.H.S. equals:

$$\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\sum_{k=0}^{\infty}\frac{(xy/q)^k}{(q;q)_k}\,g_{n-k}(x,y;q)\,g_{m-k}(x,y;q)\frac{(-1)^{n-k}t^n}{(q;q)_{n-k}}\frac{(-1)^{m-k}s^m}{(q;q)_{m-k}}\,q^{\binom{n-k}{2}+\binom{m-k}{2}}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\min\{n,m\}}\frac{(xy/q)^k}{(q;q)_k}\,g_{n-k}(x,y;q)\,g_{m-k}(x,y;q)\frac{(-1)^n t^n}{(q;q)_{n-k}}\frac{(-1)^m s^m}{(q;q)_{m-k}}\,q^{\binom{n-k}{2}+\binom{m-k}{2}}$$

By comparing the coefficients of $t^n s^m$ with the L.H.S. of (3.2), get:

$$\frac{g_{n+m}(x,y;q)}{(q;q)_n(q;q)_m}q^{\binom{n}{2}+\binom{m}{2}}=\sum_{k=0}^{\min\{n,m\}}(xy/q)^k\,g_{n-k}(x,y;q)\,g_{m-k}(x,y;q)\frac{q^{\binom{n-k}{2}+\binom{m-k}{2}}}{(q;q)_{n-k}(q;q)_{m-k}(q;q)_k}.$$

Therefore:

$$g_{n+m}(x,y;q)=\sum_{k=0}^{\min\{n,m\}}\frac{(q;q)_n(q;q)_m}{(q;q)_k(q;q)_{n-k}(q;q)_{m-k}}(xy)^k\,g_{n-k}(x,y;q)\,g_{m-k}(x,y;q)\,q^{\binom{n-k}{2}+\binom{m-k}{2}-\binom{n}{2}-\binom{m}{2}-k}$$

Hence

$$g_{n+m}(x,y;q)=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(xy)^k q^{k(k-m-n)}\,g_{n-k}(x,y;q)\,g_{m-k}(x,y;q).$$

□

In the following theorem we deriving the Rogers-type formula for the bivariate Carlitz polynomials by using identity (2.4) of the q-exponential operator $E(b\theta)$.

Theorem 2.9 **(The Rogers-Type Formula for $g_n(x,y;q)$)** We have:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}g_{n+m}(x,y;q)\frac{t^n}{(q;q)_n}\frac{(-1)^n q^{\binom{n}{2}}s^m}{(q;q)_m}=\frac{(xt,yt;q)_\infty}{(xs;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0,\ q/xt\\ q/xs\end{matrix};q,qyt\right),\ (2.13)$$

where $\max\{|xs|,|qyt|\}<1$.

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}g_{n+m}(x,y;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}(-1)^n q^{\binom{n}{2}}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E(y\theta)\{x^{n+m}\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}(-1)^n q^{\binom{n}{2}}$$

$$=E(y\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n(xt)^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\}$$

$$=E(y\theta)\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\}$$

The proof will be completed after substituting $w\to t,\ v\to s,\ b\to y$ and $a\to x$ in identity (2.4).

3. An Extended Identities for $g_n(x,y;q)$ Polynomials:

In this section, we introduce some extended identities: an extended generating function which is deriving by using the identity (1.9) of the q-exponential operator involving a ${}_2\phi_1$ sum, an extended Mehler's formula which is deriving by using the identity (1.10) of the q-exponential operator involving a ${}_3\phi_2$ sum, an extended Rogers formula which is deriving by using the identity (1.10) of the q-exponential operator involving a ${}_3\phi_2$ sum and another two extended identities for the bivariate Carlitz polynomials.

