On Fuzzy Normed Space

By

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Abstract

This paper studies the concepts of fuzzy normed space and introduces definitions to the convergence fuzzy normed space, fuzzy continuity. It also proves some theorems in this subject.

Keywords: fuzzy Cauchy sequence, fuzzy convergent sequence, fuzzy completeness in fuzzy normed space, fuzzy continuous function.

1. Introduction

Many mathematicians have studied fuzzy normed spaces from several angles. The concept of fuzzy norm was introduced by Katsaras in 1984. This paper introduced some theorems related to this concept as fuzzy convergence and fuzzy continuity.

2. Preliminaries

Definition(2.1):[1] A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a t-norm if $*$ is satisfies the following condition:

(i) $*$Is commutative and associative;

(ii) $a * 1 = a \; for \; all \; a \in [0,1]$;

(iii) $a * b \leq c * d \; whenever \; a \leq c \; and \; b \leq d \; forall \; a, b, c, d \in [0,1]$.

Definition (2.2): [4] Let X be a non-empty set, $*$ be a continuous t-norm on I= [0, 1].A function $N : X \times (0, \infty) \rightarrow [0,1]$ is called a fuzzy norm function on X if satisfies the following axioms for all $x, y \in X, t, s > 0$:

(i) $N(x,t) > o$;

(ii) $N(x,t) = 1 \Leftrightarrow x = 0$;

(iii) $N(\alpha x, t) = N(x, \frac{t}{|\alpha|})$;

(iv) $N(x,t) * N(y,s) \leq N(x + y, t + s)$;

(v) $N(x, .) : (0, \infty) \rightarrow [0,1]$ is continuous;

(vi) $\text{Lim}_{t\rightarrow\infty} N(x,t) = 1$.

$(X, N, *)$ is said to be a fuzzy normed space

Lemma (2.3): Let N be a fuzzy norm. Then:

(1) $N(x,.)$ is non-decreasing with respect to t for each $x \in X$.

(2) $N(-x,t) = N(x,t)$ hence $N(x-y,t) = N(y-x,t)$.

Proof: (1) Suppose $N(x,.)$ is not non-decreasing, for some $0 < s < t$,

$$N(x,t) < N(x,s) \quad \dots (1)$$

$$N(x,s) * N(0,t-s) \leq N(x,t)$$

$$N(x,s) * 1 \leq N(x,t) \text{ by axiom(ii)} \Longrightarrow N(x,s) \leq N(x,t)$$

Contradiction with (1) therefore $N(x,.)$ is non-decreasing.

(2) It is direct from axiom (iii).

Definition (2.4): [2] Let $(X, N, *)$ be a fuzzy normed space .Then:

(a) A sequence $\{x_n\}$ in X is said to fuzzy converges to x in X if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in Z^+$ such that

$N(x_n - x, t) > 1 - \varepsilon$

For all $n \geq n_0$ (Or equivalently $\lim_{n\to\infty} N(x_n - x, t) = 1$).

(b) A sequence $\{x_n\}$ in X is said to be fuzzy Cauchy if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in Z^+$ such that

$N(x_n - x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.(or equivalently $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$).

(c) A fuzzy normed space in which every fuzzy Cauchy sequence is fuzzy convergent is said to be complete. A complete fuzzy normed space is called a fuzzy Banach space.

3. Main results

Theorem (3.1): Let $\{x_n\}$, $\{y_n\}$ be a sequences in fuzzy normed space X and for all $\alpha_1 \in (0, 1)$ there exist $\alpha \in (0,1)$ such that $\alpha * \alpha \geq \alpha_1$

(1) Every sequence in X has a unique fuzzy convergence.

(2)If $x_n \to x$ then $c\,x_n \to c\,x$, $c \in F/\{0\}$(F is field).

(3)If $x_n \to x, y_n \to y$, then $x_n + y_n \to x + y$.

Proof: (1) Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ and $x_n \to y$ and $x \neq y$ then there exist $t, s > 0$ such that $\lim_{n\to\infty} N(x_n - x, s) = 1$,

$\lim_{n\to\infty} N(x_n - y, t - s) = 1$

$N(x - y, t) \geq N(x_n - x, s) * N(x_n - y, t - s)$

Taking limit:

$N(x - y, t) \geq 1 * 1 = 1$.But $N(x - y, t) \leq 1 \Rightarrow N(x - y, t) = 1$. Then by axiom (ii) $x - y = 0 \Rightarrow x = y$.

