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Using AL-Tememe Transform to Solve System of Linear Second Order Ordinary Differential Equations With Variable Coefficients

استخدام التحويل التميمي لحل النظام الخطي للمعادلات التفاضلية الاعتيادية من الرتبة الثانية ذات المعاملات المتغيرة

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Abstract

Our aim is to apply AL-Tememe transform to solve system of linear second order ordinary differential equations(L.O.D.E) with variable coefficients.

المستخلص

هدفنا من هذا البحث هو استخدام التحويل التميمي لحل النظام الخطي للمعادلات التفاضلية الاعتيادية من الرتبة الثانية ذات المعاملات المتغيرة.

Introduction:

We will use Al-Tememe Transform ($\mathcal{T}$.T) to solve systems of linear Second order ordinary differential equations with variable coefficients. And the method summarized by taking ($\mathcal{T}$.T) to both sides of the equations then we take ($\mathcal{T}^{-1}$. T) to both sides of the equations and by using partial fraction decomposition we find the values of values constants.

Definition 1: [1]

Let f is defined function at period (a, b) then the integral transformation for f whose it′s symbol $F(p)$ is defined as:

$$F(p) = \int_a^b k(p, x)\, f(x)\, dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or ∓∞, such that the above integral converges.

Definition 2: [2]

The Al-Tememe transformation for the function $f(x)$; $x > 1$ is defined by the following integral:

$$\mathcal{T}\,[f(x)] = \int_1^\infty x^{-p} f(x)\, dx = F(p)$$

such that this integral is convergent , p is positive constant

Property 1: [2]
This transformation is characterized by the linear property ,that is
$\mathcal{T}\,[Af(x)+Bg(x)]\;=A\mathcal{T}[f(x)]\;+B\mathcal{T}[g(x)]$ **,**
Where A ,B are constants ,the functions $f(x)$, $g(x)$ are defined when $x>1$.
The Al-Tememe transform for some fundamental functions are given in table(1) **[2]** :

ID	Function,**f(x)**	$F(p)=\int_1^\infty x^{-p} f(x)\,dx = \mathcal{T}\,[f(x)]$	Regional of convergence
1	k ; k =constant	$\frac{k}{p-1}$	**p** > **1**
2	x^n , $n\in R$	$\frac{1}{p-(n+1)}$	**p** > **n** + **1**
3	lnx	$\frac{1}{(p-1)^2}$	**p** > **1**
4	$x^n lnx$, $n\in R$	$\frac{1}{[p-(n+1)]^2}$	**p** > **n** + **1**
5	$sin(alnx)$	$\frac{a}{(p-1)^2+a^2}$	**p** > **1** a =constant
6	$cos(alnx)$	$\frac{p-1}{(p-1)^2+a^2}$	**p** > **1** a =constant
7	$sinh(alnx)$	$\frac{a}{(p-1)^2-a^2}$	$\lvert \mathbf{p-1}\rvert > \boldsymbol{a}$ a =constant
8	$cosh(alnx)$	$\frac{p-1}{(p-1)^2-a^2}$	$\lvert \mathbf{p}-1\rvert > \boldsymbol{a}$ a =constant

table(1)

From the Al-Tememe definition and the above table, we get:

Theorem 1:
If $\mathcal{T}\,f(x)\;=\;F(p)$ and a is constant, then $\mathcal{T}\,f\,(x^{-a})\;=F(p+a)$.see **[2]**

Definition 3: [2]
Let $f(x)$ be a function where ($x\;>1$) and $\mathcal{T}\,f(x)\;=F(p)$, $f(x)$ is said to be an inverse for the Al-Tememe transformation and written as
$\mathcal{T}^{-1}\,F(p)\;=f(x)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function.

