Derivation of Annular Plate Sector Element for General Bending Analysis

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Abstract

A new mathematical finite element model suitable for the general bending analysis of annular plate structure is developed in polar coordinates system depending on the strain based approach has been derived. The element is simple and contains only the essential degrees of freedom. The element has 15 degrees of freedom, three at each node and satisfies the exact representation of the rigid body modes of displacement. The results obtained by using the proposed element in several numerical problems have shown that a rapid convergence to exact solution can be obtained with acceptable degree of accuracy when only few element are used. The element has the advantage over the other available annular plate elements. The improvement obtained is due to the fact that all the displacement fields of the present element satisfy the exact representation of rigid body modes of displacements then the shape function error due to rigid body modes becomes zero. Also, the present element satisfies the full geometry of the annular plate due to this point discretization error becomes zero. Finally the error due to strain mode becomes very small because the present element satisfies the compatibility equations of strains and the 12th coefficients of strain mode derived exactly from partial differential equations of strains.

The numerical solution of several problems by using the present element proved to be powerful in the structural bending analysis of circular annular plates. Its results are better than the solution of other elements and packages with respect to analytical.

Keywords: annular sector plate, Strain based approach element

اشتقاق عنصر محدد لتحليل الانحناء العام للصفائح الدائرية بالإحداثيات القطبية

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الخلاصة:

تم في هذه الدراسة تطوير عنصر جديد للتحليل الأنحنائي للصفائح الحلقية بأستخدام نظام الأحداثيات القطبية، يعتمد على طريقة الأنفعال حيث يظم درجات الطلاقة الرئيسية. العنصر يحقق مواصفات الحركة للجسم الصلب بشكل تام (exact rigid body mode)، كما يحقق كامل الخواص للشكل الهندسي للصفائح الحلقية (full geometry of annular plate)، إضافة إلى ذلك فأن العنصر يحقق شروط التوافق لمعادلات الأنفعال (compatibility equation of strain).

يمتلك هذا العنصر خمسة عشر درجة طلاقة حرة (degree of freedom) ثلاثة في كل عقدة ركنية وثلاثة في عقدة الوسط يعتبر هذا العنصر ملاءم للتحليل الأنحنائي العام للصفائح الحلقية وهو أفضل من العناصر السابقة لتحليل المنشاءات الآنفة الذكر حيث أن الأخطاء التي تظهر في العناصر المحددة السابقة مثل أخطاء التقسيم (discritization error)، أخطاء الدوال الشكلية (shape function error) تختفي في هذا العنصر نتيجة للمواصفات التي يتمتع بها.

تم استخدام العنصر الحالي في التحليل العددي لعدد من المساءل المختلفة إذ أنها تبين وتبرهن أن العنصر الحالي ممتاز وكفوؤ في تحليل عدة أنواع من التحميل بشكل أفضل من العناصر المحددة لباحثين آخرين حيث اظهرت النتائج امكانية الاقتراب السريع الى نتائج الحل الدقيق (exact solution) وبدرجة دقة عالية وبأستخدام عدد قليل من العناصر عند تمثيل المنشاء.

Notation:

r, Θ	Polar coordinates
D	Bending rigidity.
E	Modulus of elasticity.
M_r, M_Θ	The bending moments in the directions of r and Θ axes, respectively.
$M_{r\Theta}, M_\Theta r$	The twisting moments.
P	Point load
q	The uniformly distributed load acting on the plate.
Q_r	The out of plane shearing stress resultant in the r-direction of general plate theory
Q_Θ	The out of plane shearing stress resultant in the Θ-direction of general plate theory
w	Out of plane displacement in Z-direction for cartesian coordinate, and in normal direction for polar coordinates.
w_r	The middle surface normal displacements of the annular plate element due to the rigid body part.
w_s	The middle surface normal displacements of the annular plate element due to the strain part.

$\{a\}$	is the vector of constant terms of the displacement function $\{d\}$
[A]	The transformation matrix.
[B]	The strain matrix.
[D]	The rigidity matrix.
[f]	is the matrix containing the coordinate variables
$[K^e]$	The stiffness matrix of the finite element.
$\|P\|$	The vector containing the nodal loads acting on the finite element.
$\|\delta\|$	The vector containing the degrees of freedom of the finite element.
$\|\varepsilon\|$	The vector containing the strain (and curvatures) of the finite element.
$\|\sigma\|$	The vector containing the stresses in the finite element.
ϕ_r	Rotation in the r-direction of general plate theory
ϕ_Θ	Rotation in the Θ-direction of general plate theory

1. Introduction

For the general bending analysis of annular plates based on the classical theory of thin plates. In general, the following three numerical methods have been used for the analysis of annular plates[1]:

1. The finite difference method (FDM).
2. The finite strip method (FSM).
3. The finite element method (FEM).

The experimental work about this subject were extremely limited for simple cases such as point loads and simple support [2]. The finite element method of structural analysis is now firmly established as a powerful technique for handling different problems in solid mechanics [3]. The simplest element shapes for annular plate problems are obviously a triangle element with three nodes as in work of Cheung et al [4], and rectangular element with four nodes as in the work of Rao [5]. Olson and Lindberg [3], developed an annular segment plate bending element with four corner nodes each having three external nodal degrees of freedom. However, using these elements for curved boundary problems means, that the curved boundary is being approximated by series of straight – line segments. Hence, there appears to need to develop a new element by using the polar coordinates system to get a better representation of the curved boundaries. The present element have an advantageous over the available annular plate elements. The final properties of the present element are as follows:

- The element satisfies the full geometry of annular plate segment, and due to this point the discritization errors that appear in the curved boundary becomes zero.
- The element satisfies the exact rigid body modes of annular plate segment, and due to this point the shape function error of rigid body mode part becomes zero.