(2) Since $x_n \to x$ then if for each $\varepsilon \in (0,1)$and each t>0, there exists

$n_0 \in \mathbb{Z}^+$ such that $N(x_n - x, t) > 1-\varepsilon$ for all $n \geq n_0$ put $t = \frac{t_1}{|c|}$ such that, $t_1 > 0$, $c \in F/\{0\}$.

$$N(c\,x_n - cx, t_1) = N\left(x_n - x, \frac{t_1}{|c|}\right) = N(x_n - x, t) > 1-\varepsilon .$$

Then $c\,x_n \to c\,x$.

(3) Since $x_n \to x$ then if for all $\varepsilon \in (0,1)$and all $t > 0$, there exists

$n_1 \in Z^+$ such that $N(x_n - x, \frac{t}{2}) > 1-\varepsilon$ for all $n \geq n_1$, since $y_n \to y$ then if for all $\varepsilon \in (0,1)$and all $t > 0$, there exists $n_2 \in Z^+$ such that $N(y_n - y, \frac{t}{2}) > 1-\varepsilon$ for all $n \geq n_2$.Take $n_0 = \min\{n_1, n_2\}$, for all $\varepsilon_1 \in (0,1)$such that $(1-\varepsilon) * (1-\varepsilon) > (1-\varepsilon)$ and all $t > 0$, there exists $n_0 \in Z^+$ such that

$$N((x_n + y_n) - (x + y), t) = N((x_n - x) + (y_n - y), t) \geq$$

$$N(x_n - x, \frac{t}{2}) * N(y_n - y, \frac{t}{2})$$

By taking limit as $n \to \infty \implies$

$$\lim_{n\to\infty} N\big((x_n + y_n) - (x + y), t\big) \geq 1 * 1 = 1$$

Then $\lim_{n\to\infty} N\big((x_n + y_n) - (x + y), t\big) = 1$

Therefore $x_n + y_n \to x + y$.

Theorem (3.2): Let $x_n \to x, y_n \to y$, such that $\{x_n\}$ and $\{y_n\}$ two sequences in X. $\alpha, \beta \in F / \{0\}$ then

$\alpha f(x_n) + \beta g(y_n) \to \alpha f(x) + \beta g(y)$ whenever f and g are two identity function.

Proof: Let $t > 0$, for all $\varepsilon \in (0, 1)$ there exists $\varepsilon_1 \in (0,1)$such that

$(1 - \varepsilon_1) * (1 - \varepsilon_1) > (1 - \varepsilon)$, since $x_n \to x, y_n \to y$ then for all $\varepsilon_1 \in (0, 1)$ and $t > 0$ there exist $n_1 \in Z^+$ such that

$N\,(x_n - x, \frac{t}{2|\alpha|}) > 1 - \ \varepsilon_1$ for all $n \ \geq n_1$, for all $\varepsilon_1 \in (0, 1)$ and $t > 0$

there exists $n_2 \in Z^+$ such that $N\,(y_n - y, \frac{t}{2|\beta|} > 1 - \ \varepsilon_1$ for all

$n \geq \ n_{2,}$, take $\ n_0 = \min\,\{n_1, n_2\}$,

$N\,((\alpha\, f(x_n) + \beta\ g\ (y_n) - (\alpha\, f(x)\ + \beta\ g(y)), t) =$

$N\,(\alpha\ (f(x_n) - f(x))\ +\ \beta(g(y_n) - g(y), t)\ \geq$

$N\,(f(x_n) - f(x), \frac{t}{2|\alpha|}) * N\,(g(y_n) - g(y),\ \frac{t}{2|\beta|}) =$

$N\,(x_n - x, \frac{t}{2|\alpha|}) * N\,(y_n - y, \frac{t}{2|\beta|}) > (1 - \varepsilon_1) * (1 - \varepsilon_1) > (1 - \varepsilon)$ for

all $n \ \geq n_{0,}$ therefore $\alpha\ f\ (x_n) + \beta\ g\ (y_n) \to \alpha f(x)\ + \beta\ g(y)$.

Theorem (3.3): A fuzzy normed space $(X, N, *)$ is complete fuzzy normed space if every fuzzy Cauchy sequence $\{x_n\}$ in X has fuzzy convergent subsequence.

Proof: Let $\{x_n\}$ be a fuzzy Cauchy sequence in X and $\{x_{nm}\}$ be a subsequence of $\{x_n\}$ such that $x_{nm} \to x$.

