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Perturbed Taylor expansion for bifurcation of solution of singularly parameterized perturbed ordinary differentia equations and differential algebraic equations

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Abstract:
In This paper deals with the study of singularity perturbed ordinary differential equation, and is considered the basis for obtaining the system of differential algebraic equations. In this study the we use implicit function theorem to solve for fast variable y to get a reduced model in terms of slow dynamics locally around x. It is well known that solving nonlinear algebraic equations analytical is quite difficult and numerical solution methods also face many uncertainties since nonlinear algebraic equations may have many solutions, especially around bifurcation points. We have used singularly perturbed ODE to study the bifurcation problem in Differential algebraic system. So for the first step we need to investigate the bifurcation problem in our original system
when $0 < \epsilon \leq 1$, for this purpose the known kinds bifurcations such as saddle node, transtritical and pitch fork has been studied by using Taylor expansion for one dimensional system. And for higher dimension we apply Sotomayor Theorem.
The second step is going to study bifurcation problem in DAE:

$$\dot{x} = f(x, y, \mu, 0)$$
$$0 = g(x, y, \mu, 0),$$

Where μ is bifurcation parameter. by converting such system to singularly perturbed ODE to make use the study in the first step:

$$\dot{x} = f(x, y, \mu)$$
$$\epsilon\dot{y} = g(x, y, \mu)$$

The method we used to convert DAEs to singular perturbed ODEs is PTE method. The bifurcation in index one DAEs is investigated by reduced the system to system with lower dimension by implicit function theorem. And for higher dimension index two DAEs we used Sotomayor Theorem. Also the singularity induced bifurcation for which this kind of bifurcation occurred in DAEs is studied by PTE method.

Keywords: Singularity perturbed theory, Differential-Algebraic Equations system, Bifurcation Theory.

1.INTRODUCTION:

Singular perturbation is a differential equation with another conditions having a small parameter that is multiplying the highest derivatives. The fundamental reason for studying singularity perturbed theory is to consider a problem with a small parameter ϵ and state a solution $x(\mu,\epsilon)$. A singular perturbation occurs whenever the limit of regular perturbation problem fails. Standard form of singularly perturbed ODEs:

$$\dot{x} = f(x,y,\mu,\epsilon)$$
$$\epsilon\dot{y} = g(x,y,\mu,\epsilon)$$

Where $f: R^n \times R^m \times R \to R^n, g: R^n \times R^m \times R \to R^m$
Then the differential-algebraic equation will be obtained by setting ϵ

$$\dot{x} = f(x,y,\mu,0)$$
$$0 = g(x,y,\mu,0),$$

Identify applicable funding agency here. If none, delete this. We apply PTE technique to convert a DAEs to singularly perturbed ODEs, time domain simulation is easily performed to this ODEs to observe dynamic responses, a simplified unreduced Jacobian matrix of the ODEs is introduced to perform eigenvalue analysis, the results shown that PTE technique satisfies preserves the bifurcation properties of the original DAEs, in the following the described method PTE.

$$\dot{x} = f(x,y,\mu,0)$$
$$0 = g(x,y,\mu,0)$$

is corresponding to singularly perturbed ODE:

$$\dot{x} = f(x,y,\mu,\epsilon)$$
$$\epsilon\dot{y} = g(x,y,\mu,\epsilon)$$

under this condition, the simulation results obtained from the ODEs will have similar behaviors as the original DAEs. Since $g(x,y,\mu) = 0$ of DAE, we simply perturbed it around the algebraic constraint by a small positive scalar ϵ