- The strain mode of element is obtained from integrating of assumed strain functions satisfying the compatibility equation of annular plate segment; due to this point the shape function error of strain mode part becomes very small.
- The explicit integration is used to derive this element; due to this point the error in numerical integration becomes zero.
- According to the central node of element the real value of stresses in the centre of element are found, not approximate stresses due to the mean of four corner nodes as in the other available annular plate elements.

2. Derivation of Annular Plate Element Using Strain Based Approach

2.1 Theoretical Consideration

Figure (1) shows an annular plate sector. To idealize this annular plate sector, an element is chosen as shown in **Figure (2).** For the general bending analysis of annular plate sector under arbitrary loading, the strains (direct strains and changes in curvature) of the middle surface are derived from several theories of annular plate sector [13]:

The lateral deflection (w) is a function of (r) and (Θ), then the laplacian operator becomes[13]:

$$\nabla w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial q^2} \qquad \text{………………(1)}$$

Then the deflection surface of a laterally loaded plate transforms is becomes:

$$\nabla\nabla w = \left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial q^2}\right\}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial q^2}\right\} = \frac{q}{D} \qquad \text{………………(2)}$$

Due to load is symmetrically distributed with respect to the center of the plate, the lateral deflection (w) is independent of (θ) and the above equation becomes[12]:

$$\nabla\nabla w = \left\{\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right\}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right\} = \frac{q}{D} \qquad \text{………………(3)}$$

Where q is the applied load and it is given as a function of (r) and (θ),

D is the flexural rigidity, and equals $D = \dfrac{Et^3}{12(1-n^2)}$

The out of plane components of the strains curvatures are follows:

$$c_r = \frac{\partial^2 w}{\partial r^2} \qquad \text{........................ (4a)}$$

$$c_q = \frac{\partial w}{\partial r}\frac{1}{r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial q^2} \qquad \text{........................ (4b)}$$

$$c_{rq} = \frac{1}{r}\frac{\partial^2 w}{\partial r \partial q} + \frac{1}{r^2}\frac{\partial w}{\partial q} \qquad \text{........................ (4c)}$$

where: (r) is the radial coordinate measured from the apex of the annular plate sector, and (q) is the angular coordinate measured around the circumference.

The equation of radial, angular and twisting moments, M_r, M_Θ, and $M_{r\Theta}$ respectively becomes:

$$M_r = -D\{c_r + u c_q\} \qquad \text{..................... (5a)}$$

$$M_q = -D\{c_q + u c_r\} \qquad \text{..................... (5b)}$$

$$M_{rq} = -(1-n)D\{c_{rq}\} \qquad \text{..................... (5c)}$$

and the vertical shearing forces Q_r, and Q_Θ respectively becomes:

$$Q_r = -D\frac{\partial}{\partial r}[\nabla w] \qquad \text{..................... (6a)}$$

$$Q_q = -D\frac{1}{r}\frac{\partial}{\partial q}[\nabla w] \qquad \text{..................... (6b)}$$

The above three components of strains cannot be considered independent as they are in terms of the lateral displacement (w) and hence, the strains must satisfy additional equations called the compatibility equations. These equations are obtained by eliminating the lateral displacement (w) from equations (4). The final results of compatibility equations are as follows:

$$\frac{\partial(r^2 c_{rq})}{\partial r} - r\frac{\partial c_r}{\partial q} = 0 \qquad \text{............................ (7a)}$$

$$\frac{\partial(r c_q)}{\partial q} - \frac{\partial(c_{rq})}{\partial q} - c_r = 0 \qquad \text{............................ (7b)}$$

2.2 Displacement Functions:

The proposed element is allowed to have only the essential external degrees of freedom (w, ϕ_r, and ϕ_θ) at each node. Proceeding as with the usual strain-based approach, the first major component of the displacement function is due to (strain-free) rigid body modes of displacement and can be obtained by equating all the components of strains [8], equations (4), to zero and integrating the resulting partial differential equations becomes:

$$w_R = a_1 + a_2 r\cos q + a_3 r\sin q \qquad \text{.....................} \ \mathbf{(8)}$$

In this equation w_R are the rigid body component of the displacement field w, and is expressed in terms of the three independent constants $(a_1, —, a_3)$. The second major component of the displacement function is due to straining of the element. If the element is to have fifteen degrees of freedom (three at each node) then the strains of the element must be associated with twelve additional constants $(a_4, —, a_{15})$. Assuming strain polynomial functions of $(a_4 - - a_{15})$ constants.

$$c_r = \frac{\partial^2 w}{\partial r^2} = A_4 r + A_5 q + A_6 r^2 q + A_7 r q^2 + A_8 r^2 q^2 \qquad \text{.....................} \ \mathbf{(9a)}$$

$$c_q = \frac{\partial w}{\partial r}\frac{1}{r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial q^2} = A_9 r + A_{10} q + A_{11} r^2 q + A_{12} r q^2 + A_{13} r^2 q^2 \qquad \text{.....................} \ \mathbf{(9b)}$$

$$c_{rq} = \frac{1}{r}\frac{\partial^2 w}{\partial r \partial q} + \frac{1}{r^2}\frac{\partial w}{\partial q} = A_{14} r + A_{15} q \qquad \text{.....................} \ \mathbf{(9c)}$$

Checking the above polynomials of strain for compatibility equations of annular plate sector (7a, and 7b). Finally, the assumed strain functions of annular plate sector element which satisfy the requirement of compatibility equations are the same of equations 9a, and 9b except equation 9c become:

$$c_{Sq} = \left[(2A_9 - A_4) r q + (A_{10} - A_5)\frac{q^2}{2} + (3A_{11} - A_6)\frac{r^2 q^2}{2} + (2A_{12} - A_7)\frac{r q^3}{3} \right] + A_{14} r + A_{15} q \qquad \text{............} \ \mathbf{(10)}$$