Now to prove $x_n \to x$. For all $\varepsilon_1 \in (0, 1)$ there exist $\varepsilon \in (0, 1)$ such that $(1 - \ \varepsilon_1) * (1 - \ \varepsilon_1) \geq (1 - \varepsilon)$. Since $\{x_n\}$ is a fuzzy Cauchy sequence then for all $t > 0$ and $\varepsilon_1 \in (0, 1)$ there exists $n_0 \in Z^+$ such that: $N\,(x_n - x_m, \frac{t}{2}) > 1 - \ \varepsilon_1$, for all $n, m \geq n_0$.

Since $\{x_{nm}\}$ fuzzy converges to x, there exists $\ im \ \geq n_0$ such that

$$N(x_{im} - x, \tfrac{t}{2}) > 1 - \varepsilon_1$$

$$N(x_n - x, t) = N((x_n - x_{im}) + (x_{im} - x), \tfrac{t}{2} + \tfrac{t}{2}) \geq$$

$$N(x_n - x_{im}, \tfrac{t}{2}) * N(x_{im} - x, \tfrac{t}{2}) > (1 - \varepsilon_1) * (1 - \varepsilon_1) \geq (1 - \varepsilon).$$

Therefore $x_n \to x$, $\{x_n\}$ is fuzzy converges to x.

Hence $(X, N, *)$ is complete fuzzy normed space.

Definition (3.4): [2] Let $(X, N, *)$ and $(Y, N, *)$ be a fuzzy normed spaces. A mapping $f: X \to Y$ is said to be fuzzy continuous at $x_0 \in X$ is for every $\varepsilon \in (0,1)$, $t > 0$ there exist $\delta \in (0,1), s > 0$ such that $x \in X$

$$N(x - x_0, s) > 1 - \delta \text{ implies } N(f(x) - f(x_0), t) > 1 - \varepsilon.$$

Then f is continuous on X if it is fuzzy continuous at each point of X.

Theorem (3.5): Every identity fuzzy function is a fuzzy continuous function in fuzzy normed space.

Proof: For all $\varepsilon \in (0,1), t > 0$ there exist $\delta \in (0,1), s = t > 0$ such that

$\varepsilon > \delta, x \in X : N(x_n - x, s) > 1 - \delta$

$$N(f(x_n) - f(x), t) = N(f(x_n - x), t) = N(x_n - x, s) > 1 - \delta > 1 - \varepsilon$$

therefore f is a fuzzy continuous at $x \in X$, since x is arbitrary point in X then f is a fuzzy continuous function.

Theorem (3.6): Let X be a fuzzy normed over $\mathbb{F}$. Then the functions

$f: X \times X \to X, f(x, y) = x + y$ and $g: \mathbb{F} \times X \to X, g(\lambda, x) = \lambda x$

are fuzzy continuous functions.

Proof: (1) Let ε ∈ (0, 1) then there exists $\varepsilon_1 \in (0,1)$ such that

$(1-\varepsilon_1) * (1-\varepsilon_1) \geq (1-\varepsilon)$.

let $x, y \in X$ and $\{x_n\}$, $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to$ y as $n \to \infty$, then for each $\varepsilon_1 \in (0,1)$ and each $\frac{t}{2} > 0$ there exists $n_1 \in Z^+$such that $N(x_n - x, \frac{t}{2}) > 1-\varepsilon_1$ for all $n \geq n_1$, and for each $\varepsilon_1 > 0$ and each $\frac{t}{2} > 0$ there exists $n_2 \in Z^+$ such that $N\left(y_n - y, \frac{t}{2}\right) > 1-\varepsilon_1$ for all $n \geq n_2$, put $n_0 = min\{n_1, n_2\}$

$N(f(x_n, y_n) - f(x,y), t) = N((x_n + y_n) - (x+y), t) =$

$N((x_n - x) + (y_n - y), t) \geq N\left(x_n - x, \frac{t}{2}\right) * N\left(y_n - y, \frac{t}{2}\right) >$

$(1-\varepsilon_1) * (1-\varepsilon_1) \geq 1-\varepsilon$ for all $n \geq n_0$, therefore

$f(x_n, y_n) \to f(x,y)\ as\ n \to \infty, f$ is fuzzy continuous function at $(x,y)\ and\ (x,y)$ is any point in $X \times X$, hence f is fuzzy continuous function.

