

Applying Boundary Element Method to Solve the Heat Conduction Problems Using Constant Element

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Abstract

In this work two-dimensional boundary element method (BEM) using constant element is mathematically and numerically investigated and applied to solve differential equations of the heat conduction problems and temperature distribution of solid bodies to be found . A computer program in Fortran 90 language is constructed to solve heat conduction problems in steady-state regime by using boundary element technique with constant element . The effects of elements number (ne) and internal points number (np) of the boundary model on the temperature distribution have been investigated . A more accuracy of results and more near with exact solution are obtained by increasing number of boundary elements and internal points of the boundary model .

Keywords: Boundary Element Method (BEM) , Heat Conduction , Constant Element , Steady-State .

تطبيق طريقة العناصر المحيطية لحل مشاكل التوصيل الحراري بأستخدام العنصر الثابت

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الخلاصة

في هذا البحث أجريت دراسة رياضية وعددية لطريقة العناصر المحيطية (BEM) بأستخدام العنصر الثابت وتطبيقها في حل المعادلات التفاضلية لمشاكل التوصيل الحراري ثنائية الأبعاد وأيجاد التوزيع الحراري للأجسام الصلبة . تم بناء برنامج حاسوبي بلغة فورتران 90 لحل مشاكل التوصيل الحراري في الحالة المستقرة بأستخدام تقنية العناصر المحيطية و العنصر الثابت. تم دراسة تأثير كل من عدد العناصر (ne) وعدد النقاط الداخلية (np) للنموذج المحيطي على التوزيع الحراري . لقد أظهرت النتائج أن زيادة عدد العناصر المحيطية والنقاط الداخلية للنموذج المحيطي تعطي نتائج أكثر دقة وتقاربا مع الحل المثالي .

1. Introduction

The boundary element method (BEM) is a numerical computational technique for solving partial differential equations (PDE) which have been formulated as integral equations based on the boundary integral equation (BIE). It is a mature technique for using in the numerical analysis of a large variety of problems in science and engineering and it is suited to problem solving with infinite domains such as soil mechanics , hydraulics , fractures mechanics , stress analysis , electromagnetic , acoustics , fluid mechanics and potential problems .

The boundary element method (BEM) is a technique which offers important advantages over domain-type solutions like finite difference method (FDM) and finite element method (FEM). A domain-based methods like finite element method (FEM) requires discretization the entire domain of the model while the boundary element method (BEM) requires discretization of boundary only. Therefore the boundary element method (BEM) is advantageous when the domain extends to infinity or the shape of the boundary is complex. The main advantage of the boundary element method (BEM) are much smaller systems of equations and considerable reduction in the data required to run a problem , as well as the numerical accuracy of boundary elements is generally greater than that of finite elements.

Nakayama, T. (1983) ^[1] used a boundary element technique to solve problems of nonlinear two-dimensional water wave and compared the numerical results with exact solution. **Grilli et al. (1989)** ^[2] derived a computational model for highly nonlinear two-dimensional water waves based on boundary element method, and investigated problems of the wave generation and absorption. They found very good agreement with the analytical solutions. **Lesnic et al. (1996)** ^[3] applied the boundary element method to solve the inverse heat conduction problems. They supplemented the theoretical considerations and verified the correctness of algorithms proposed by the numerical examples . **Zhu et al. (1998)** ^[4] applied a meshless local boundary integral equation method using the moving least squares approach to solve nonlinear boundary value problems. **Sladek et al. (2004)** ^[5] used a meshless local boundary integral equation method (LBIE) to solve problems of two-dimensional heat conduction in non-homogeneous solids. They obtained high accuracy and efficiency of numerical results. **Ramsak, M. and Skerget, L. (2005)** ^[6] developed two-dimensional model for complex turbulent flow based on boundary element method. They found excellent agreement of the numerical results with the results of commercial CFD package (CFX 4.0)

based on finite volume method (FVM). **Abreu et al. (2010)** ^[7] solved the basic integral equation of time-domain boundary element method for transient heat conduction problems using a convolution quadrature method. They found a good agreement of numerical results of the test examples with analytical solutions . **Sarbu and Popina (2011)** ^[8] solved the differential equations of heat conduction and determined the temperature distribution in orthotropic body and in pipe insulation using finite element and boundary element methods. They implemented the models in two developed computer programs. They showed good performance of the proposed numerical models. **Werner-Juszczuk A. J. and Sorko S. A. (2012)** ^[9] developed and verified new mathematical (BEM) algorithm for two-dimensional transient heat conduction problem with periodic boundary condition. They obtained a good agreement of the numerical simulation of flat plate under non-zero initial condition with analytical method using a computer program written in Fortran language.

In the present investigation , the boundary element method (BEM) with constant element is applied to solve two-dimensional steady state heat conduction problems . The boundary integral equation (BIE) and boundary element equation (BEE) are derived for steady state heat conduction. Constant boundary element type is analyzed and used in the solution . The influence of elements number (ne) and internal points number (np) of the boundary model on the temperature distribution and accuracy of the results are investigated and analyzed .

2. Derivation of the Boundary Integral Equation for Heat Conduction Problem

The governing differential equation of two-dimensional steady-state heat conduction can be written as follows ^[10,11] :

$$\nabla^2 u(x, y) + q(x, y) \equiv \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} + q(x, y) = 0 \quad \text{in } \Omega \quad \dots\dots(1)$$

where , $u(x, y)$ is known potential function $\in \Omega$, Ω is a bounded domain or region as shown in Fig.1 , $q(x, y) = q'(x, y) / k$, $q'(x, y)$ is a heat generation , and k is a thermal conductivity and equal constant for isotropic materials . The boundary conditions applicable are as follows ^[12,13] :

i. Dirichlet boundary condition :

$$u(x, y) = \bar{u}(x, y) \quad \text{on } \Gamma_1 \quad \dots\dots\dots(2)$$

ii. Neumann boundary condition :

$$q(x, y) = \bar{q}(x, y) = l \frac{\partial u}{\partial x} + m \frac{\partial u}{\partial y} \quad \text{on } \Gamma_2 \quad \dots\dots\dots(3)$$

where, \bar{u} and \bar{q} are known functions , l and m are the directional cosines of tangential unit vector (\hat{t}), Γ_1 and Γ_2 are the parts of the boundary Γ of the domain (Ω) .