$$0 = g(x, y + \epsilon\dot{y}, \mu).$$

By Taylor expansion we have

$$0 = g(x,y,\mu) + g_y(x,y,\mu)\epsilon\dot{y} + o(\epsilon)$$

Ignoring higher order terms, we have

$$g_y(x,y,\mu)\epsilon\dot{y} = -g(x,y,\mu).$$

If g_y is nonsingular, then

$$\epsilon\dot{y} = -g_y^{-1}(x,y,\mu)g(x,y,\mu)$$

Therefor, the singularly perturbed ODEs obtained by PTE is as below

$$\dot{x} = f(x,y,\mu,\epsilon)$$
$$\epsilon\dot{y} = -{g_y}^{-1}(x,y,\mu)g(x,y,\mu)$$

the corresponding Jacobian matrix is given

$$J_u = \begin{pmatrix} f_x & f_y \\ G_x & G_y \end{pmatrix} \quad (1)$$

where $G(x,y,\mu) = \frac{-1}{\epsilon}{g_y}^{-1}g(x,y,\mu)$ and J_u (unreduced Jacobian matrix). This technique is used to compute eigenvalues at each equilibrium point [1]. and to apply PTE of simple power system voltage dynamic generator voltage control transmission, the dynamic behavior of singularly perturbed ODEs will match that of the DAEs and We will observe the time responses and trace the bifurcation directly through the singularly perturbed ODEs [2].

Bifurcation is A change in the qualitative properties could mean a change in stability of the original system and ,thus The system must assumed a stat different form the original, in vague terms, the values of the parameters where this change take place [3],The ideas of [4] 1988 this discus the bifurcation of fixed point occur is near non hyperbolic fixed and periodic orbit of singularly perturbed delay differential equation, [5] 1994 It is shown that when singularly perturbed models of ODEs are considered, the singularity induced bifurcation in the slow DAEs corresponds to oscillatory behavior in the singularly perturbed models. it is proved that oscillations in the singularity induced bifurcation in the slowly DAEs which inturn corresponds to the occurrence of supercritical Hopf bifurcation in the singularly perturbed models, [6] (2005) we discus computation singular point and singularity induced bifurcation point of the differential algebraic equation power system model, [7] (2012) we will study The different types of bifurcation That can arise in ordinary differential equation, the necessary condition for each type of bifurcation to occur and The normal form. Then consider system of ordinary differential equation with parameter and study Hopf bifurcation, [8] (2012) we discuss several types of bifurcation saddle node, transcritiecal, pitchfork and Hoph bifurcation. Among these types we especially focus one Hoph bifurcation, the first there types of bifurcation occur in scalar and in the systems of differential equations, the fourth type called Hoph bifurcation does not occur in scalar differential equation because this type of bifurcation involves a change to a periodic solution A special type of bifurcation called singularity-induced bifurcation(SIB) exists in DAE system , The state space of DAE system are divided in to open surface Where the Jacobian matrix of algebraic equation are singular The surface defined by:

$$S = \{(x, y, \mu) \in R^n \times R^m \times R^r : g(x, y, \mu) = 0, det g_y(x, y, \mu) = 0\}$$

this hypersurface is call impasses surfaces, because it cannot be crossed by the trajectories of the system. An SIB is defined by both singularity and equilibrium condition:

$$f(x, y, \mu) = 0$$
$$g(x, y, \mu) = 0$$
$$det g_y(x, y, \mu) = 0$$

The SIB occurs when k-dimensional equilibrium manifold, which lie one The constraints manifold of a DAE systems intersect The n + k -1, dimensional impasses surface. These singular points is not equilibrium point because the systems cannot be defined on The impasse surface, the equilibrium point may exists arbitrarily closed to, both side of such a singularity.