Property 2: [2]
If $\mathcal{T}^{-1}\,F_1(p)\;=f_1(x)$, $\mathcal{T}^{-1}\,F_2(p)\;=f_2(x)$,…, $\mathcal{T}^{-1}\,F_n(p)\;=f_n(x)$ and a_1 , a_2 , ... a_n are constants, then
$\mathcal{T}^{-1}[a_1F_1(p)+a_2F_2(p)+\cdots+a_nF_n(p)]=a_1f_1(x)+a_2f_2(x)+\cdots+a_n\,f_n(x)$

Theorem 2: [2]
If the function $f(x)$ is defined for $x > 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \dots, f^{(n)}(x)$ are exist then:

$$\mathcal{T}\left[x^n f^{(n)}(x)\right] = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \cdots$$
$$-(p-n)\,(p-(n-1))\ \dots (\,p-2)\,f(1)\ + (p-n)!\,F(p)$$

Definition 4: [3]
A function $f(x)$ is piecewise continuous on an interval $[a,b]$ if the interval can be partitioned by a finite number of points
$a = x_0 < x_1 < \cdots < x_n = b$ such that:
1. $f(x)$is continuous on each subinterval (x_i, x_{i+1}), for $i = 0, 1, 2, \dots, n-1$
2. The function f has jump discontinuity at x_i , thus
$\left|\lim_{x\to x_i^+} f(x)\right| < \infty\ , i = 0, 1, 2, \dots, n-1;$
$\left|\lim_{x\to x_i^-} f(x)\right| < \infty\ , i = 0, 1, 2, \dots, n$

Al-Tememe Transform Method for Solving linear Systems of Ordinary Differential Equations:

Let us consider, we have a linear system of ordinary differential equation of second order with variable coefficients which we can write it by :

$$x^2 y_1'' + b_1 x y_1' = a_{11} y_1 + a_{12} y_2 + g_1(x) \qquad \cdots(1)$$
$$x^2 y_2'' + b_2 x y_2' = a_{21} y_1 + a_{22} y_2 + g_2(x)$$

Subject to some initial conditions $y_1(1), y_1'(1), y_2(1)$ *and* $y_2'(1)$.
Where b_1 , b_2, a_{11} , a_{12} , a_{21} and a_{22} are constants , y_1' *and* y_1'' are derivatives of function $y_1(x)$, and y_2' *and* y_2'' are derivatives of function $y_2(x)$, such that $y_1(x)$ and $y_2(x)$ are continuous functions and the ($\mathcal{T}$.T) of $g_1(x)$ and $g_2(x)$are known.
For solving the system (1) we take ($\mathcal{T}$.T) to both sides of it , and after simplification we put $Y_1=\mathcal{T}(y_1)$, $Y_2=\mathcal{T}(y_2)$,$G_1=\mathcal{T}(g_1)$, $G_2=\mathcal{T}(g_2)$ so,we get:

$$(p-2)(p-1)Y_1 - (p-2)y_1(1) - y_1'(1) + b_1(p-1)Y_1 - b_1 y_1(1)$$
$$= a_{11}Y_1 + a_{12}Y_2 + G_1\,(p)$$

$$(p-2)(p-1)Y_2 - (p-2)y_2(1) - y_2'(1) + b_2(p-1)Y_2 - b_2 y_2(1)$$
$$= a_{21}Y_1 + a_{22}Y_2 + G_2\,(p)$$

Hence,

$$(p^2 + c_1 p + c_2)Y_1 - a_{12}Y_2$$

$$= (p-2)y_1(1) + y_1'(1) + b_1 y_1(1) + G_1\,(p) \qquad \cdots(2)$$

Also,

$$-a_{21}Y_1 + (p^2 + d_1 p + d_1)Y_2$$
$$= (p-2)y_2(1) + y_2'(1) + b_2 y_2(1) + G_2\,(p) \qquad \cdots(3)$$

Where $c_1 = b_1 - 3$, $c_2 = 2 - b_1 - a_{11}$, $d_1 = b_2 - 3$
$d_2 = 2 - b_2 - a_{22}$ and

By multiplying eq. (2) by $(p^2 + d_1 p + d_1)$ and (3) by a_{12} .
and collecting the result terms we have :

$$Y_1 = \frac{q_1(p)}{h_1(p)} \quad ; \quad h_1(p) \neq 0 \qquad \text{... (4)}$$