The above three equations 9a, 9b, and 10 are integrated in the same procedure that was used to derive the rigid body modes of annular plate sector element, then the final polynomial function of strain mode becomes:

$$w_S = a_4 q + a_5 r^2 + a_6 q^2 + a_7 r^2 q + a_8 r q^2 + a_9 r^3 + a_{10} q^3 + a_{11} r^3 q \\ a_{12} r q^3 + a_{13} r^3 q^2 + a_{14} q^2 r^2 + a_{15} r^2 q^3 \qquad \text{..................} \ \mathbf{(11)}$$

The complete displacement field functions for the proposed element is becomes.

$w = w_r + w_s$

$$w = a_1 + a_2 r\cos q + a_3 r\sin q + a_4 q + a_5 r^2 + a_6 q^2 + a_7 r^2 q + a_8 r q^2 + a_9 r^3 + a_{10} q^3 + a_{11} r^3 q + a_{12} r q^3 + a_{13} r^3 q^2 + a_{14} q^2 r^2 + a_{15} r^2 q^3 \quad \text{……..………..… (12)}$$

The rotation ϕ_r, and ϕ_θ are also given below:

$$f_r = \frac{\partial w}{\partial r} = a_2 \cos q + a_3 \sin q + 2a_5 r + 2a_7 rq + a_8 q^2 + 3a_9 r^2 + 3a_{11} r^2 q + a_{12} q^3 + 3a_{13} r^2 q^2 + 2a_{14} q^2 r + 2a_{15} r q^3 \quad \text{………..………….(13)}$$

$$f_q = \frac{\partial w}{r\partial q} = -a_2 \sin q + a_3 \cos q + \frac{a_4}{r} + \frac{2a_6 q}{r} + a_7 r + 2a_8 q + \frac{3a_{10} q^2}{r} + a_{11} r^2 + 3a_{12} q^2 + 2a_{13} r^2 q + 2a_{14} rq + 3a_{15} rq^2 \quad \text{………..………..… (14)}$$

2.3 The Element Stiffness Matrix

Figure (1) shows that the origin of the redial coordinate of the element (r) is located at the apex of the annular plate sector. Consequently, the origin of the angular coordinate (θ) is located at the center of the element. Since, the calculation of the stiffness matrix is carried out explicitly; this choice of the origin will simplify the task of integration, thus:

$$\left[k^e\right] = \left[A^{-1}\right]^T \left[\int_{-b}^{b}\int_{r_1}^{r_2} [B]^T [D][B] r dr dq\right] [A]^{-1} \quad \text{………..………..… (15)}$$

where [A], [B] and [D] are the transformation, strain and rigidity matrices of the element, respectively. The (15X15) element stiffness matrix [k^e], can now be calculated using the displacement functions (12) and the strain displacement relationships (4).

On the other hand, to keep the storage memory small the stiffness matrix is condensated from (15x15) to (12x12) by removing the influence of the central point (node 5) to the four corner points (nodes 1, 2, 3, and 4) as follows:

$$\left[K^n{}_{12x12}\right]\left\{d^n{}_{12x1}\right\} = \left\{P^n{}_{12x1}\right\} \quad \text{………..………..… (16)}$$

where:

$$\left[K^n{}_{12x12}\right] = \left[K^e{}_{12x12}\right] - \left[K^e{}_{12x3}\right]\left[K^e{}_{3x3}\right]^{-1}\left[K^e{}_{3x12}\right]$$

$$\left\{d^n{}_{12x1}\right\} = \left\{d^e{}_{12x1}\right\}$$

$$\left\{P^n{}_{12x1}\right\} = \left\{P^e{}_{12x1}\right\} - \left(\left[K^e{}_{12x3}\right]\left[K^e{}_{3x3}\right]^{-1}\left\{P^e{}_{3x1}\right\}\right)$$

2.4 Consistent Load Vector

The external applied nodal loads considered in the present finite element analysis are calculated by using a consistent load vector. The consistent load vector is obtained by equating the work done by the nodal loads on the nodal displacements to the work done by the external applied load on the assumed displacement function of the element. The annular plate element shown in **Figure (2)** is considered. If the element is subjected to several loading types such as, distributed normal pressure (q), concentrated load (P_i), uniform distributed moment (M), and concentrated moment (M_i) the load vector becomes[11]:

$$\{P^e{}_{15x1}\} = \int_{-b}^{b}\int_{r_1}^{r_2} [A]^{-1}[f]\begin{Bmatrix} q \\ m \end{Bmatrix} drdq + \int_{-b}^{b}\int_{r_1}^{r_2} [A]^{-1}[f]_{at\ node\ (i)}\begin{Bmatrix} P_i \\ M_i \end{Bmatrix} drdq \qquad \text{.......................} \textbf{(17)}$$

For the element stiffness matrix after condensation, the load vector is taken as follows:

$$\{P^n{}_{12x1}\} = \{P^e{}_{12x1}\} - \left([K^e{}_{12x5}][K^e{}_{3x3}]^{-1}\{P^e{}_{3x1}\}\right) \qquad \text{.......................} \textbf{(18)}$$