Considering the source point $\mathbf{i} (x_i, y_i) \in \Omega$, boundary points $\mathbf{a} (x_a, y_a)$ and $\mathbf{b} (x_b, y_b) \in \Gamma$, \hat{i} and \hat{j} are a unit vectors . The length of line \mathbf{ab} can be expressed as follows :

$$\vec{ab} = \vec{ds} = (x_b - x_a) \hat{i} + (y_b - y_a) \hat{j} \quad \dots\dots\dots(4)$$

or ,

$$\vec{ds} = (\Delta x) \hat{i} + (\Delta y) \hat{j} \quad \dots\dots\dots(5)$$

Hence ,

$$ds = |\vec{ds}| = (dx^2 + dy^2)^{1/2} \quad \dots\dots\dots(6)$$

The normal and tangential unit vectors , \hat{n} and \hat{t} respectively are :

$$\hat{n} = \frac{dx}{dn} \hat{i} + \frac{dy}{dn} \hat{j} = l \hat{i} + m \hat{j} \quad \dots\dots\dots(7)$$

$$\hat{t} = \hat{s} = \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} = -m \hat{i} + l \hat{j} \quad \dots\dots\dots(8)$$

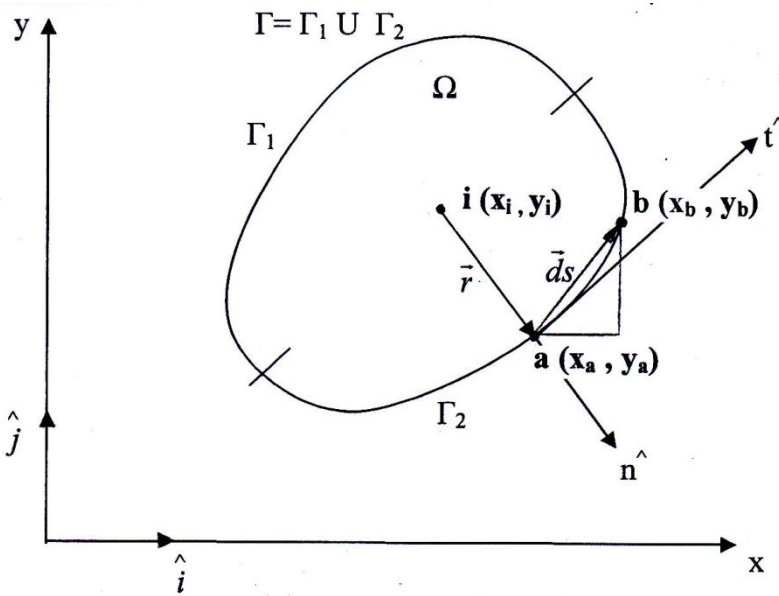


Fig. 1 Discretization of the Domain (Ω) and Boundary Conditions

Applying the weighted residual theorem , and integrated by parts (Green-Gauss theorem)^[12,13,14] of equation (1), leads to inverse weighted residual expression (IWRE) :

$$\iint_{\Omega} (\nabla^2 u^*) u dx dy + \iint_{\Omega} u^* q dx dy + B.I.terms = 0 \dots\dots\dots(9)$$

where ,

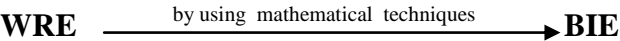
$$u^* = \frac{1}{2\pi} \log\left(\frac{1}{r}\right) = \text{fundamental solution}^{[8,12,13]}$$

$$r = [(x - x_i)^2 + (y - y_i)^2]^{1/2}$$

$$q = \frac{\partial u}{\partial n} = \text{potential derivative}$$

$$B.I. terms = \oint_{\Gamma} u^* q ds - \oint_{\Gamma} q^* u ds = \text{boundary integral terms}$$

The boundary integral equation (BIE) is obtained from weighted residual expression (WRE) , i.e. :



Using the fundamental equation $\nabla^2 u^* + \delta(x - x_i, y - y_i) = 0$ ^[8,9] which contains a weighting function (u^*) . Eq. (9) can be reduced to :

$$- \iint_{\Omega} \delta(x - x_i, y - y_i) u dx dy + \iint_{\Omega} u^* q dx dy + B.I.terms = 0 \dots\dots\dots(10)$$

Using the following theorem ^[12,15,16] :

$$\iint_{\Omega} \delta(x - x_i, y - y_i) u dx dy = c_i u(x_i, y_i) \dots\dots\dots(11)$$

where ,

$c_i = 1$ when source point $\mathbf{i}(x_i, y_i)$ lies inside the domain Ω .

$c_i = 0$ when source point $\mathbf{i}(x_i, y_i)$ lies offset of the domain Ω .

$c_i = 0.5$ when source point $\mathbf{i}(x_i, y_i)$ lies on the boundary of the domain Ω .

$\delta(x - x_i, y - y_i) = \text{Dirac delta function}$.

