On modified approximation properties by q- analogue summation – integral type operators

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Abstract

The purpose of this paper is to introduce a summation-integral q-Beta-Szàsz operators denoted by $M_{n,v}^{q}(f(t),x)$. We use the method of Korovik-type statistical approximation to prove our operators is approximate . then, we establish a Voronovkaja-type asymptotic formula for the q-operators. Finely,we obtain an error estimate in terms of modulus of continuity being approximated.

Key words: Koroviktheorem, Voronovkaja-type asymptotic formula**,** modulus of continuity, Beta-Szàsz operators.

1. Introduction.

Due to the importance of Beta –Szasz operators a variety of their generalizations and related topics have been studied (see[5])

Then, Gupta and Yadav [6] introduced the family of summation-integral type operators q-Beta-Szàsz type operators for $q \in (0,1)$ as

$$B_n^q(f,x) = \sum_{k=0}^{\infty} p_{n,k}^q(x) q^{-k-1} \int_0^{q/(1-q^n)} s_{n,k}^q(t) f(tq^{-k-1}) dqt, x \in [0,\infty). \qquad (1.1)$$

Where

$$p_{n,k}^q(x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(1+x)_q^{n+k+1}}, s_{n,k}^q(t) = E_q(-[n]_q t) \frac{([n]_q t)^k}{[k]_q!}, \qquad (1.2)$$

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\dots(1+q^{n-1}x), & n = 1,2,\dots \\ 1, & n = 0 \end{cases}.$$

For a fixed $v \in N_0 = \{0,1,2,\dots\}$ we denote by C_B^v the set of all $f \in C_B$ having derivatives $f^{(k)} \in C_B$ such that $k = 1,2,\dots,v$. Rempulska and Walczak defined the new sequence see[3].

We use a similar idea to introduce a generalization for the q-Beta- Szàsz operators as

For $f \in C_B^v[0,\infty)$and $q \in (0,1)$, we propose the q-Beta –Szàsz operators as

$$M_{n,v}^q(f(t),x) = \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{q/(1-q^n)} s_{n,k}^q(t) q^{-k-1} \sum_{j=0}^{v} \frac{f^{(j)}(t)}{[j]!} (tq^{-k-1} - x)^j d_q t, (1.3)$$

Where $p_{n,k}^q(x)$ and $s_{n,k}^q(t)$is as defined by (1.2).

In the first we recall some notation of q-calculus, which can also be found in [1] and[2]. Throughout the present article q be a real number satisfying the inequality $0 < q < 1$. For any $n \in N \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n]_q = 1 + q + \dots + q^{n-1},$$

$$[0]_q = 0$$

And the q-factional $[n]! = [n]_q!$ by

$$[n]_q! = [1][2]\dots[n]$$

$$[0]_q! = 1.$$

For integers $0 \le k \le n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!},$$

Also, from [2] we use the following notation:

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} a^{n-j} x^j.$$

And the q-derivative $D_q f$ of a function f is given by

$$D_q\big(f(x)\big) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0.$$

The q analogue of product rule by defined as $D_q\big(f(x)g(x)\big) = g(qx)D_q\big(f(x)\big) - f(x)D_q\big(g(x)\big)$.

In this paper, we investigate the rate of convergence for the sequence $M_{n,v}^q(f(t),x)$ by the modules of continuity. We discuss Voronovskaja-type theorems for our operators for arbitrary fixed q > 0. Moreover, we establish the weighted approximation for this operators.