Firstly, an extended generating function identity is deriving by using the identity (1.9) of the q-exponential operator as follows:

Theorem 3.1 (Extended Generating Function for $g_n(x,y;q)$) We have:

$$\sum_{n=0}^{\infty} g_{n+k}(x,y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= x^k\,(yt,xt;q)_\infty\ {}_2\phi_1\left(\begin{matrix}q^{-k}, & q/xt\\ & 0\end{matrix};q,yt\right), \qquad \ldots(3.1)$$

$where\ |yt|<1.$

Proof:

$$\sum_{n=0}^{\infty} g_{n+k}(x,y;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty} E(y\theta)\{x^{n+k}\}\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$=E(y\theta)\left\{x^k\sum_{n=0}^{\infty}\frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$=E(y\theta)\{x^k\,(xt;q)_\infty\}$$

$$=x^k\,(yt,xt;q)_\infty\ {}_2\phi_1\left(\begin{matrix}q^{-k}, & q/xt\\ & 0\end{matrix};q,yt\right).$$

□

Secondly, we give an extended Mehler's formula by using the identity (1.10) of the q-exponential operator.

Theorem 3.2 (Extended Mehler's Formula for $g_n(x,y;q)$) We have:

$$\sum_{n=0}^{\infty} g_{n+k}(x,y;q)\,g_n(z,w;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$=\frac{x^k(xwt,xzt,ywt,yzt;q)_\infty}{(xywzt^2/q;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}q^{-k}, & q/xwt, & q/xzt\\ & q^2/xywzt^2, & 0\end{matrix};q,q\right), \qquad \ldots(3.2)$$

$where\ |xyzwt^2/q|<1.$

Proof:

$$\sum_{n=0}^{\infty} g_{n+k}(x,y;q)\,g_n(z,w;q)\frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} E_x(y\theta)\{x^{n+k}\}\, g_n(z,w;q)\, \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}$$
$$= E_x(y\theta)\left\{ x^k \sum_{n=0}^{\infty} g_n(z,w;q)\, \frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n} \right\}$$
$$= E_x(y\theta)\{x^k (xwt, xzt;q)_\infty\}$$
$$= \frac{x^k (xwt, xzt, ywt, yzt;q)_\infty}{(xywzt^2/q;q)_\infty}\; {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/xwt, & q/xzt \\ & q^2/xywzt^2, & 0 \end{matrix}; q, q \right).$$

□

Thirdly, we derive an extended Rogers formula by using the identity (1.10) of the q-exponential operator.

Theorem 3.3 **(Extended Rogers Formula for $g_n(x,y;q)$)** We have:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m+k}(x,y;q)\, \frac{t^n}{(q;q)_n}\, \frac{s^m}{(q;q)_m}\, (-1)^{n+m} q^{\binom{n}{2}+\binom{m}{2}}$$

$$= \frac{x^k (yt, ys, xt, xs;q)_\infty}{(xyts/q;q)_\infty}\; {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/xt, & q/xs \\ & q^2/xyts, & 0 \end{matrix}; q, q \right), \qquad \dots (3.3)$$

$where\ |xyts/q| < 1.$

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n+m+k}(x,y;q)\, \frac{t^n}{(q;q)_n}\, \frac{s^m}{(q;q)_m}\, (-1)^{n+m} q^{\binom{n}{2}+\binom{m}{2}}$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E(y\theta)\{x^{n+m+k}\}\, \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q;q)_n}\, \frac{(-1)^m s^m q^{\binom{m}{2}}}{(q;q)_m}$$
$$= E(y\theta)\left\{ x^k \sum_{n=0}^{\infty} \frac{(-1)^n (xt)^n q^{\binom{n}{2}}}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m (xs)^m}{(q;q)_m}\, q^{\binom{m}{2}} \right\}$$
$$= E(y\theta)\{\, x^k (xt, xs;q)_\infty\}$$
$$= \frac{x^k (yt, ys, xt, xs;q)_\infty}{(xyts/q;q)_\infty}\; {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/xt, & q/xs \\ & q^2/xyts, & 0 \end{matrix}; q, q \right).$$

□

Now we give the following extended identity with triple summations for the bivariate Carlitz polynomials by using **Theorem 2.1**.