(2) Let $x \in X$, $\lambda \in \mathbb{F}$ and $\{x_n\}$ in X, $\{\lambda_n\}$ in $\mathbb{F}$ such that $x_n \to x$ and $\lambda_n \to \lambda$ as $n \to \infty$, i.e. for each ε ∈ (0, 1) and each $\frac{t}{2|\lambda_n|} > 0$ there exists $n_1 \in Z^+$ such that $N(x_n - x, \frac{t}{2|\lambda_n|}) > 1-\varepsilon$ for all $n \geq n_1$,

$|\lambda_n - \lambda| \to 0$ as $n \to \infty$,

$N(g(\lambda_n, x_n) - g(\lambda, x), t) = N(\lambda_n x_n - \lambda x, t) =$

$N((\lambda_n x_n - \lambda_n x) + (\lambda_n x - x\lambda), t)$

$\geq N\ (\lambda_n(x_n - x), \frac{t}{2}) * N\ (x\ (\lambda_n - \lambda), \frac{t}{2})\ =$

$N\ \left(x_n - x, \frac{t}{2|\lambda_n|}\right) * N\ (x, \frac{t}{2|\lambda_n - \lambda|})\ =\ N\ (x_n - x, \frac{t}{2|\lambda_n|}) * 1\ > 1 - \varepsilon$

for all $n\ \geq n_1$,

$g(\lambda_n, x_n) \rightarrow\ g(\lambda, x)\ as\ n \rightarrow \infty, g$ is fuzzy continuous at (λ, x) and (λ, x) is any point in $\mathbb{F} \times X$, hence g is fuzzy continuous.

Definition (3.7): [3] Let $(X, N,*)$ and $(Y, N,*)$ be vector spaces over the same $\mathbb{F}$. A function $f: X \rightarrow Y$ is called a linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$.

Theorem (3.8): Let $(X, N,*), (Y, N,*)$ be fuzzy normed spaces and let $f : X \rightarrow Y$ be a linear function. Then f is a fuzzy continuous either at every point of X or at no point of X.

Proof: Let $x_1$and x_2 be any two points of X and suppose f is fuzzy continuous at x_1.Then for each $\varepsilon \in (0,1)$, $t > 0$ there exist $\delta \in (0,1)$ such that $x \in X$,

$N(x -\ x_1,s) > 1 - \delta \Longrightarrow N\ (f(x) - f(x_1),t) > 1 - \varepsilon$

Now:

$N\ (x - x_2,s) > 1 - \delta\ \Longrightarrow N\ ((x + x_1 - x_2) - x_1,$

$s) > 1 - \delta\ \Longrightarrow N\ (f(x + x_1 - x_2) - f(x_1), t) > 1 - \varepsilon$

$\Longrightarrow N\ (f(x) + f(x_1) - f(x_2) - f(x_1$

$), t) > 1 - \varepsilon \Longrightarrow N\ (f(x) - f(x_2), t) > 1 - \varepsilon$, f is a fuzzy continuous at x_2, since x_2 is arbitrary point. Hence f is a fuzzy continuous.

Corollary (3.9): Let X, Y be fuzzy normed spaces and let $f : X \to Y$ be a linear function. If f is a fuzzy continuous at 0 then it is fuzzy continuous at every point.

Proof: Let $\{x_n\}$ be a sequence in X such that there exist x_0 , and $x_n \to x_0$, since f is continuous at 0 then:

For all $\varepsilon \in (0,1), t > 0$ there exist $\delta \in (0,1), s > 0: (x_n - x) \in X,$

$N((x_n - x_0) - 0, s) > 1 - \delta \Rightarrow N(f(x_n - x_0) - f(0), t) > 1 - \varepsilon,$

$$N(x_n - x_0, s) > 1 - \delta \Rightarrow N(f(x_n - x_0) - f(0), t) > 1 - \varepsilon$$

$$N(x_n - x_0, s) > 1 - \delta \Rightarrow N(f(x_n) - f(x_0) - f(0), t) > 1 - \varepsilon$$

$$N(x_n - x_0, s) > 1 - \delta \Rightarrow N(f(x_n) - f(x_0), t) > 1 - \varepsilon$$

$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$ therefore f is fuzzy continuous at x_0 since x_0 is arbitrary point then f is fuzzy continuous.