Lemma 1. [6] for $0 < q < 1, x \in [0,\infty)$, the following equalities are true:

1) $B_n^q(1,x) = 1,$ 2) $B_n^q(t,x) = x\left(1 + \frac{1}{q[n]_q}\right) + \frac{1}{[n]_q},$

3) $$B_n^q(t^2,x) = \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^2 + \frac{[n+1]_q}{q^2[n]_q^2}(1+2q+q^2)x + \frac{[2]_q}{[n]_q^2}.$$

Lemma 2. [6] for all $x \in [0,\infty), n \in N$ and $q \in (0,1)$, the moment of the operators $B_n^q(f,x)$ are given by

$$B_n^q(t-x,x) = \frac{x}{q[n]_q} + \frac{1}{[n]_q},$$

$$B_n^q((t-x)^2,x) = \frac{q[n]_q + [2]_q}{q^3[n]_q^2}x^2 + \frac{q(1+q^2)[n]_q + (1+q)^2}{q^2[n]_q^2}x + \frac{[2]_q}{[n]_q^2}.$$

Lemma 3.[6] for $q \in (0,1),\ x \in [0,\infty)$, the following identity is true

$$1)\ qx(1+x)D_q p_{n,k}^q(x) = \left(\frac{[k]}{q^{k-1}[n+1]_q} - qx\right)[n+1]_q p_{n,k}^q(qx).$$

$$2)\ tD_q\left(s_{n,k}^q\left(\frac{t}{q}\right)\right) = \left([k]_q - \frac{[n]_q t}{q}\right) q^{-k} s_{n,k}^q(t)$$

Lemma 4.if we define the m-th order moment of the operators (1.1) as

$$T_{n,m}(x) = B_n^q(t^m,x) = \sum_{k=0}^{\infty} p_{n,k}^q(x) q^{-k-1} \int_0^{q/(1-q^n)} s_{n,k}^q(t)\left(\frac{t}{q^{k+1}} - x\right)^m dqt,$$

then we have,

$$[n]T_{n,m+1}(qx) = x(1+x)D_q T_{n,m}(x) + ([n+1]-[n])xT_{n,m}(qx) + [m]x(1+x)T_{n,m-1}(x) + [m+1]T_{n,m}(qx).$$

Proof. By using Lemma 3, we get

$D_q\left(T_{n,m}(x)\right)$

$= -[m]\sum_{k=o}^{\infty} p_{n,k}^{q}(x)q^{-k-1}\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)\left(\frac{t}{q^{k+1}} - x\right)^{m-1} dqt$

$+\sum_{k=0}^{\infty} D_q\left(p_{n,k}^{q}(x)\right)q^{-k-1}\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)\left(\frac{t}{q^{k+1}} - x\right)^{m} dqt,$

$qx(1+x)D_q\left(T_{n,m}(x)\right)$

$= -[m]qx(1+x)T_{n,m-1}(x)$

$+\sum_{k=0}^{\infty}\left(\frac{[k]}{q^{k-1}[n+1]_q} - qx\right)[n+1]_q p_{n,k}^{q}(qx)q^{-k-1}\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)\left(\frac{t}{q^{k+1}} - x\right)^{m} dqt,$

$= -[m]_q qx(1+x)T_{n,m-1}(x) - qx[n+1]_q T_{n,m}(qx)$

$+q\sum_{k=0}^{\infty}\left([k] - \frac{[n]_q t}{q} + \frac{[n]_q t}{q} + [n]_q x - [n]_q x\right)q^{-k}p_{n,k}^{q}(qx)q^{-k-1}\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)\left(\frac{t}{q^{k+1}} - x\right)^{m} dqt,$

$= -[m]_q qx(1+x)T_{n,m-1}(x) - qx[n+1]_q T_{n,m}(x) + q[n]_q T_{n,m+1}(qx) + [n]_q xT_{n,m}(qx)$

$+q\sum_{k=0}^{\infty} p_{n,k}^{q}(qx)q^{-k-1}\int_0^{q/(1-q^n)} tD_q\left(s_{n,k}^{q}(t)\right)\left(\frac{t}{q^{k+1}} - x\right)^{m} dqt,$

This completes the proof of recurrence relation.