Equation (10) becomes :

$$- c_i u(x_i, y_i) + \iint_{\Omega} u^* q dx dy + B.I.terms = 0 \dots\dots\dots(12)$$

or , in another expression :

$$c_i u_i + \oint_{\Gamma} q^* u ds - \oint_{\Gamma} u^* q ds = \iint_{\Omega} u^* q dx dy \dots\dots\dots(13)$$

Finally , the boundary integral equation (BIE) can be expressed as follows :

$$c_i u_i + \oint_{\Gamma} q^* u ds - \oint_{\Gamma} u^* q ds = b_i \dots\dots\dots(14)$$

where , $b_i = b(x_i, y_i) = \iint_{\Omega} u^* (x - x_i, y - y_i) q(x, y) dx dy$ = heat generation term

$$q^* = \frac{\partial u^*}{\partial n} = \frac{\partial u^*}{\partial r} \cdot \frac{\partial r}{\partial n} = \text{normal derivative of fundamental solution}$$

$$\frac{\partial u^*}{\partial r} = \frac{-1}{2\pi r}$$

$$\frac{\partial r}{\partial n} = l \frac{\partial r}{\partial x} + m \frac{\partial r}{\partial y} = \frac{l(x - x_i) + m(y - y_i)}{r}$$

3. Derivation of the Boundary Element Equation with Constant Element

The boundary Γ , can be divided into a number of sub-boundaries called boundary elements (Γ_e) connects by boundary nodes. Consider $f(x,y)$ is the continuous field function over boundary Γ , then can be deduced :

$$\oint f ds = \sum_{e=1}^{ne} \int_{\Gamma_e} f ds \dots\dots\dots(15)$$

Equation (14) can be rewritten as follows :

$$c_i u_i + \sum_{e=1}^{ne} \left[\int_{\Gamma_e} q^* (x - x_i, y - y_i) u(\Gamma_e) ds \right] = \sum_{e=1}^{ne} \left[\int_{\Gamma_e} u^* (x - x_i, y - y_i) q(\Gamma_e) ds \right] + b_i \dots(16)$$

where , $u(\Gamma_e)$ and $q(\Gamma_e)$ represents of potential values $u(x,y)$ and flux $q(x,y)$ over sub-boundary Γ_e . The parameters $u(\Gamma_e)$ and $q(\Gamma_e)$ may be approximated by means of interpolation expression in term of nodal values at some boundary nodes specified over boundary element Γ_e , hence :

$$u(\Gamma_e) = \sum_{j=1}^n N_j(s) u_j \dots\dots\dots(17)$$

$$q(\Gamma_e) = \sum_{j=1}^n N_j(s) q_j \dots\dots\dots(18)$$

where , u_j and q_j are nodal values at j^{th} nodes , $N_j(s)$ are shape functions . For constant boundary element , the number of boundary elements (ne) are equal of boundary nodes number (n) , then $u(\Gamma_e) \approx u_e$, and $q(\Gamma_e) \approx q_e$.

Define the matrices $(g_i)_e$ and $(h_i)_e$:

$$(g_i)_e = \int_{\Gamma_e} u^* ds = \int_{\Gamma_e} u^*(x - x_i, y - y_i) ds \dots\dots\dots(19)$$

$$(h_i)_e = \int_{\Gamma_e} q^* ds = \int_{\Gamma_e} q^*(x - x_i, y - y_i) ds \dots\dots\dots(20)$$

Then, boundary element equation (BEE) for heat conduction can be expressed as follows :

$$c_i u_i + \sum_{e=1}^{ne} (h_i)_e u_e = \sum_{e=1}^{ne} (g_i)_e q_e + b_i \dots\dots\dots(21)$$

3.1 Geometry of the Constant Boundary Element

Considering the source point $i (x_i, y_i)$ is a midpoint of the line \mathbf{ab} , then , $\mathbf{x}_i = (\mathbf{x}_a + \mathbf{x}_b) / 2$ and $\mathbf{y}_i = (\mathbf{y}_a + \mathbf{y}_b) / 2$. If the field point (x, y) moves on sub-boundary Γ_e , the intrinsic parameter ζ will be employed such that ^[16,17] :

$$\zeta = 0 \quad \text{at point a } (x_a, y_a)$$

$$\zeta = 1 \quad \text{at point b } (x_b, y_b)$$

$$\zeta = 0.5 \quad \text{at midpoint i } (x_i, y_i) \text{ when using constant element .}$$

$$x(\zeta) = x_a + \zeta(x_b - x_a) \dots\dots\dots(22)$$

$$y(\zeta) = y_a + \zeta(y_b - y_a) \dots\dots\dots(23)$$

Then , Eq.(6) becomes :

$$ds = \left(\left(\frac{dx}{d\zeta} \right)^2 + \left(\frac{dy}{d\zeta} \right)^2 d\zeta \right)^{1/2} = J d\zeta \dots\dots\dots(24)$$

where, J is the boundary element length .

$$J = [(x_b - x_a)^2 + (y_b - y_a)^2]^{1/2} \dots\dots\dots(25)$$

Hence , equations (19) and (20) becomes :

$$g_{ie} = \frac{-1}{2\pi} \int_0^1 \ln[(x_a + \zeta(x_b - x_a) - x_i)^2 + (y_a + \zeta(y_b - y_a) - y_i)^2]^{1/2} J d\zeta \dots\dots\dots(26)$$

$$h_{ie} = \frac{-1}{2\pi} \int_0^1 \frac{[l(x_a + \zeta(x_b - x_a) - x_i) + m(y_a + \zeta(y_b - y_a) - y_i)] J d\zeta}{(x_a + \zeta(x_b - x_a) - x_i)^2 + (y_a + \zeta(y_b - y_a) - y_i)^2} \dots\dots\dots(27)$$

Rewrite equation (21) :

$$c_i u_i + \sum_{e=1}^{ne} (h_i)_e u_e - \sum_{e=1}^{ne} (g_i)_e q_e = b_i \dots\dots\dots(28)$$