By similar method ,we find

$$Y_2 = \frac{q_2(p)}{h_2(p)} \quad ; \quad h_2(p) \neq 0 \quad , \qquad \text{... (5)}$$

where q_1 , q_2 , h_1 and h_2 are polynomials of p , such that the degree of q_1 is less than the degree of h_1 and the degree of q_2 is less than the degree of h_2 .
By taking the inverse of Al-Tememe transform ($\mathcal{T}^{-1}$.T) to both sides of equations (4) and (5) we get:

$$y_1 = \mathcal{T}^{-1}\left[\frac{q_1(p)}{h_1(p)}\right]$$

$$\cdots(6)$$

$$y_2 = \mathcal{T}^{-1}\left[\frac{q_2(p)}{h_2(p)}\right]$$

Equations (6) represents the general solution of system (1) which we can be written it as follows:

$$y_1 = A_1 k_1(x) + A_2 k_2(x) + \cdots + A_m k_m(x)$$

$$y_2 = B_1 \rho_1(x) + B_2 \rho_2(x) + \cdots + B_m \rho_m(x)$$

Where λ_1 , λ_2 ,…, λ_m and ρ_1 , ρ_2 ,…, ρ_m are functions of x , and A_1 , A_2 ,…, A_m are constants , which is number equals to the degree of $h_1(p)$ also B_1 , B_2 ,…, B_m are constants, which is , number equals to the
degree of $h_2(p)$.
To find the values of constants of A_1 , A_2 ,…, A_m and B_1 , B_2 ,…, B_m we used partial fraction decomposition .

***Example*(1):**For solving the system

$$x^2 y_1'' - x y_1' = -y_1 + y_2 + x^{-3} \quad ; \; y_1(1) = 0 \;, y'_1(1) = 0$$

$$x^2 y_2'' + 7x y_2' = y_1 - 9y_2 + x \quad ; \; y_2(1) = 0 \;, y'_2(1) = 0$$

Sol:

We take Al-Tememe transform to both sides of above system and we get :

$$(p-2)(p-1)Y_1 - (p-2)y_1(1) - y_1'(1) - (p-1)Y_1 + y_1(1)$$

$$= -Y_1 + Y_2 + \frac{1}{p+2} \qquad ; \; p > -2 \qquad \cdots (7)$$

$$(p-2)(p-1)Y_2 - (p-2)y_2(1) - y_1'(1) + 7(p-1)Y_2 - 7y_2(1)$$

$$= Y_1 - 9Y_2 + \frac{1}{p-2} \qquad ; \; p > -2 \qquad \cdots (8)$$

After simplification eq. (7) and eq. (8) we get

$$(p-2)^2 Y_1 - Y_2 = \frac{1}{p+2} \qquad \cdots (9)$$

$$(p+2)^2 Y_2 - Y_1 = \frac{1}{p-2} \qquad \cdots (10)$$

By multiplying eq. (9) by $(p+2)^2$ and eq. (10) by 1 we get:

$$(p-2)^2(p+2)^2Y_1-(p+2)^2Y_2 \;=\; (p+2)$$
$$-Y_1 \quad + \; (p+2)^2\,Y_2 \;=\frac{1}{p-2}$$

we get :

$$Y_1=\frac{1}{(p-2)\;(p^2-5)}$$

and

$$Y_2=\frac{1}{(p+2)\;(p^2-5)}$$

Therefore , after using $\mathcal{T}^{\,-1}\,.T$ we get:

$$y_1=\mathcal{T}^{\,-1}\left[\frac{A_1}{p-2}+\frac{B_1p+C_1}{p^2-5}\right]$$
$$y_2=\mathcal{T}^{\,-1}\left[\frac{A_2}{p+2}+\frac{B_2p+C_2}{p^2-5}\right]$$