Theorem 3.4 We have:

$$\sum_{n=o}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} g_{n+m+k}(x,y;q)\, \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\, \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m}\, \frac{v^k}{(q;q)_k}$$

$$= \frac{(xt, xs, ys;q)_\infty}{(xv;q)_\infty}\; {}_2\phi_1\left(\begin{matrix} t/v, & q/xs \\ & q/xv \end{matrix}; q, qys \right), \qquad \dots (3.4)$$

$where$ $\max\{\,|xv|, |qys|\,\} < 1.$

Proof:

$$\sum_{n=o}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} g_{n+m+k}(x,y;q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}$$
$$=\sum_{n=o}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} E(y\theta)\{x^{n+m+k}\}\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}s^m}{(q;q)_m}\frac{v^k}{(q;q)_k}$$
$$=E(y\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(xs)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(xv)^k}{(q;q)_k}\right\}$$
$$=E(y\theta)\left\{\frac{(xt,xs;q)_\infty}{(xv;q)_\infty}\right\}.$$

The required identity will be followed when we setting $a\to x$, $w\to s$ and $b\to y$ in identity (2.1). □

Here, we derive the following extended identity with four summations for the bivariate Carlitz polynomial by using identity (1.8) of the q-exponential operator.

Theorem 3.5 **We** have:

$$\sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{n+m+k+l}(x,y;q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}s^m}{(q;q)_m}\frac{(-1)^k q^{\binom{k}{2}}w^k}{(q;q)_k}\frac{v^l}{(q;q)_l}$$

$$=\frac{(xt,xs,xw,ys,yw;q)_\infty}{(xv,xysw/q\,;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}t/v,\ q/xs\ ,q/xw\\ q/xv,\ \ q^2/xysw\end{matrix}\ ;q,q\right), \qquad \dots(3.5)$$

where $\max\{\,|xv|,|xysw/q|\,\}<1$.

Proof:

$$\sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{n+m+k+l}(x,y;q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}s^m}{(q;q)_m}\frac{(-1)^k q^{\binom{k}{2}}w^k}{(q;q)_k}\frac{v^l}{(q;q)_l}$$

$$\sum_{n=0}^{\infty}\sum_{m=o}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} E(y\theta)\{x^{n+m+k+l}\}\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}\frac{(-1)^m q^{\binom{m}{2}}s^m}{(q;q)_m}\frac{(-1)^k q^{\binom{k}{2}}w^k}{(q;q)_k}\frac{v^l}{(q;q)_l}$$
$$=E(y\theta)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(xs)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(xw)^k}{(q;q)_k}\sum_{l=0}^{\infty}\frac{(xv)^l}{(q;q)_l}\right\}$$
$$=E(y\theta)\left\{\frac{(xt,xs,xw;q)_\infty}{(xv;q)_\infty}\right\}$$
$$=\frac{(xt,xs,xw,ys,yw;q)_\infty}{(xv,xysw/q;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}t/v,\ q/xs\ ,q/xw\\ q/xv,\ \ q^2/xysw\end{matrix}\ ;q,q\right). \qquad □$$

4. Numerical Applications:

In this section, we shall explain the convergence condition of the operator results (2.3) and (2.4), also for the generating function (2.6), Mehler's formula (2.8), Rogers formula (2.9), Rogers–type formula (2.13)**,** Theorem (3.4) and Theorem (3.5) for the bivariate Carlitz polynomial $g_n(x,y;q)$ by testing convergence intervals of those formulas, where we take different values for $x,y,z\,,w,t,s$ and $0<q<1$, we notice that the formulas are converge when their convergence condition less than one and undefined or infinity when the condition is greater than one. The values tables and figures below shows the numerical results .(The symbol NaN mean that the result is not a number).

Notice that there is a similarity in the convergence condition between generating function and it's extended identity, also between Mehler's formula and it's extended identity, so that between Rogers formula and it's extended identity. We use the numerical approximations of infinite values by using $(a\,;q)_\infty = \lim_{n\to\infty}(a\,;q)_n$.