Theorem (3.10): Let $(X, N, *), (Y, N, *)$ be a fuzzy normed spaces then the function $f : X \to Y$ is fuzzy continuous in $x_0 \in X$ if and only if for all a sequence $\{x_n\}$ is convergent to x_0 in X then the sequence $\{f(x_n)\}$ is convergent to $f(x_0)$ in Y.

Proof: suppose the function f is fuzzy continuous in x_0 and let $\{x_n\}$ is a sequence in X such that $x_n \to x_0$ .Let $\varepsilon \in (0,1), t > 0,$ since f is fuzzy continuous in x_0 $\Rightarrow$there exist $\delta \in (0,1), s > 0,$ such that for all $x \in X$ $N\,(x - x_0, s) > 1 - \delta \Rightarrow$

$N\,(f(x) - f(\,x_0), t) > 1 - \varepsilon$

Since $x_n \to x_0, \delta \in (0,1), s > 0 \Rightarrow K \in Z^+$ such that

$N\,(x_n - x_0, s) > 1 - \delta$ for all $n > K$ hence

$N\,(f(x_n) - f(\,x_0), t) > 1 - \varepsilon$ for all $n > K$ therefore $f(x_n) \to f(\,x_0)$.

Conversely suppose the condition in the theorem is true.

Suppose f is not fuzzy continuous at x_0.

There exist $\varepsilon \in (0,1), t > 0$ such that for all $\delta \in (0,1), s > 0$ there exist $x \in X$ and $N\,(x - x_0, s) > 1 - \delta \Rightarrow$

$N\,(f(x) - f(x_0)\,t) \leq 1 - \varepsilon \Rightarrow$for all $n \in Z^+$ there exist $x_n \in X$ such that

$N\,(x_n - x_0\,, s) > 1 - \frac{1}{n}$ $N\,(f(x_n) - f(\,x_0\,), t) \leq 1 - \varepsilon$ that is mean $x_n \to x_0$ in X, but $f(x_n) \nrightarrow f(x_0)$ in Y this contradiction , thus f is fuzzy continuous at x_0.

Theorem (3.11): Let $(X, N_1, *)$ $(Y, N_2, *)$ be tow fuzzy normed spaces. If the functions $f : X \to Y$, $g: X \to Y$ are two fuzzy continuous functions and with $a * a = a \; for \; all \; a \in [0,1]$ then

$f + g, Kf \; where \; K \in F/\{0\}$, are also fuzzy continuous functions over the same filed F.

Proof :(1) Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ .Thus for all $t > 0$, there exist $s > 0 \ni \lim_{n\to\infty} N_1(x_n - x, s) = 1$ ... (1)

Since f, g are fuzzy continuous at x then (1) implies

$\lim_{n\to\infty} N_2(f(x_n) - f(x), \frac{t}{2}) = 1$

, $\lim_{n\to\infty} N_2(g(x_n) - g(x), \frac{t}{2}) = 1$ for all $t > 0$.

Now: $N_2\big((f+g)(x_n) - (f+g)(x), t\big) =$

$$N_2(f(x_n) + g(x_n) - f(x) - g(x), t)$$
$$\geq N_2\left(f(x_n) - f(x), \frac{t}{2}\right) * N_2\left(g(x_n) - g(x), \frac{t}{2}\right)$$

Taking limit: $\lim_{n\to\infty} N_2((f+g)(x_n) - (f+g)(x), t) \geq$

$$\lim_{n\to\infty} N_2(f(x_n) - f(x), \tfrac{t}{2}) * \lim_{n\to\infty} N_2(g(x_n) - g(x), \tfrac{t}{2}) = 1 * 1 = 1$$

$\Rightarrow \lim_{n\to\infty} N_2((f+g)(x_n) - (f+g)(x), t) = 1$

then $f + g$ is fuzzy continuous function.

(2) Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ .Thus for all $t > 0$, there exist $s > 0 \ni \lim_{n\to\infty} N_1(x_n - x, s) = 1$ implies

$\lim_{n\to\infty} N_2(f(x_n) - f(x), t) = 1$ take $0 < t_1 = t|k|$

$$\mathrm{N}_2((k f)(x_n) - (kf)(x), t_1) = N_2(k(f(x_n) - f(x)), t_1) =$$
$$N_2(f(x_n) - f(x), t)$$

Taking limit:

$$\lim_{n\to\infty} \mathrm{N}_2((k f)(x_n) - (kf)(x), t_1) = 1$$

Then kf is a fuzzy continuous function.

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