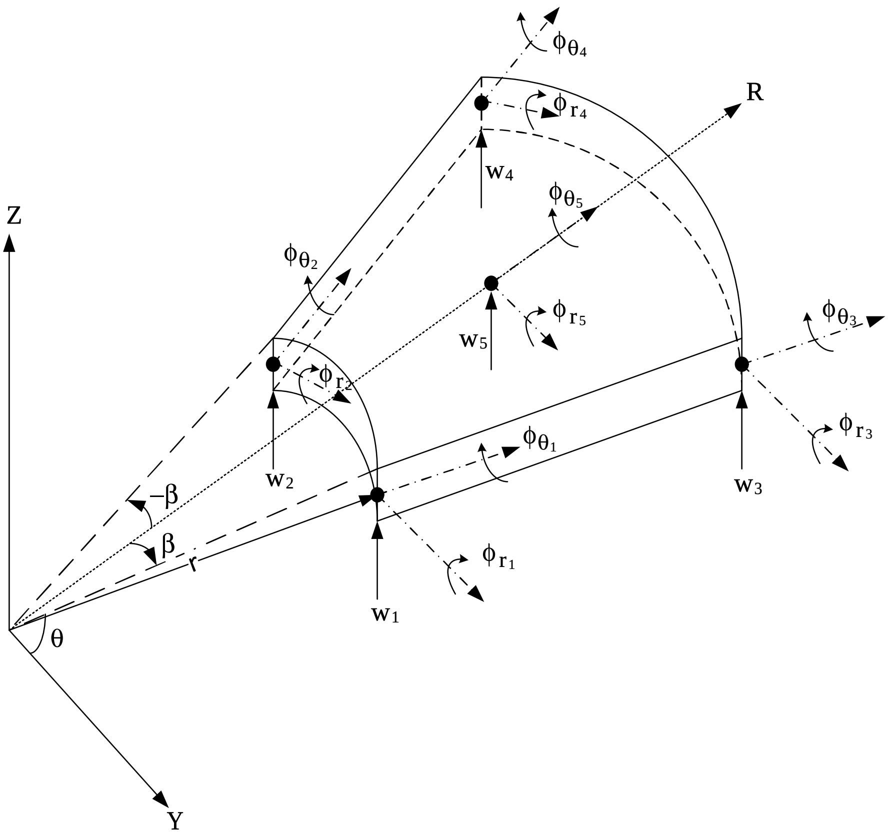


Fig .(1) An annular segment element with the coordinates system for plate bending problems.

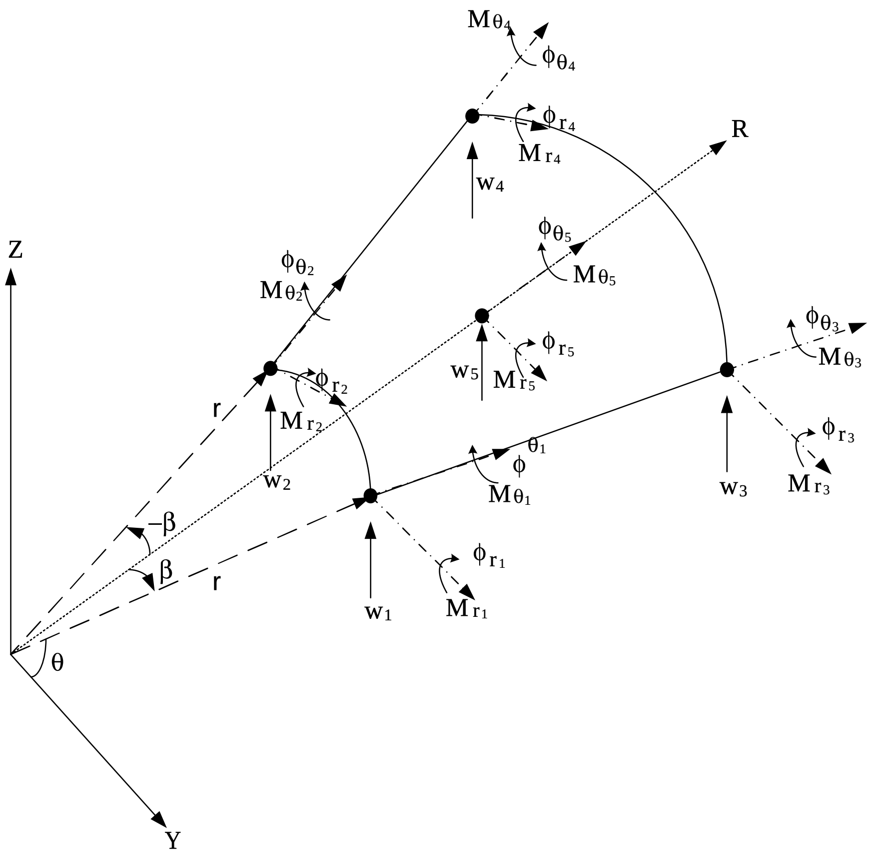


Fig .(2) Displacement and their corresponding forces for annular plate element.

3. An annular plate under a point load

The problem considered is that of an annular plate deforming unsymmetrically as shown in **Figure (3).** The plate is clamped along the inner edge (r_1=b), and loaded by a concentrated load at the outer edge (r_2=a). An exact solution for this problem is given in Timoshenko and Woinowsky-Krieger[6] for b/a = 1/1.5 and μ=0.3. Olson and Lindberg[3], Turki[7] and Al-khafaji and Mehdi[8] presented an annular finite element, to analyze this problem. Symmetry about the diameter containing the load was used, so that, only one half of the plate was to be modeled. The results for the lateral displacement (w) under the load for different mesh sizes are given in **Table (1).** When this problem is analyzed with (5x24) mesh using both elements, the error is about 0.45% for Olson and Lindberg[3], Turki[7], and 0.3% for Al-khafaji and Mehdi[8], but the error is about 0.1% for the present element and with (8x24) grid for both elements the error is about 0.7% for Olson and Lindberg[3], Turki[7], and 0.089% for Al-khafaji and Mehdi[8], but the error is about 0.026% for the present element. It is clear in **Figure (3)** that the lateral deflections along the inner edge converge rapidly to the exact solution when the mesh size is refined.