Theorem 1. By applying Koroviktheorem[4] on our operators the following equalities are hold:

1)$M_{n,v}^{q}(1,x) = 1,$

2)$M_{n,v}^{q}(t,x) = x + 2\left(\frac{x}{q[n]_q} + \frac{1}{[n]_q}\right),$

3)$M_{n,v}^{q}(t^2,x) = (2+[2])\left(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^2 + \frac{[n+1]_q}{q^2[n]_q^2}(1+2q+q^2)x + \frac{[2]_q}{[n]_q^2}\right) - \left(2[2]x^2\left(1+\frac{1}{q[n]_q}\right) + \frac{2[2]x}{[n]_q}\right) + x^2.$

Proof. By using the definition of the $B_n^q(f(t),x)$ and Lemma 1, we have

$M_{n,v}^{q}(1,x) = 1$

$M_{n,v}^{q}(t,x)$

$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x)\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)q^{-k-1}\sum_{j=0}^{v}\frac{(tq^{-k-1}-x)^j}{[j]!}D_q^j(tq^{-k-1})d_q t$

$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x)\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)q^{-k-1}\left\{tq^{-k-1} + \left(tq^{-k-1} - x\right) + 0 + 0 + \cdots\right\}d_q t$

$= 2B_n^q(t,x) - xB_n^q(1,x)$

Where $B_n^q(t,x)$ is the operators defined by (1.1), then we have

$M_{n,v}^{q}(t,x) = x + 2\left(\frac{1}{q[n]_q} + \frac{1}{[n]_q}\right).$

Now,

$M_{n,v}^{q}(t^2,x)$

$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x)\int_0^{q/(1-q^n)} s_{n,k}^{q}(t)q^{-k-1}\sum_{j=0}^{v}\frac{(tq^{-k-1}-x)^j}{[j]!}D_q^j(tq^{-k-1})^2 d_q t$

$$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_{0}^{q/(1-q^n)} s_{n,k}^{q}(t) q^{-k-1} \left\{ t^2 q^{-2k-2} + [2] t q^{-k-1} \left(t q^{-k-1} - x\right)_q + \left(t q^{-k-1} - x\right)_q^2 \right\}.$$

$$= B_n^q(t^2, x)(2 + [2]) - B_n^q(t, x)(2[2]x) + B_n^q(1, x) x^2.$$

$$M_{n,v}^{q}(t^2, x) = (2 + [2]) \left(\frac{[n+1]_q [n+2]_q}{q^3 [n]_q^2} x^2 + \frac{[n+1]_q}{q^2 [n]_q^2} (1 + 2q + q^2) x + \frac{[2]_q}{[n]_q^2} \right) - \left(2[2] x^2 \left(1 + \frac{1}{q[n]_q}\right) + \frac{2[2]x}{[n]_q} \right) + x^2$$

Then we calculate that our operators $M_{n,v}^{q}$ is approximate to $f(x) \in C_B^v[0, \infty)$as $n \to \infty$.

Corollary 1. If we defined the center moment as

$$\tilde{T}_{n,m}(x) = M_{n,v}^{q}((t-x)^m, x) = \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_{0}^{q/(1-q^n)} s_{n,k}^{q}(t) q^{-k-1} \sum_{j=0}^{v} \frac{(tq^{-k-1} - x)^j}{[j]!} D_q^j (tq^{-k-1} - x)^m d_q t,$$

$x \in [0, \infty)$

Then,

$$\tilde{T}_{n,0}(x) = 1,$$

$$\tilde{T}_{n,1}(x) = 2 \left(\frac{x}{q[n]_q} + \frac{1}{[n]_q} \right),$$

$$\tilde{T}_{n,2}(x) = (2 + [2]) \left(\frac{q[n]_q + [2]_q}{q^3 [n]_q^2} x^2 + \frac{q(1+q^2)[n]_q + (1+q)^2}{q^2 [n]_q^2} x + \frac{[2]_q}{[n]_q^2} \right).$$

And for $n > m$, we have the following recurrences relation:

$$\tilde{T}_{n,m}(x) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} B_n^q\left(\left(tq^{-k-1} - x\right)^m, x\right).$$

Proof.By simple computation, we can find the central moments. Now

$$\tilde{T}_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_{0}^{q/(1-q^n)} s_{n,k}^{q}(t) q^{-k-1} \sum_{j=0}^{v} \frac{(tq^{-k-1} - x)^j}{[j]!} D_q^j (tq^{-k-1} - x)^m d_q t$$

$$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_{0}^{q/(1-q^n)} s_{n,k}^{q}(t) q^{-k-1} \sum_{j=0}^{m} \frac{(tq^{-k-1} - x)^j}{[j]!} D_q^j (tq^{-k-1} - x)^m d_q t + 0,$$

This of completes the proof Lemma.