Theorem(1): (Implicit Function Theorem) Let $G(x, \mu)$ be a C^1 function on $R \times R^n, G: R \times R^n \to R^n$, such that $G(0,0) = 0$ and $\det[D_x G(0,0)] \neq 0$ then there exists a unique differentiable function $X(\mu)$ defined on a neighborhood of $\mu = 0, X: M \to R^n$, such that $X(0) = 0$ and

$$G\big(\mu, X(\mu)\big) = 0.$$

In words the theorem says the following, given $G(x, \mu)$ assume that zero set, i.e., the set of $(\mu, x) such\ that\ G(\mu, x) = 0$, contains at least one point (0,0)

$$(D_x G(0,0))_{i,j} = \frac{\partial G_i}{\partial x_j}(0,0)_{i,j=1,\cdots,n.}$$

has nonzero determinant, then we can solve the equation $G(\mu, x) = 0$ uniquely for x as a function of μ at least for values of μ sufficiently near $\mu = 0$,this mean that, $(x, \mu) = (0,0)$, the zero set of $G(\mu, x)$ consists of a singular or branch.

I. TAYLOR EXPANSION METHOD FOR BIFURCATION OF SOLUTION IN SINGULARLY PERTURBED ODE:

The bifurcation of solution in singularly perturbed ODEs when the perturbed parameter $0 < \epsilon \leq 1$ is studied by using Taylor expansion. The bifurcation refers to a qualitative change in the behavior of dynamical system, as some parameter on which the system depends varies continuous. Our study will conclude the case of one dimensional and the higher dimensional systems. In case of higher dimensional systems, we will apply Sotomayor Theorem, directly to singular perturbed ordinary differential equations without reduction. For one dimensional systems Taylor expansion will play a role in determining which kind of bifurcation in such systems will occurred.

A. Taylor expansion method for bifurcation of one dimensional singularly perturbed ODEs:

Singularly perturbed system is a special class of nonlinear control system. We begin our study of bifurcation with simple kind that occured in dynamical system at non hyperbolic equilibrium point. We consider the one dimensional singularly perturbed ODE:

$$\epsilon \dot{x} = f(x, \mu) \quad , x \in R, \mu \in R,\ 0 < \epsilon \leq 1 \qquad (2)$$

where the small positive scalar ϵ called perturbation parameter and μ is bifurcation parameter, and f is smooth function. This will permit us to apply the bifurcation theory on singularly parameterized ODE. The three simple types of bifurcation that occurred at a nonhyperbolic critical point are saddle-node, transcritical, and pitchfork bifurcation.

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1) *Saddle-node Bifurcation:* A saddle-node bifurcation is a collision and disappearance of two equilibrium point in dynamical system. This occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation. In the following theorem we formulate the sufficient conditions for the saddle node bifurcation to be occurred in one dimensional singularly perturbed ODEs by using Taylor expansion.

Theorem (2): consider the singularly perturbed ODE

$$\epsilon \dot{x} = f(x, \mu) \quad , x \in R, \mu \in R,\ 0 < \epsilon \ll 1$$

Assume that following condition holds:

1-$f_\mu(x_0, \mu_0) \neq 0$

2-$f_{xx}(x_0, \mu_0) \neq 0$

where (x_0, μ_0) is critical point. Then the ODE undergoes saddle node bifurcation and (x_0, μ_0) is a bifurcation point and the system is qualitatively equivalent to

$$\epsilon \dot{x} = \mu \mp x^2 \qquad (3)$$

Proof: Consider the equation:

$$\epsilon \dot{x} = f(x, \mu) \qquad (4)$$

By Taylor expansion about non hyperbolic equilibrium point (x_0, μ_0) we have

$$\dot{x} = \frac{1}{\epsilon} f(x_0, \mu_0) + \frac{1}{\epsilon} f_x(x_0, \mu_0)x + \frac{1}{\epsilon} f_\mu(x_0, \mu_0)\mu + \frac{1}{2\epsilon}[f_{xx}(x_0, \mu_0)x^2 + 2f_{x\mu}(x_0, \mu_0)x\mu + f_{\mu\mu}(x_0, \mu_0)\mu^2] + \cdots \quad (5)$$

Now since (x_0, μ_0) is non hyperbolic so the first two terms are vanished and by conditions 1,2 given in theorem the equation (5) becomes:

$$\dot{x} = \frac{1}{\epsilon} f_\mu(x_0, \mu_0) + \frac{1}{2\epsilon} f_{xx}(x_0, \mu_0)x^2 + O(|x\mu|, |\mu^2|, \dots). \quad (6)$$