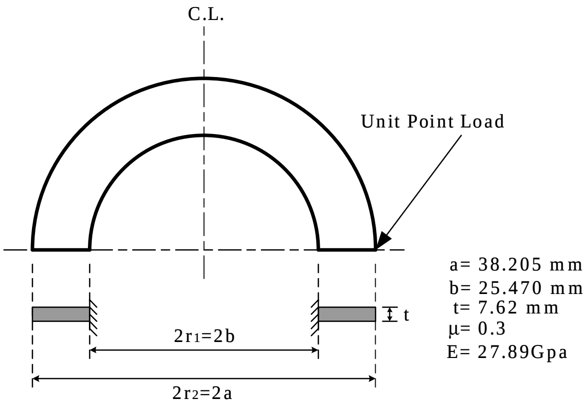
Olson and Lindberg[3] confined the results in their published work to deflections only (no results were given for the moments). **Figure (3)** shows the distribution of radial bending moment in a non-dimensional form of (M_r/2P) along the inner edge for (12x24) mesh. It is evident from this figure that the proposed element gives satisfactory results for moments when compared with the exact solution given by Timoshenko and Woinowsky-Krieger[6]. **Table (2)** shows the convergence error of four elements, the result appears that convergence of the present element is stable as when element mesh is increasing the error of solution is decreasing continuously, but in the other elements the convergence of solution is unstable with increasing mesh size as appearing in **Table (2).** In general, the two types of convergence are shown in **Figure (4).** The reason of two types of errors are concluded by the rigid body mode and strain mode errors, the present element satisfies the rigid body and strain modes as this element's result is stable, the error is decreasing when mesh size is increasing, but other elements do not satisfy the above conditions.

Table .(1) Convergence of the lateral deflection under the applied point load

Mesh size	**Deflection * P/D**			
	Olson's and Lindberg Element	**Turki's Element**	**Mehdi's Element**	**Present Element**
1x6	0.050896	0.050859	0.051675	0.050834
2x8	0.051372	0.051370	0.051456	0.050817
2x12	0.051027	0.051027	0.051256	0.050795
4x24	0.050885	0.050885	0.050932	0.050781
5x24	0.050956	0.050945	0.050873	0.050773
7x24	0.050997	0.051007	0.050824	0.050758
8x24	0.051078	0.051050	0.050763	0.050731
12x24	0.051032	0.050992	0.0507424	0.050722
Exact	0.0507180 (Timoshenko and Woinowsky-Krieger 1981)			

Table .(2) Convergence error of several meshes of the lateral deflection under the applied point load.

Mesh size	**Deflection * P/D**			
	Olson's Element	**Turki's Element**	**Mehdi's Element**	**Present Element**
1x6	0.35%	0.30%	1.900%	0.23%
2x8	1.30%	1.30%	1.500%	0.195%
2x12	0.61%	0.61%	1.060%	0.152%
4x24	0.33%	0.33%	0.420%	0.124%
5x24	0.47%	0.45%	0.300%	0.1%
7x24	0.55%	0.57%	0.200%	0.0788%
8x24	0.71%	0.66%	0.089%	0.026%
12x24	0.62%	0.54%	0.048%	0.0079%



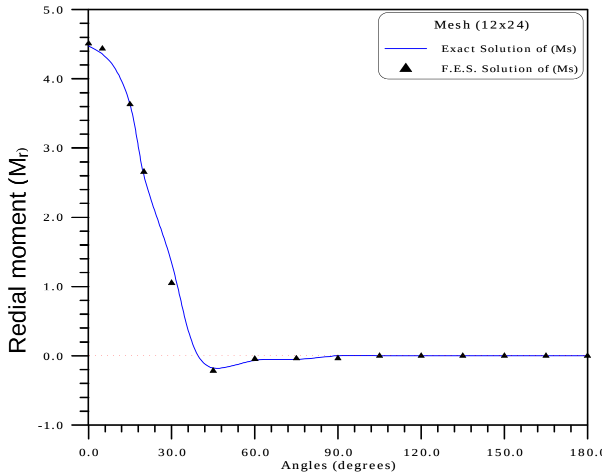
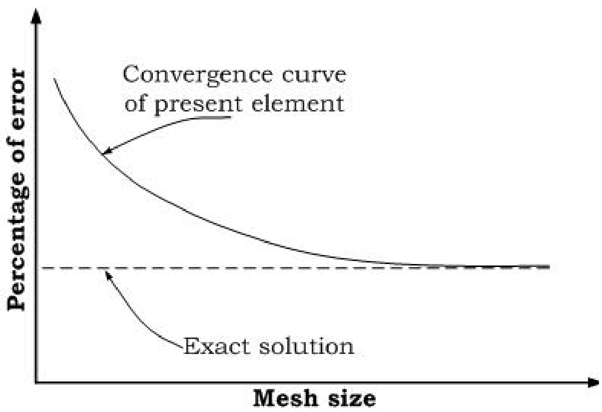
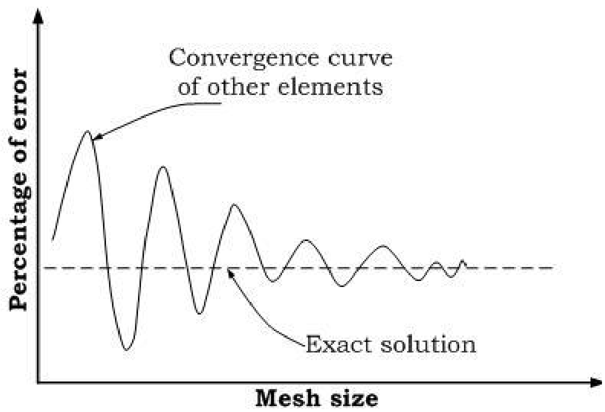


Fig .(3) An annular plate under a point load in problem.