a=b=w=v x=y	Theorem 2.1 $\times 10^{-6}$	Max { $\lvert av\rvert\,,\lvert qbw\rvert$ }	The generating function	$\lvert yt\rvert$
٠,١	٠,٠٠٠١	٠,٠١٠٠	٠,٩٧٧٩	٠,٠١٠٠
٠,٢	٠,٠٠٤٥	٠,٠٤٠٠	٠,٩٠٣٣	٠,٠٤٠٠
٠,٣	٠,٠٣٧٠	٠,٠٩٠٠	٠,٧٦٦٠	٠,٠٩٠٠
٠,٤	٠,١٢٩٧	٠,١٦٠٠	٠,٥٦٧٢	٠,١٦٠٠
٠,٥	٠,٢٤٦٠	٠,٢٥٠٠	٠,٣٣٣٦	٠,٢٥٠٠
٠,٦	٠,٢٤٣٠	٠,٣٦٠٠	٠,١٢٨٠	٠,٣٦٠٠
٠,٧	٠,٠٨٦٣	٠,٤٩٠٠	٠,٠١٩٩	٠,٤٩٠٠
٠,٨	٠,٠٠٢٨	٠,٦٤٠٠	٠,٠٠٠٣	٠,٦٤٠٠
٠,٩	٠,٠٠٠٠	٠,٨١٠٠	٠,٠٠٠٠	٠,٨١٠٠
١	NaN	١,٠٠٠٠	NaN	١,٠٠٠٠
١,١	NaN	١,٣٣١٠	∞	١,٢١٠٠
١,٢	NaN	١,٧٢٨٠	∞	١,٤٤٠٠
١,٣	NaN	٢,١٩٧٠	∞	١,٦٩٠٠
١,٤	NaN	٢,٧٤٤٠	∞	١,٩٦٠٠
١,٥	NaN	٣,٣٧٥٠	∞	٢,٢٥٠٠
١,٦	NaN	٤,٠٩٦٠	∞	٢,٥٦٠٠
١,٧	NaN	٤,٩١٣٠	∞	٢,٨٩٠٠
١,٨	NaN	٥,٨٣٢٠	∞	٣,٢٤٠٠
١,٩	NaN	٦,٨٥٩٠	∞	٣,٦١٠٠
٢	NaN	٨,٠٠٠٠	∞	٤,٠٠٠٠

Table 1: Theorem (2.1) of operator $E(b\theta)$ and the generating function formula (2.6)

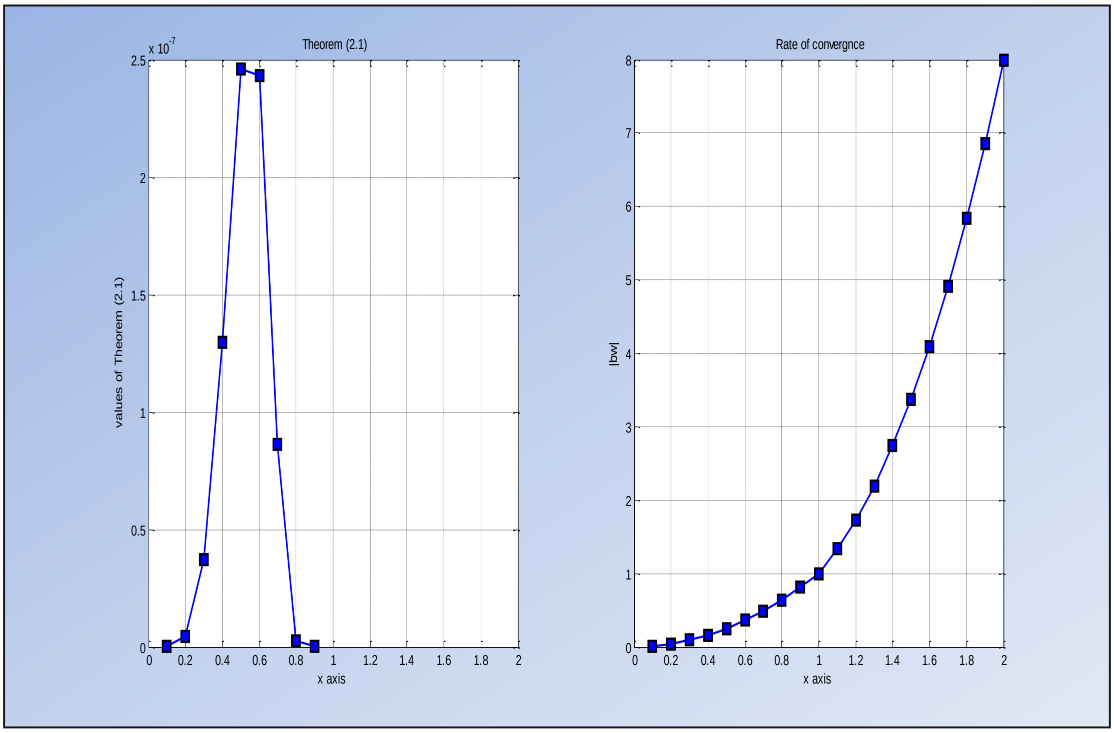

Figure 1: Rate of convergence for Theorem (2.1)