Theorem 2. Let $f \in C_B^v[0, \infty)$ be a bounded function and q_n denote a sequence such that $0 < q_n < 1$ and $q_n = q \to 1$ as $n \to \infty$. Then we have for a point $x \in (0, \infty)$

$$\lim_{n\to\infty} [n]_q \left(M_{n,v}^{q}(f(x), x) - f(x) \right) = 2(1 + x) f'(x) + (2 + [2]) \left(\frac{x^2}{2} + x \right) f''(x). \quad (1.4)$$

proof. In order to prove this identitywe use Taylor'sexpansion on f,

$$f(t) - f(x) = (t - x) f'(x) + (t - x)^2 \left(\frac{1}{2} f''(x) + \vartheta(x; t) \right),$$

Where, ϑ is bounded and $\lim_{n\to\infty} \vartheta(t) = 0$. By applying the operators $M_{n,v}^{q}$to the above relation obtains

$$M_{n,v}^{q}(f(t), x) - f(x) = M_{n,v}^{q}\left((t - x), x\right) f'(x) + M_{n,v}^{q}((t - x)^2, x) \left(\frac{1}{2} f''(x) + \vartheta(x; t) \right)$$

$$= 2(1+x)f'(x) + 2\left(\frac{x^2}{2}+x\right)f''(x) + M_{n,v}^q(\vartheta(x;t)(t-x)^2,x).$$

Now, by using Cauchy- Schwarz inequality, we get

$$[n]_q M_{n,v}^q(\vartheta(x;t)(t-x)^2,x) \leq \left(M_{n,v}^q(\vartheta(x;t)^2,x)\right)^{1/2}\left([n]_q^2 M_{n,v}^q((t-x)^4,x)\right)^{1/2}.$$

And by application Corollary 1, we can show that

$$\lim_{n\to\infty}[n]_q^2\, M_{n,v}^q((t-x)^4,x) = 0,$$

From the above we have desired result.

Theorem 3. Let $f \in C_B^v[0,\infty)$, then for every $x \in [0,\infty)$ and for $n > 1$, we have

$$\left|M_{n,v}^q(f,x) - f(x)\right| \leq 2\omega(f,\sqrt{\delta_n}),$$

Where $\delta_n = M_{n,v}^q((t-x)^2,x)$ and ω the modulus of continuity.

Proof. By the linearity and monotonicity of $M_{n,v}^q(f,x)$, we have

$$\left|M_{n,v}^q(f(t),x) - f(x)\right| \leq M_{n,v}^q(|f(t)-f(x)|;x)$$

$$= \sum_{k=0}^{\infty} p_{n,k}^q(x)\int_0^{q/(1-q^n)} s_{n,k}^q(t)q^{-k-1}\sum_{j=0}^{v}\frac{(tq^{-k-1}-x)^j}{[j]!}D_q^j\,|f(t)-f(x)|d_qt.$$

The modulus of continuity possesses the following properties

$$\forall\lambda,\delta>0, \omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$$

$$\forall\delta>0, |f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta).$$

Then by using this properties we have:

$$\left|M_{n,v}^q(f(t),x) - f(x)\right| \leq \sum_{k=0}^{\infty} p_{n,k}^q(x)\int_0^{q/(1-q^n)} s_{n,k}^q(t)q^{-k-1}\sum_{j=0}^{v}\frac{(tq^{-k-1}-x)^j}{[j]!}D_q^j\left(\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta)\right)d_qt.$$

$$\leq \omega(f,\delta)\left(M_{n,v}^q(1;x) + \frac{1}{\delta^2}M_{n,v}^q((t-x)^2;x)\right),$$

By usingCorollary 1 and choosing $\delta_n = M_{n,v}^q((t-x)^2,x), \delta = \sqrt{\delta_n}$,we get the result.