Fig .(4) Distribution of radial bending moment (Ms) along the inner edge in problem.



(a) Al-khafaji and Mehdi, and Present element behavior.



(b) Olson's and Lindberg element and Turki's element behavior.

Fig .(5) General types of convergence errors of finite element solution.

4. An annular plate under uniformly distributed load

As an axisymmetric problem, a circular plate with a central circular hole is analyzed. Four cases are considered as shown in **Figure (6),** and in each case, one element in the angular direction is used with an included angle of 4.5^{o}. Inserting zero value the normal rotation ϕ_θ at all nodes along both the radial edges, thus satisfying the circular symmetry criterion.

Table (3) shows the maximum deflections and moments which, are obtained by using the proposed element together with those obtained by the exact solution given by Timoshenko and Woinowsky-Krieger(6) and Sawko,s element[9].

Table (4) shows the convergence of maximum deflections and moments. Referring to this table, it is concluded that the difference in the results is very small and in most cases tends to be stable, specially, for deflections. Generally, satisfactory results can be obtained with less than 0.1% difference by using 24 degrees of freedom for deflections and 60 degrees of freedom for moments, in all considered cases.

The accuracy of this approximation is illustrated in **Figures (7)** for various loading and boundary conditions. This table also shows the convergence of the maximum deflection for all considered cases.

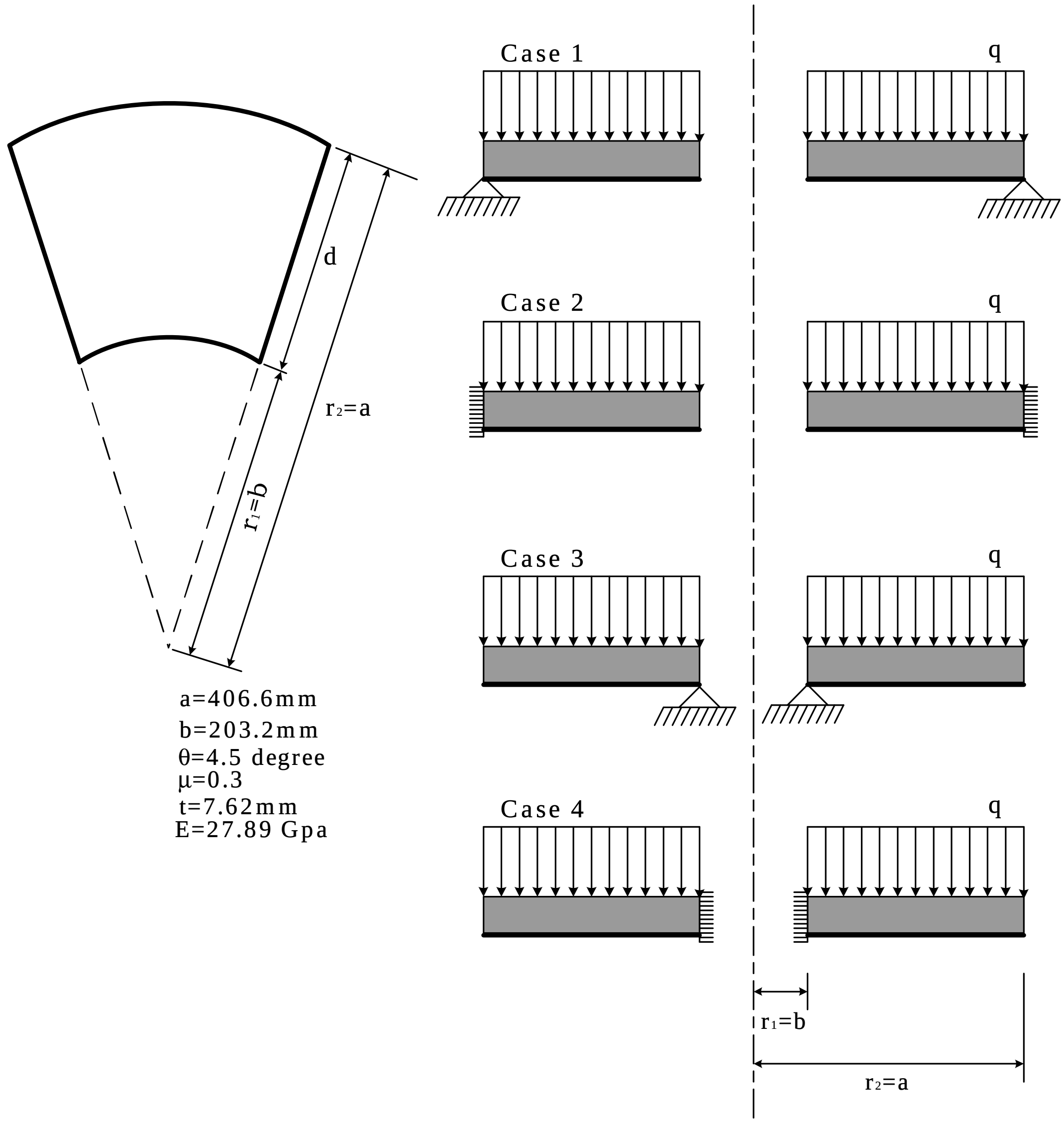


Fig .(6) Uniformly distributed load annular plates sector.