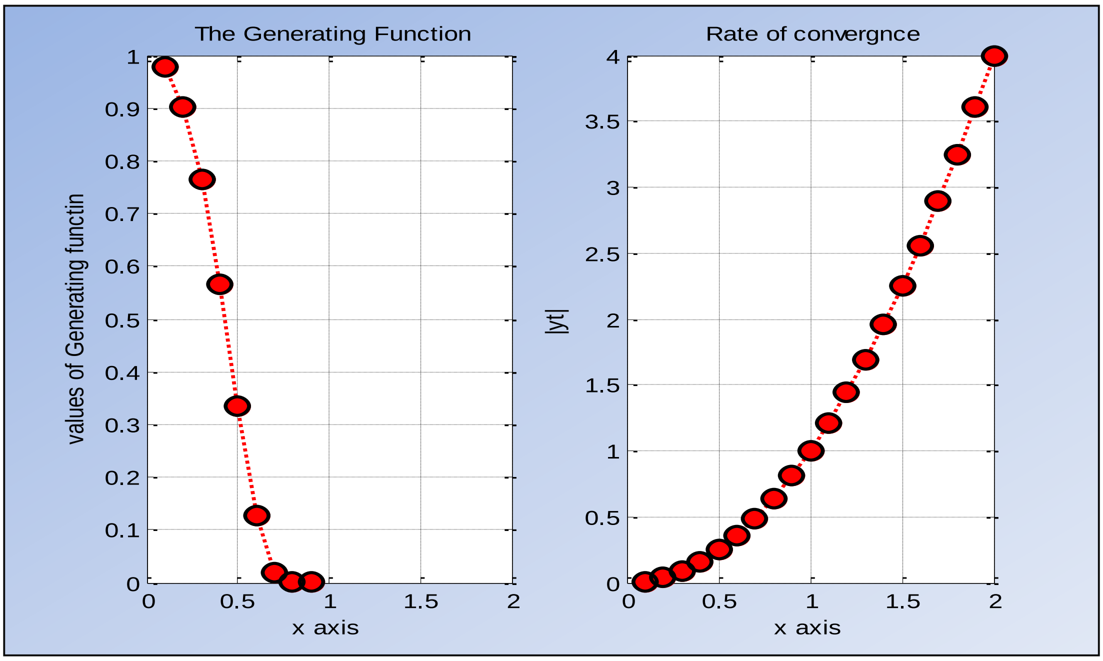

Figure 2: Rate of convergence for the generating function

x=y=w=z	Mehler's formula	$\|xywzt^2/q\|$	Rogers formula	$\|xyts/q\|$	Rogers-type formula	Max { $\|xs\|,\|qyt\|$ }
٠,١	٠,٩٩٥٦	٠,٠٠٠٠	٠,٩٥٧٤	٠,٠٠١٠	-0.0090	٠,٠١٠٠
٠,٢	٠,٩٦١٠	٠,٠٠٠٣	٠,٨٢٤٢	٠,٠٠٨٠	-0.0038	٠,٠٤٠٠
٠,٣	٠,٨٥٨٦	٠,٠٠٢٤	٠,٦١٠١	٠,٠٢٧٠	-0.0020	٠,٠٩٠٠
٠,٤	٠,٦٥٧٣	٠,٠١٠٢	٠,٣٥٨٨	٠,٠٦٤٠	-0.0011	٠,١٦٠٠
٠,٥	٠,٣٧٤٦	٠,٠٣١٣	٠,١٤٤٥	٠,١٢٥٠	-0.0006	٠,٢٥٠٠
٠,٦	٠,١١٩٢	٠,٠٧٧٨	٠,٠٢٩٣	٠,٢١٦٠	-0.0002	٠,٣٦٠٠
٠,٧	٠,٠١٠٦	٠,١٦٨١	٠,٠٠١٤	٠,٣٤٣٠	-0.0001	٠,٤٩٠٠
٠,٨	٠,٠٠٠٠	٠,٣٢٧٧	٠,٠٠٠٠	٠,٥١٢٠	-0.000	٠,٦٤٠٠
٠,٩	٠,٠٠٠٠	٠,٥٩٠٥	٠,٠٠٠٠	٠,٧٢٩٠	0.000	٠,٨١٠٠
١	NaN	١,٠٠٠٠	NaN	١,٠٠٠٠	NaN	١,٠٠٠٠
١,١	NaN	١,٦١٠٥	NaN	١,٣٣١٠	NaN	١,٣٣١٠
١,٢	NaN	٢,٤٨٨٣	NaN	١,٧٢٨٠	NaN	١,٧٢٨٠
١,٣	NaN	٣,٧١٢٩	NaN	٢,١٩٧٠	NaN	٢,١٩٧٠
١,٤	NaN	٥,٣٧٨٢	NaN	٢,٧٤٤٠	NaN	٢,٧٤٤٠
١,٥	NaN	٧,٥٩٣٨	NaN	٣,٣٧٥٠	NaN	٣,٣٧٥٠
١,٦	NaN	١٠,٤٨٥٨	NaN	٤,٠٩٦٠	NaN	٤,٠٩٦٠
١,٧	NaN	١٤,١٩٨٦	NaN	٤,٩١٣٠	NaN	٤,٩١٣٠
١,٨	NaN	١٨,٨٩٥٧	NaN	٥,٨٣٢٠	NaN	٥,٨٣٢٠
١,٩	NaN	٢٤,٧٦١٠	NaN	٦,٨٥٩٠	NaN	٦,٨٥٩٠
٢	NaN	٣٢,٠٠٠٠	NaN	٨,٠٠٠٠	NaN	٨,٠٠٠٠

Table 2 : Mehler's formula (2.8), Rogers formula (2.9) and Rogers–type (2.13).