And after using partial fractions decomposition we get the equations:

$$A_1+B_1=0$$
$$-2B_1+C_1=0$$
$$-5A_1-2C_1=1$$

And hence ,

$A_1=-1\;,\;B_1=1\,,C_1=2$

$$\Longrightarrow\; y_1=-x+x^{-1}cosh\sqrt{5}\;lnx+\frac{2\sqrt{5}}{5}x^{-1}sinh\sqrt{5}\;lnx$$

$$A_2+B_2=0$$
$$2B_2+C_2=0$$
$$-5A_2+2C_2=1$$

Also we get:

$A_2=-1\;,\;B_2=1\,,C_2=-2$

$$\Longrightarrow\; y_2=-x^3+x^{-1}cosh\sqrt{5}\;lnx-\frac{2\sqrt{5}}{5}x^{-1}sinh\sqrt{5}\;lnx$$

***Example* (2):** For solving the system

$x^2y_1''+xy_1'=y_2+x^{-2}$ $\qquad y_1(1)=0\;,y'_1(1)=0$

$x^2y_2''+5xy_2'=y_1-4y_2+lnx$ $\qquad y_2(1)=0\;,y'_2(1)=0$

Sol: We take Al-Tememe transform to both sides of above system and we get :

$$(p-2)(p-1)Y_1-(p-2)y_1(1)-y_1'(1)+(p-1)Y_1-y_1(1)$$
$$=Y_2+\frac{1}{p+1} \qquad \cdots(11)$$
$$(p-2)(p-1)Y_2-(p-2)y_2(1)-y_1'(1)+5(p-1)Y_2-5y_2(1)$$
$$=Y_1-4Y_2+\frac{1}{(p-1)^2} \qquad \cdots(12)$$

After simplification eq. (11) and eq. (12) we get

$$(p-1)^2Y_1 - Y_2 = \frac{1}{p+1} \quad ; \quad p > -1 \qquad \cdots (13)$$

$$(p+1)^2Y_2 - Y_1 = \frac{1}{(p-1)^2} \quad ; \quad p > 1 \qquad \cdots (14)$$

By multiplying eq. (13) by $(p+1)^2$ and eq. (14) by 1 we get:

$$(p-1)^2(p+1)^2Y_1 - (p+1)^2Y_2 = (p+1)$$

$$-Y_1 + (p+2)^2\, Y_2 = \frac{1}{(p-1)^2}$$

we get:

$$Y_1 = \frac{p^3 - p^2 - p + 2}{p^2\ (p-1)^2(p^2-2)}$$

and

$$Y_2 = \frac{p+2}{p^2\ (p+1)(p^2-2)}$$

Therefore , after using $\mathcal{T}^{-1}.T$ and partial fractions decomposition

we get:

$$y_1 = \mathcal{T}^{-1}\left[\frac{A_1}{p} + \frac{B_1}{p^2} + \frac{C_1}{p-1} + \frac{D_1}{(p-1)^2} + \frac{E_1p + F_1}{p^2-2}\right]$$

$A_1 + C_1 + E_1 = 0$

$-2A_1 + B_1 - C_1 + D_1 - 2E_1 + F_1 = 0$

$-A_1 - 2B_1 - 2C_1 + E_1 - 2F_1 = 1$

$4A_1 - B_1 + 2C_1 - 2D_1 + F_1 = -1$

$-2A_1 + 4B_1 = -1$

$-2B_1 = 2$

Hence,

$A_1 = {}^{-3}/_2$, $B_1 = -1$, $C_1 = 0$, $D_1 = -1$, $E_1 = {}^{3}/_2$, $F_1 = 2$

$\Rightarrow\ y_1 = {}^{-3}/_2\, x^{-1} - x^{-1}\, lnx - lnx + {}^{3}/_2\, x^{-1} cosh\sqrt{2}\, lnx + \sqrt{2}x^{-1} sinh\sqrt{2}\, lnx$

By the same method we find :

$A_2 = {}^{1}/_2$, $B_2 = -1$, $C_2 = -1$, $D_2 = {}^{1}/_2$, $E_2 = 0$

$\Rightarrow\ y_2 = {}^{1}/_2\, x^{-1} cosh\sqrt{2}\, lnx + {}^{1}/_2\, x^{-1} - x^{-1}\, lnx - x^{-2}$

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