Equation (28) can be expressed in a matrix form :

$$(\underline{c} + \underline{h})\underline{u} - \underline{g}q = \underline{b} \tag{29}$$

or

$$\begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & c_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} (h_1)_1 & (h_1)_2 & \dots & (h_1)_{ne} \\ (h_2)_1 & (h_2)_2 & \dots & (h_2)_{ne} \\ \dots & \dots & \dots & \dots \\ (h_n)_1 & (h_n)_2 & \dots & (h_n)_{ne} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} - \begin{bmatrix} (g_1)_1 & (g_1)_2 & \dots & (g_1)_{ne} \\ (g_2)_1 & (g_2)_2 & \dots & (g_2)_{ne} \\ \dots & \dots & \dots & \dots \\ (g_n)_1 & (g_n)_2 & \dots & (g_n)_{ne} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \tag{30}$$

Define the matrix , $\underline{H} = \underline{c} + \underline{h}$ then equation (29) becomes :

$$\underline{H}\underline{u} - \underline{g}q = \underline{b} \tag{31}$$

or ,

$$\underline{H}\underline{u} = \underline{g}q + \underline{b} \tag{32}$$

Equation (32) generates system of n-equations with 2n-unknowns (\underline{u} and q) . After application the boundary conditions of problem , the number of unknowns is reduced to (n). Values of the off-diagonal coefficients \underline{h} and \underline{g} (when $n \neq ne$) are computed using four-point Gaussian quadrature formula^[16,18] as follows :

$$\underline{g} = \frac{J_{ne}}{2} \sum_{p=1}^4 u_p^* w_p \tag{33}$$

$$\underline{h} = \frac{J_{ne}}{2} \sum_{p=1}^4 q_p^* w_p \tag{34}$$

where, w_p = weight coefficients .

While the diagonal values of \underline{h} and \underline{g} (\underline{h}_d and \underline{g}_d respectively) , i.e. when $n = ne$ are calculated as follows :

$$\underline{g}_d = \frac{J_{ne}}{2\pi} \left[1 - \ln \frac{J_{ne}}{2} \right] \tag{35}$$

$$\underline{h}_d = \underline{h} + \frac{1}{2} \tag{36}$$

Rearranging equation (32) with unknowns vector \underline{X} and knows vector \underline{F} yields system of equations whose matrix form is :

$$\underline{A}\underline{X} = \underline{F} \tag{37}$$

where ,

$$\underline{A} = \begin{bmatrix} \underline{H}_{uu} & -\underline{g}_{ku} \\ \underline{H}_{ku} & -\underline{g}_{uu} \end{bmatrix} , \quad \underline{X} = \begin{bmatrix} \underline{u}_u \\ \underline{q}_u \end{bmatrix}$$

\underline{F} is the known vector obtained by multiplying matrix elements \underline{B} by the vector of known boundary values \underline{Y} and then adding with heat generation vector \underline{b} , i.e. :

$$\underline{F} = \underline{B}\underline{Y} + \underline{b} \quad \dots\dots\dots(38)$$

where,

$$\underline{B} = \begin{bmatrix} \underline{g}_{kk} & -\underline{H}_{uk} \\ \underline{g}_{uk} & -\underline{H}_{kk} \end{bmatrix} , \quad \underline{Y} = \begin{bmatrix} \underline{q}_k \\ \underline{u}_k \end{bmatrix}$$

\underline{b} is the heat generation vector, $\underline{b} = 0$ (when no heat generation). The subscripts (k) denoted known parameter and (u) denoted unknown parameter .

Equations from (28) to (38) are represents the numerical model based on boundary element method (BEM) using constant element for solution of two-dimensional steady-state heat conduction problems (potential problems). The whole set of equations is solved using Gauss elimination method .

4. Numerical Applications

A computer program was built in Fortran 90 language to solve the developed numerical model based on boundary element technique , which was implemented two applications , the first thick insulation beam and the second square plate with heat generation.

4.1 Thick Insulation Beam

The engineering applications for this problem is in electrical equipments , fins. Consider the heat conduction problem inside a thick insulation beam of dimensions $\mathbf{L} \times \mathbf{H}$, where $L= 10.0$ cm and $H= 2.0$ cm, the sides of length L are thermally isolated, and the lateral sides of plate thick H are subject to the temperatures 250 °C and 140 °C respectively. Different cases of the boundary elements number n_e and internal points n_p for boundary model are analyzed and computed as shown in Table 1 and Fig. 2 .

Table 1 Cases of the Model Analysis

Case	Analysis of the Model	
	Boundary Elements Number (ne)	Internal Points Number (np)
a	6	0
b	10	2
c	14	5
d	24	15