Table .(3) Maximum deflections of problem in Figure (6)

Case No.	$W_{max}*qa^4/D*10^{-3}$			
	Exact	Sawko's Element	Turki's Element	Present Element
Case 1	60.806	60.272	60.440	60.793
Case 2	5.266	5.081	5.269	5.2655
Case 3	82.600	79.650	82.568	82.603
Case 4	8.59	8.300	8.589	8.5893

Table .(4) Maximum moments of problem in Figure (6)

Case No.	$M_{max}*qa^2/D*10^{-2}$			
	Exact	Sawko's Element	Turki's Element	Present Element
Case 1	24	24.023	24.112	24.007
Case 2	8	7.969	7.984	8.002
Case 3	34	34.18	34.177	33.986
Case 4	17.33	17.36	17.179	17.346

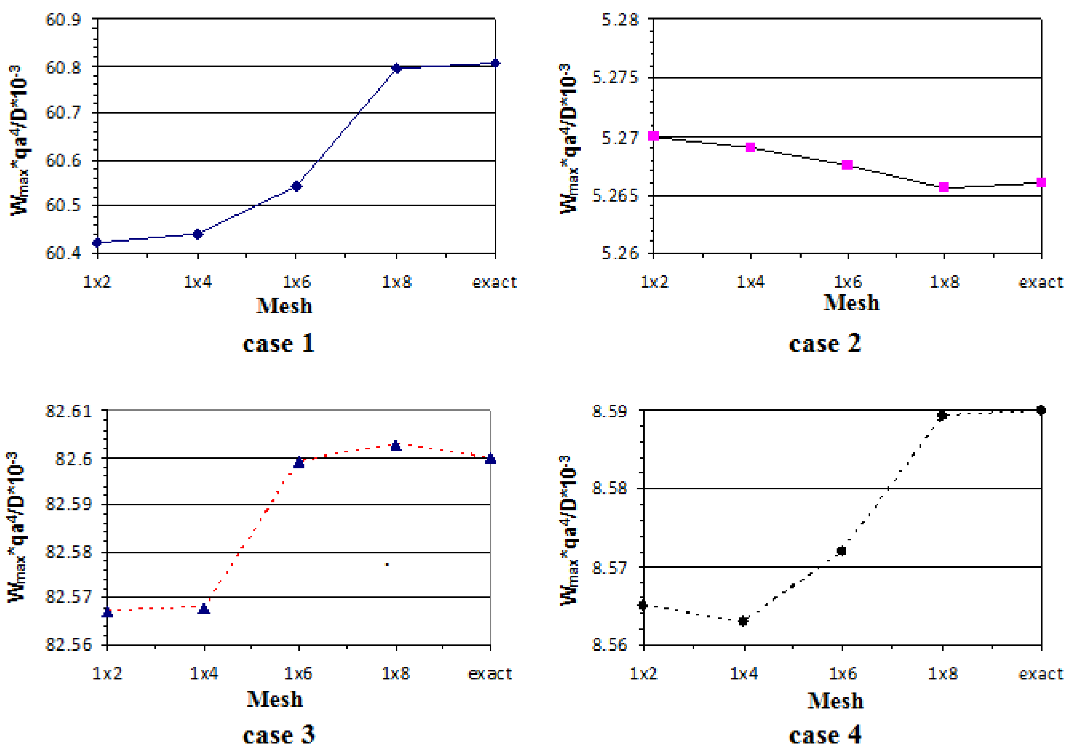


Fig .(7) Mesh study for convergence of four cases problem in Figure (6) to exact solution for ($W_{max}*qa^4/D*10^{-3}$)

5. Uniformly loaded annular plate sector with all edges clamped.

The final problem considered is that of a uniformly loaded annular plate sector with all edges clamped. The sketch of this annular plate and its properties are given in the following figure.

This problem was first analysed by Cheung and Chan[10], using the finite strip method. Harik[1] developed an analytical solution to represent this problem. The maximum deflection and the maximum positive moments (radial and angular moments) along the centeral redial line are calculated by the proposed element for (9*9) mesh, and for different values of the (b/a) ratio. The results are given in **Tables (5a, B, And C),** together with those of the other methods.

Convergence tests were carried out using the proposed element for the deflections and moments when the annular plate is divided into meshes ranging from (4x4) to (16x16). **Table (6)** shows the convergence of maximum deflections and maximum positive moments along the central radial line and when the (b/a) ratio equals (0.75).

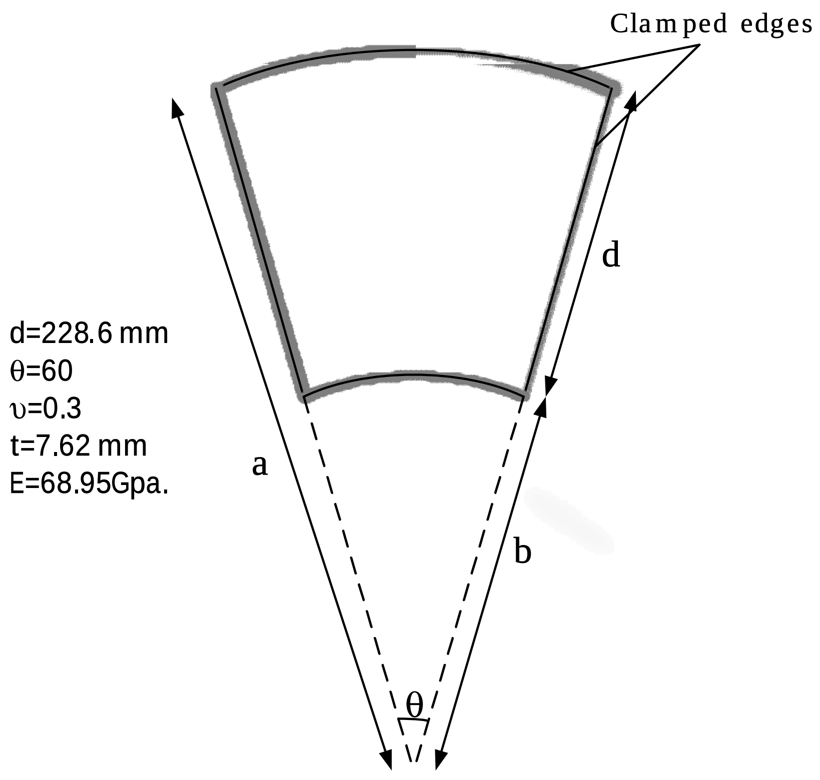


Fig .(8) Uniformly loaded annular plate sector with all edge clamped and its properties.