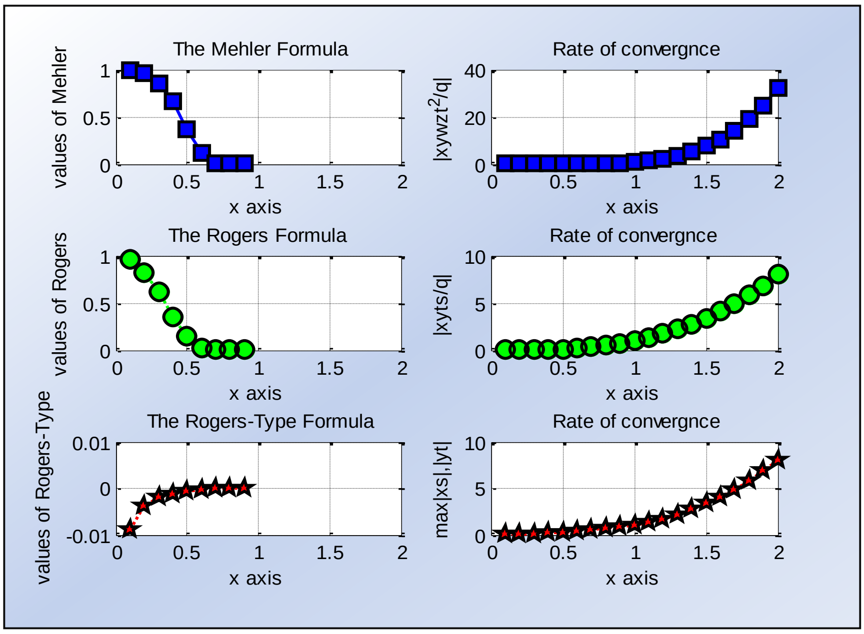

Figure 3 : Rate of convergence for Mehler's, Rogers and Rogers–type formulas

x=y=w=v	Theorem (3.4) $\times 10^{-5}$	Max $\{ \|xv\| , \|qys\| \}$	Theorem (3.5)	Max $\{ \|xv\| , \|xysw/q\| \}$
٠,١	٠,٠٠٠١	٠,٠١٠٠	١,٠٧٣٢	٠,٠١٠٠
٠,٢	٠,٠٠٤٥	٠,٠٤٠٠	٨,٥٧١٥	٠,٠٤٠٠
٠,٣	٠,٠٣٧١	٠,٠٩٠٠	٢٥,٧٨٠٨	٠,٠٩٠٠
٠,٤	٠,١٣٠٠	٠,١٦٠٠	٤٥,٠٦٥١	٠,١٦٠٠
٠,٥	٠,٢٤٦٦	٠,٢٥٠٠	٤٥,٨٤٦٠	٠,٢٥٠٠
٠,٦	٠,٢٤٣٧	٠,٣٦٠٠	١٩,٩٩٣٥	٠,٣٦٠٠
٠,٧	٠,٠٨٦٦	٠,٤٩٠٠	١,٤٧٠٤	٠,٤٩٠٠
٠,٨	٠,٠٠٢٨	٠,٦٤٠٠	٠,٠٠١٤	٠,٦٤٠٠