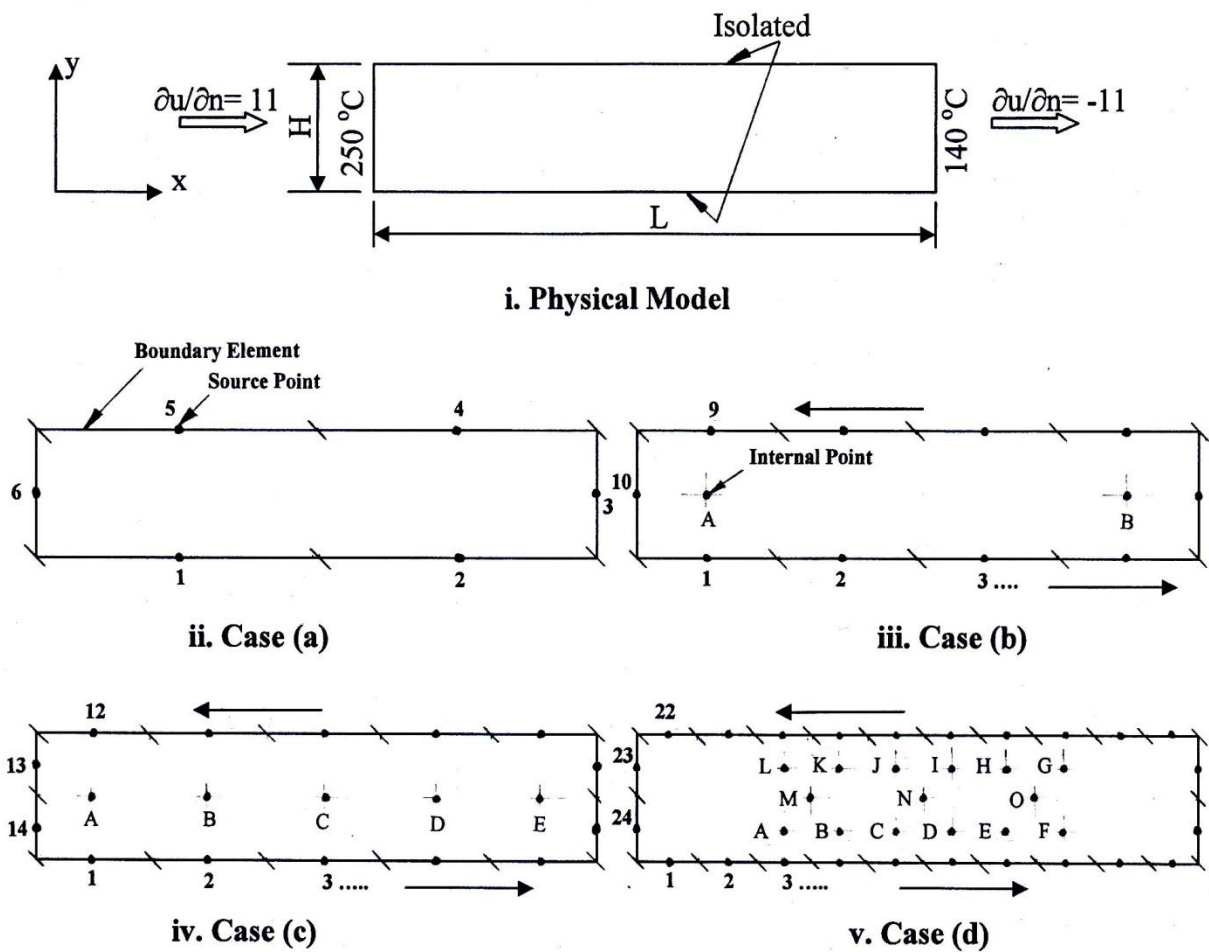


Fig. 2 Model Geometry and Boundary Element Idealization
 i. Physical Model ii. Case (a) iii. Case (b) iv. Case (c) v. Case (d)

The obtained numerical results from the constructed program are presented in Figs. 3 and 4 for the mentioned cases in Table 1 and comparatively with exact solution ^[19]. Fig. 5 illustrate the effects of boundary elements number (e) in the boundary model on values of the maximum potential derivative ($\partial u/\partial n$).

The absolute percentage relative (absolute error) of temperature % ϵ_{ui} and its derivative % $\epsilon_{(\partial u/\partial n)_i}$ is calculated as follows :

$$\% \epsilon_{ui} = \left(\frac{u_{iExact} - u_{iBEM}}{u_{iExact}} \right) \times 100 \dots\dots\dots(39)$$

$$\% \epsilon_{(\partial u/\partial n)_i} = \left(\frac{u_{(\partial u/\partial n)_iExact} - u_{(\partial u/\partial n)_iBEM}}{u_{(\partial u/\partial n)_iExact}} \right) \times 100 \dots\dots\dots(40)$$

For case **d** (24 boundary elements and 15 internal points) , the absolute percentage relative of temperature between exact solution and numerical solution based on boundary element method (BEM) is (% $\epsilon_{ui} < 0.8\%$) and its derivative (% $\epsilon_{(\partial u/\partial n)_i} < 3.2\%$) .