Table .(5a) Maximum deflections of problem in Figure (8)

b/a	$W_{max}*qa^4/D*10^{-4}$		
	Cheung and Chan (19)	Harik (2)	Present Element
0.00	2.82	2.91	2.88
0.25	2.76	2.85	2.84
0.50	1.41	1.43	1.4223
0.75	0.09	0.10	0.9845

Table .(5b) Maximum moments of problem in Figure (8)

b/a	$Mr_{max}*qa^2/D*10^{-3}$		
	Cheung and Chan (19)	Harik (2)	Present Element
0.00	11.287	10.948	10.9973
0.25	11.643	11.119	11.0352
0.50	10.173	9.531	9.6873
0.75	2.912	2.568	2.5698

Table .(5c) Maximum moments of problem in Figure (8)

b/a	$Mq_{max}*qa^2/D*10^{-3}$		
	Cheung and Chan (19)	Harik (2)	Present Element
0.00	13.011	11.674	11.783
0.25	12.638	11.459	10.956
0.50	6.203	5.249	5.127
0.75	0.907	0.698	0.807

Table .(6) Convergence of the maximum deflections and positive moments along the central redial line in problem in Figure (8) for b/a = 0.75.

mesh	$W_{max}*qa^4/D*10^{-4}$	$Mr_{max}*qa^2/D*10^{-3}$	$Mq_{max}*qa^2/D*10^{-3}$
2x2	1.04205	2.1235	0.8634
4x4	1.01810	2.4046	0.8587
6x6	1.00609	2.4867	0.8432
8x8	1.00110	2.5485	0.8345
10x10	0.98781	2.5587	0.8197
12x12	0.98456	2.5688	0.8083
16x16	0.98450	2.5698	0.8070

6. Conclusion:

Annular plate finite element suitable for the general bending analysis of annular plate sector has been developed. The element is simple and contains only the essential degrees of freedom. The element has the advantage over the other available annular plate element. The improvement obtained is due to the fact that all the displacement fields of the present element satisfy the exact representation of rigid body modes of displacements. Also, the displacement fields due to straining of the element are based on independent strains and satisfy the exact compatibility equations of strain modes.

The present element is used to analyze several types of problem. The numerical results of the present element are compared with the analytical, and numerical results of other researchers. The results of the present element showed good and rapid convergence of displacements and stresses with the use a few elements. The errors of output results is less than 0.5% of mesh size (1x6) and less than 0.03% of mesh size (8x24) for static analysis of plate problems.

7. References

1. **Harik, I. E. "Analytical solution to orthotropic sectors", J. Eng. Mech., ASCE, Vol. 110, No. 4, April 1984, pp. 554 – 568.**
2. **Coull, A. and Das, P. C., "Analysis of curved bridge desks", Proc. Inst. Civ. Eng., Vol. 37, May, 1967, pp. 75 – 85.**
3. **Olson, M. D. and Lindberg, G. M., “Annular and circular sector finite element of bending analysis”, Int. J. Mech. Sci., Vol. 12, 1970, pp. 17-33.**
4. **Cheung, Y. K., King, I. P., and Zienkiewiez, O. C, "Slab Bridges with arbitrary shape and support conditions: A general method of analysis based on finite elements" Proc. Inst. Civ. Engrs., Vol. 40, May, 1968., pp. 9 – 36.**
5. **Rao, S. S., "The finite element method in engineering", 2nd Ed., Pergaman Press, 1984.**
6. **Timoshenko, S. P. and Krieger, S. W., “Theory of plates and shells”, copyright ©, 1981, international edition McGraw-Hill book co, New York.**
7. **Turki B., “A finite element model for bending analysis of orthogonally stiffened annular plate sectors”, MSc. Thesis submitted to Department of Civil Engineering, College of Engineering, Al-Mustansiriyah University 1999.**
8. **Al-Khafaji, J. M., and Mehdi, H. A., " Strain Based Finite Element Model for Conical Shells", International journal for engineering and technology, India, 2012, pp. 475 – 495.**
9. **Sawko, F., and Merriman, P. A., "An annular segment finite element for plate bending", Int. J. Num. Meth. In Eng., Vol. 3, 1971, pp. 199 – 129.**

10. Cheung, M. S., and Chan, M. Y. T., "Static and dynamic analysis of thin and thick structural plates by the finite strip method", Comp. and Struc., Vol. 14, No. 1 – 2, 1988, pp. 79 – 88.

11. El_Erris, H. F., “Finite element analysis of shell structures”, Ph.D Thesis, 1989, submitted to Cardiff University, U. K.

12. Szilard, R., "Theory and analysis of plate", Printice-Hall, Inc. Englewood Cliffc, N. J., 1974.

13. Zienkiwics, O. C., and Taylor, R. L., "Finite element method for solid and structural mechanics", Sixth edition, 2005, Elsevier Inc, Oxford, London, U.K.