2.Weighted approximation

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on the interval $[0,\infty)$satisfying the condition

$$|f(x) \leq \mathcal{M}_f(1+x^2),|$$

Where $\mathcal{M}_f$ is a constant depending on f.$B_{x^2}[0,\infty)$ is a normal space with the norm

$$\|f\|_{x^2} = \min_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}, f \in B_{x^2}[0,\infty).$$

$C_{x^2}[0,\infty)$ is the subspace of all continuous function in $B_{x^2}[0,\infty)$ and $C_{x^2}^*[0,\infty)$ denotes the subspace of all function $f \in C_{x^2}[0,\infty)$with $\lim_{x\to\infty}\frac{|f(x)|}{1+x^2} = K$.

Theorem 4. Let $q = q_n \in (0,1)$ such that $q_n \to 1$as $n \to \infty$, then for each $f \in C_{x^2}^*[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|M_{n,v}^q(f(x),x) - f(x)\right\|_{x^2} = 0. \qquad (2.1)$$

Proof. By theorem (1), and for$f \in C_{x^2}^*[0,\infty)$.

As,

$$\left\|M_{n,v}^q(f(x),x) - 1\right\|_{x^2} = 0. \qquad (2.2)$$

By Lemma, for $n > 1$,

$$\lim_{n\to\infty}\left\|M_{n,v}^q(t,x) - x\right\|_{x^2} = \lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|M_{n,v}^q(t,x)-x\right|}{1+x^2}$$

$$\leq \lim_{n\to\infty} \frac{\left(1+q^{-k-1}\right)}{[n]_q} + \left(\left(1+q^{-k-1}\right)\left(1+\frac{1}{q[n]_q}\right) - q^{-k-1}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2}$$

$$\leq \lim_{n\to\infty} \frac{(1+q^{-k-1})}{[n]_q} + \left(\left(1+q^{-k-1}\right)\left(1+\frac{1}{q[n]_q}\right) - q^{-k-1}\right),$$

Then we have

$$\left\|M_{n,v}^q(t,x) - x\right\|_{x^2} \to 0. \qquad (2.3\ )$$

Similarly for $n > 1$, we have

$$\lim_{n\to\infty} \left\|M_{n,v}^q(t^2,x) - x^2\right\|_{x^2} = \lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|M_{n,v}^q(t,x) - x^2\right|}{1+x^2},$$

$$\leq \lim_{n\to\infty} \left(\left(\left(1+[2]+q^{-k-1}\right)\left(\frac{[n+1]_q[n+2]_q}{q^2[n]_q^2}\right) - \left([2]+q^{-k-1}+q^{-k}\right) + q^{-k}\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} - \left(\left(1+2q+q^2\right)\left(\frac{[n+1]_q}{q^2[n]_q^2}\right) + \frac{[2]+q^{-k-1}+q^{-k}}{[n]_q}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\left(1+[2]+q^{-k-1}\right)}{q^2[n]_q^2}\right).$$

$$\leq \lim_{n\to\infty} \left(\left(\left(1+[2]+q^{-k-1}\right)\left(\frac{[n+1]_q[n+2]_q}{q^2[n]_q^2}\right) - \left([2]+q^{-k-1}+q^{-k}\right) + q^{-k}\right) - \left(\left(1+2q+q^2\right)\left(\frac{[n+1]_q}{q^2[n]_q^2}\right) + \frac{[2]+q^{-k-1}+q^{-k}}{[n]_q}\right) + \frac{\left(1+[2]+q^{-k-1}\right)}{q^2[n]_q^2}\right)$$

Now, we have

$$\left\|M_{n,v}^q(t^2,x) - x^2\right\|_{x^2} \to 0. \qquad (2.4)$$

By (2.2),(2.3),(2.4) and by Korovkin's theorem , we get the desired result.

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