Then we have

$$\dot{x} \approx \frac{1}{\epsilon}\mu f_\mu(x_0, \mu_0) + \frac{x^2}{2\epsilon} f_{xx}(x_0, \mu_0). \quad (7)$$

Analyzing (7) we have

$$x_* = \pm\left(\frac{-2\mu f_\mu(x_0, \mu_0)}{f_{xx}(x_0, \mu_0)}\right)^{\frac{1}{2}}$$

As critical points. The Jacobain at x_* is

$$Df(x, \mu)|_{x_*} = \pm\frac{1}{\epsilon}\left(\frac{-2f_\mu(x_0, \mu_0)}{f_{xx}(x_0, \mu_0)}\right)^{\frac{1}{2}} f_{xx}(x_0, \mu_0)$$

If $\frac{\mu f_\mu(x_0,\mu_0)}{f_{xx}(x_0,\mu_0)} < 0$ and $f_{xx}(x_0, \mu_0) < 0$ then x_* is stable and is unstable if $f_{xx}(x_0, \mu_0) > 0$. Which means that (x_0, μ_0) is saddle node bifurcation point and the system $\epsilon\dot{x} = f(x, \mu)$ undergoes a saddle node bifurcation when the conditions 1,2 given in the theorem satisfied. Then we conclude from the proof of the theorem that $\epsilon\dot{x} = \mu \pm x^2$ is the normal form for this kind of bifurcation. Finally, for saddle node bifurcation to be occured the following should satisfied:

- $\frac{\partial f}{\partial \mu}(x_0, \mu_0) = \frac{1}{\epsilon} \neq 0$
- $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) = \frac{2}{\epsilon} \neq 0$

2) Transcritical bifurcation: In transcritical bifurcation two families of equilibrium point collide and exchange their stability properties the family, that was stable before the bifurcation is unstable after it . The other equilibrium point goes from begin unstable to begin stable.

Theorem (3): Consider the singularly perturbed ODEs

$$\epsilon\dot{x} = f(x, \mu) \quad , x \in R, \mu \in R, \; 0 < \epsilon \ll 1$$

Assumed that following conditions are satisfied:

1-$f_\mu(x_0, \mu_0) = 0$

1-$f_{xx}(x_0, \mu_0) \neq 0$

2-$f_{x\mu}(x_0, \mu_0) \neq 0$

where (x_0, μ_0) is critical point. Then the ODE undergoes transcritical bifurcation and (x_0, μ_0) is bifurcation point and the system is qualitativly equivalent to

$$\epsilon\dot{x} = \mu \mp x^2. \quad (8)$$

Proof:

Assume (x_0, μ_0) is non hyperbolic equilibrium point, so $f(x_0, \mu_0) = 0$ and $f_x(x_0, \mu_0) = 0$. Then from conditions 1,2,3 given in theorem Taylor expansion gives:

$$\dot{x} = \frac{1}{2\epsilon}(f_{xx}(x_0, \mu_0)x^2 + 2f_{x\mu}(x_0, \mu_0)x\mu + f_{\mu\mu}(x_0, \mu_0)\,\mu^2) + O(|x|^3, |\mu|^3, \dots) \quad (9)$$

then we have:

$$\dot{x} \approx \frac{1}{2\epsilon}(f_{xx}(x_0,\mu_0)x^2 + 2f_{x\mu}(x_0,\mu_0)x\mu)$$

The critical points are $x_{*1} = 0$, and $x_{*2} = \frac{-2\mu f_{x\mu}(x_0,\mu_0)}{f_{xx}(x_0,\mu_0)}$. And the Jacobian is:

$$Df(x,\mu) = \frac{1}{\epsilon}(xf_{xx}(x_0,\mu_0) + \mu f_{x\mu}(x_0,\mu_0)).$$