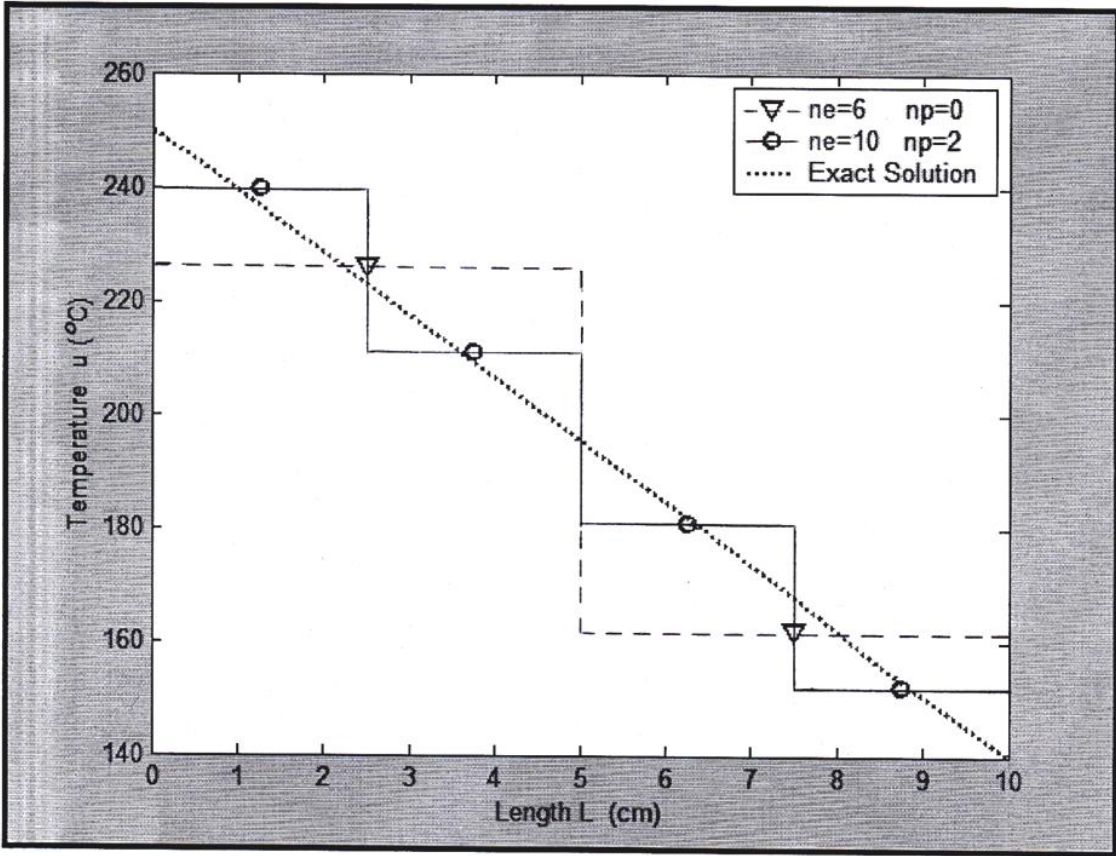


Fig. 3 Temperature Distribution of Case a (ne=6 , np=0) and Case b (ne=10 , np=2) with Exact Solution^[19]

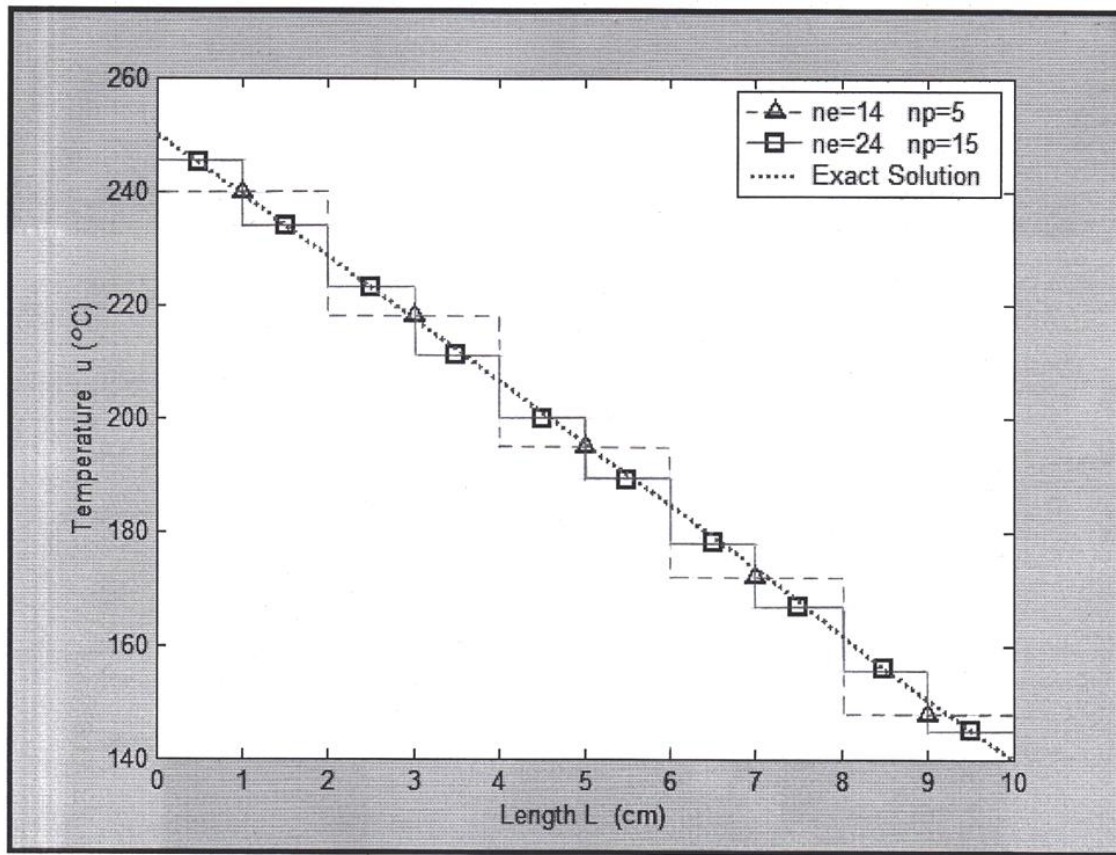


Fig. 4 Temperature Distribution of Case c (ne=14 , np=5) and Case d (ne=24 , np=15) with Exact Solution^[19]

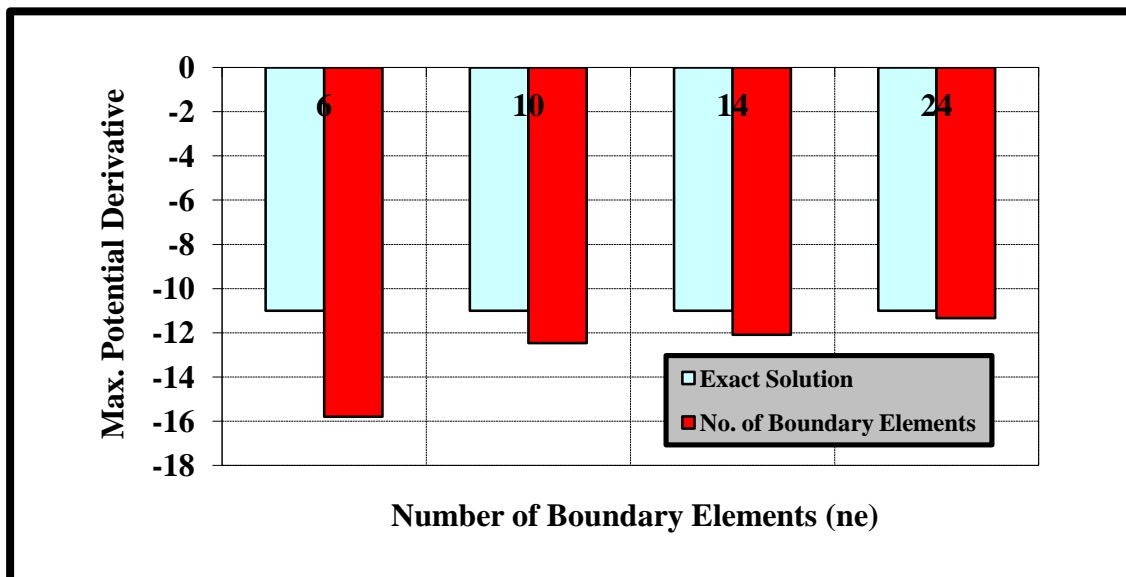


Fig. 5 Maximum Potential Derivative ($\partial u/\partial n$) Versus Boundary Elements Number (ne)

The numerical results of boundary element method (BEM) for the case **d** ($n_e = 24$ and $n_p = 15$) with exact solution are tabulated in Table 2 .