٠,٩	٠,٠٠٠٠	٠,٨١٠٠	٠,٠٠٠٠	٠,٨١٠٠
١	NaN	١,٠٠٠٠	NaN	١,٠٠٠٠
١,١	NaN	١,٣٣١٠	NaN	١,٣٣١٠
١,٢	NaN	١,٧٢٨٠	NaN	١,٧٢٨٠
١,٣	NaN	٢,١٩٧٠	NaN	٢,١٩٧٠
١,٤	NaN	٢,٧٤٤٠	NaN	٢,٧٤٤٠
١,٥	NaN	٣,٣٧٥٠	NaN	٣,٣٧٥٠
١,٦	NaN	٤,٠٩٦٠	NaN	٤,٠٩٦٠
١,٧	NaN	٤,٩١٣٠	NaN	٤,٩١٣٠
١,٨	NaN	٥,٨٣٢٠	NaN	٥,٨٣٢٠
١,٩	NaN	٦,٨٥٩٠	NaN	٦,٨٥٩٠
٢	NaN	٨,٠٠٠٠	NaN	٨,٠٠٠٠

Table 3: Theorem (3.4) and Theorem (3.5)

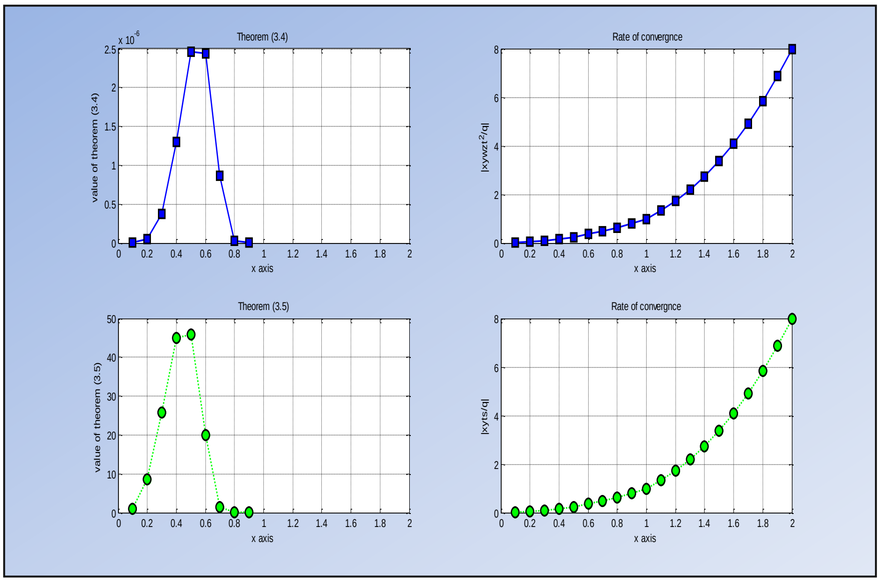

Figure 4 : Rate of convergence for Theorem (3.4) and Theorem (3.5

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