Table 2 Numerical Results of Boundary Element Method (BEM) for Case (d)

Source Point <i>i</i>	Coordinates		BEM Solution		Exact Solution		Absolute Error	
	x_i	y_i	u_i (°C)	$(\partial u/\partial n)_i$	u_i (°C)	$(\partial u/\partial n)_i$	% ϵ_{ui}	% $\epsilon_{(\partial u/\partial n)_i}$
1	0.5	0	244.88	0	245	0	0.048	0
2	1.5	0	233.88	0	234.25	0	0.157	0
3	2.5	0	222.84	0	223.25	0	0.183	0
4	3.5	0	211.76	0	212.5	0	0.348	0
5	4.5	0	200.67	0	201.5	0	0.412	0
6	5.5	0	189.57	0	191	0	0.748	0
7	6.5	0	178.46	0	179.75	0	0.717	0
8	7.5	0	167.36	0	168.25	0	0.528	0
9	8.5	0	156.26	0	156.5	0	0.153	0
10	9.5	0.5	145.15	0	145.25	0	0.068	0
11	10	1.5	140	-11.35	140	-11	0	3.181
12	10	2	140	-11.35	140	-11	0	3.181
13	9.5	2	145.15	0	145.25	0	0.068	0
14	8.5	2	156.27	0	156.5	0	0.146	0
15	7.5	2	167.36	0	168.25	0	0.528	0
16	6.5	2	178.46	0	179.75	0	0.717	0
17	5.5	2	189.55	0	191	0	0.76	0
18	4.5	2	200.66	0	201.5	0	0.416	0
19	3.5	2	211.76	0	212.5	0	0.348	0
20	2.5	2	222.83	0	223.25	0	0.187	0
21	1.5	2	233.88	0	234.25	0	0.157	0
22	0.5	2	244.89	0	245	0	0.044	0
23	0	1.5	250	11.23	250	11	0	2.090
24	0	0.5	250	11.23	250	11	0	2.090

A. Source (Boundary) Points *i*

Internal Point <i>p</i>	Coordinates		BEM Solution
	x_p	y_p	u_p (°C)
A	2.5	0.5	222.70
B	3.5	0.5	211.65
C	4.5	0.5	200.56
D	5.5	0.5	189.47
E	6.6	0.5	178.37
F	7.5	0.5	167.28
G	7.5	1.5	167.28
H	6.5	1.5	178.36
I	5.5	1.5	189.47
J	4.5	1.5	200.56
K	3.5	1.5	211.65
L	2.5	1.5	222.70
M	3	1	217.25
N	5	1	195.08
O	7	1	172.88

B. Internal Points *p*

4.2 Square Plate with Heat Generation

Many engineering applications for this problem like electronic components . Consider the heat conduction with uniform heat generation (q') problem of a square plate with dimensions $L \times L$, where $L= 10.0$ cm , the left and lower sides are thermally isolated , and the other sides of plate are subject to the temperatures 50 °C . Two cases of the boundary model are analyzed and compute , case **a** ($n_e= 8$, $n_p= 0$) and case **b** ($n_e= 40$, $n_p= 10$) as shown in Figs. 6, and 7.

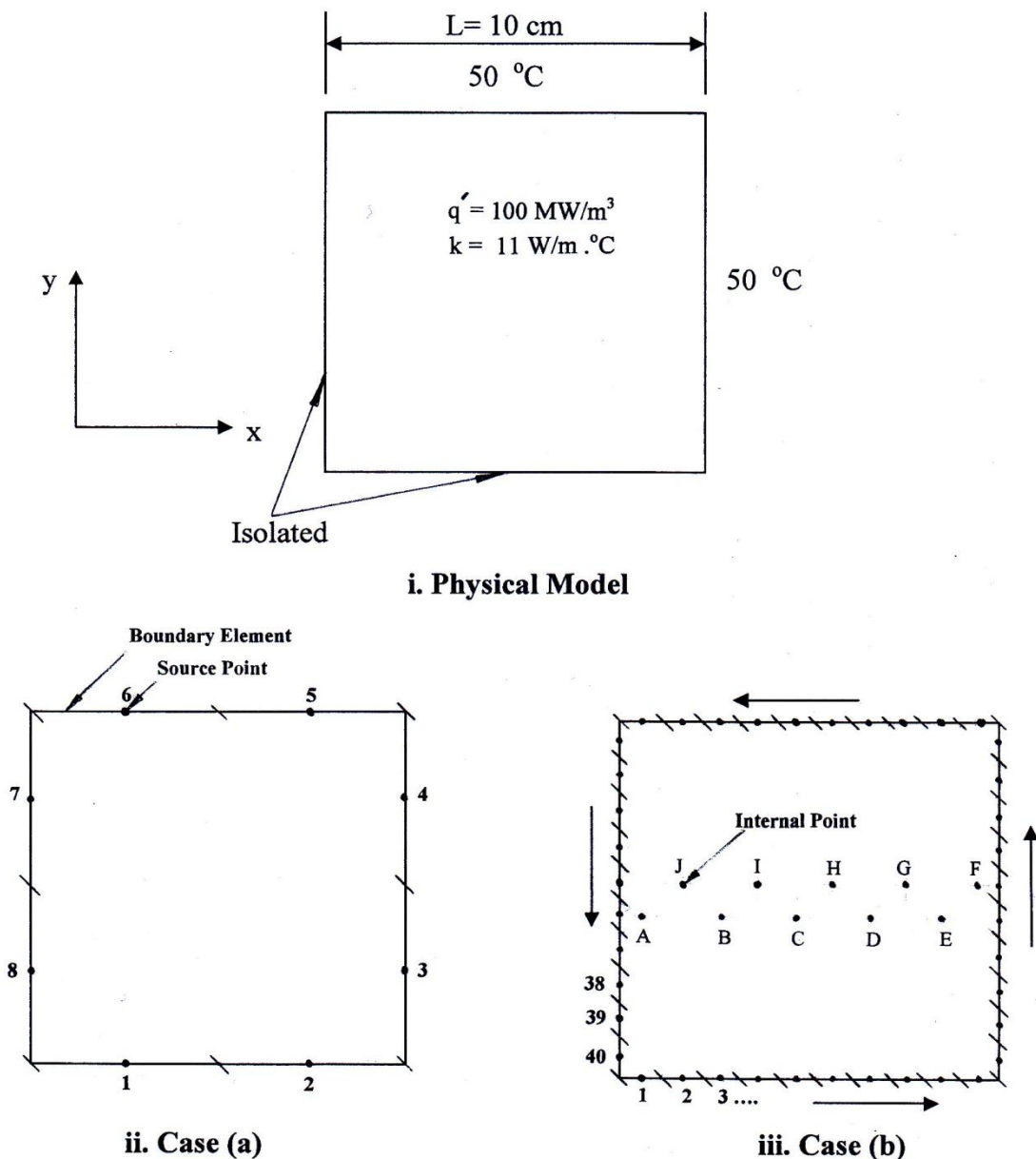


Fig. 6 Physical Model and Discretization

i. Physical Model ii. Case (a) iii. Case (b)

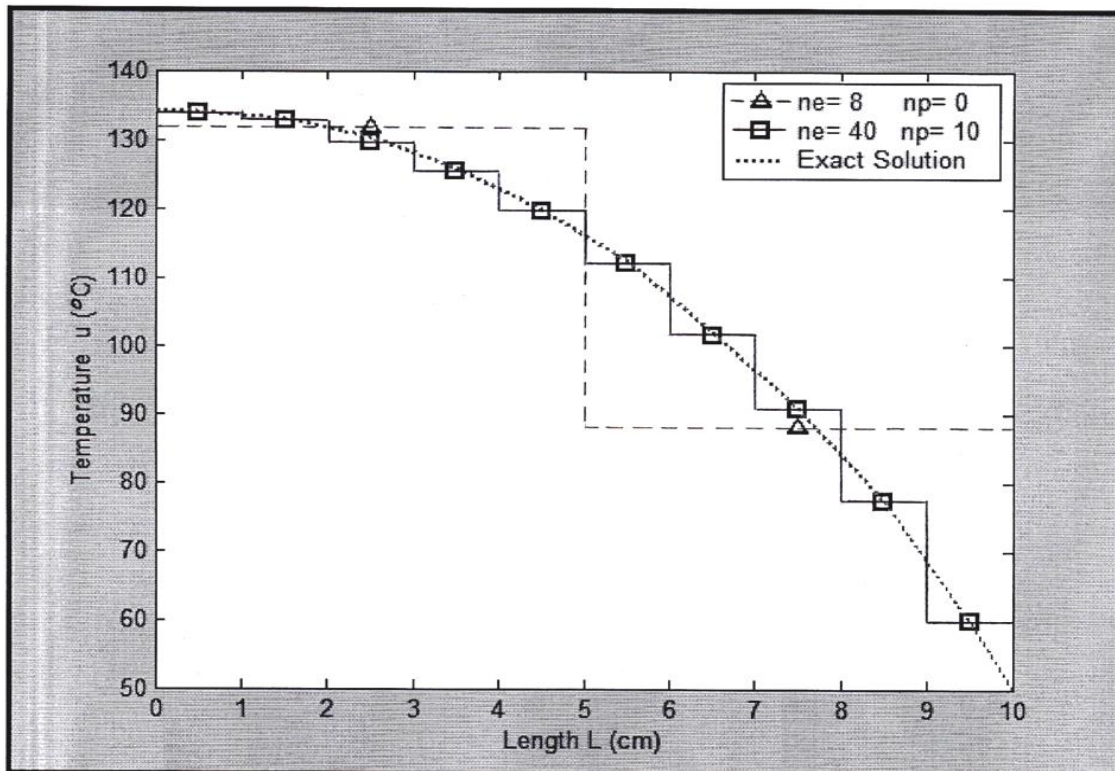


Fig. 7 Temperature Distribution Using Boundary Element Method for Two Cases a and b with Exact Solution^[19]

For case **b** (ne= 40 and np=10), the absolute percentage relative of temperature ($\% \epsilon_{ui}$) between exact solution^[19] and numerical solution using the built program based on boundary element algorithm is ($\% \epsilon_{ui} < 0.5 \%$).

5. Conclusions

In this work the boundary element technique is used to solve two-dimensional steady-state heat conduction problems. The relationship between the boundary element method (BEM) and the governing equation of two-dimensional heat conduction has been developed. Two numerical applications with different cases of the boundary model are investigated. The following conclusions can be drawn :

Increasing the boundary elements number (ne) and internal points number (np) results more accuracy and more convergence with exact solution. Maximum potential derivative ($\partial u / \partial n$) is very close with exact solution in case d of thick insulation beam when ne= 24 and np= 15. The comparison between exact and numerical solutions of studied applications proves the great accuracy of numerical model based on boundary element technique and excellent

convergence with exact solution. The boundary element method (BEM) can be successfully employed for analyzing and solving of many heat conduction problems .

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