Then we have

$$Df(x,\mu)|_{x_{*1}} = \frac{1}{\epsilon}\mu f_{x\mu}(x_0,\mu_0).$$

So if $\mu f_{x\mu}(x_0,\mu_0) > 0$ then x_{*1} is unstable, and is stable if $\mu f_{x\mu}(x_0,\mu_0) < 0$. Also we have

$$Df(x,\mu)|_{x_{*1}} = \frac{-1}{\epsilon}\mu f_{x\mu}(x_0,\mu_0)$$

Then for $\mu > 0$ *we have* x_{*2} is stable, unstable if $\mu f_{x\mu}(x_0,\mu_0) > 0$, $\mu f_{x\mu}(x_0,\mu_0) < 0$ respectively. Then we conclude from above that this kind of bifurcation is transcritical bifurcation and the normal form is equivalent to $\epsilon\dot{x} = \mu x \pm x^2$. For transcritical bifurcation to be occurred at (0,0) according to the theorem we have:

- $\frac{\partial f}{\partial \mu}(0,0) = 0.$
- $\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{2}{\epsilon} \neq 0.$
- $\frac{\partial^2 f}{\partial x \partial \mu}(0,0) = \frac{1}{\epsilon} \neq 0.$

3) *Pitchfork bifurcation:* In pitchfork bifurcation one family of equilibrium points transfers it stability properties to two families after or before the bifurcation point. If this occurs after bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial equilibrium point occur for values of parameter lower than the bifurcation value. In other words, the cases in which the emerging nontrivial equilibrium are stable are called supercritical whereas the cases in which the equilibrium are called subcritical.

Theorem (4): consider the singularly perturbed ODEs

$$\epsilon\dot{x} = f(x,\mu), x \in R, \mu \in R, 0 < \epsilon \ll 1$$

Assume that following conditions satisfied:

1-$f_\mu(x_0,\mu_0) = 0$

2-$f_{xx}(x_0,\mu_0) = 0$

3-$f_{x\mu}(x_0,\mu_0) \neq 0$

4-$f_{xxx}(x_0,\mu_0) \neq 0$

where (x_0,μ_0) is critical point. The undergoes pitchfork bifurcation and (x_0,μ_0) is bifurcation point and system qualitatviely equivale to

$$\epsilon\dot{x} = \mu \mp x^3 \qquad (10)$$

Proof: Assume (x_0,μ_0) non hyperbolic equilibrium point, since $f(x_0,\mu_0) = 0$ and $f_x(x_0,\mu_0) = 0$. Then with conditions above Taylor expansion becomes:

$$\dot{x} \approx \frac{1}{2\epsilon}\left(2f_{x\mu}(x_0,\mu_0)x\mu + f_{\mu\mu}(x_0,\mu_0)\mu^2\right) + \frac{1}{6\epsilon}\left(f_{xxx}(x_0,\mu_0)x^3 + f_{\mu\mu\mu}(x_0,\mu_0)\mu^3 + \cdots\right) \qquad (11)$$

By balancing the first term and third term we get:
$\frac{1}{\epsilon} f_{x\mu}(x_0, \mu_0)x\mu \frac{1}{6\epsilon} f_{xxx}(x_0, \mu_0)x^3 = 0$
Then $x_{*1} = 0$ and $x_{*2} = (\frac{-6f_{x\mu}(x_0,\mu_0)}{f_{xxx}(x_0,\mu_0)})^{\frac{1}{2}}$ are critical points. And the Jacobian is:

$$Df(x, \mu) = \frac{1}{\epsilon} f_{x\mu}(x_0, \mu_0)\mu + \frac{1}{2\epsilon} f_{xxx}(x_0, \mu_0)x^2 + \cdots$$

Then $x_{*1} = 0$ is stable if $\mu f_{x\mu}(0,0\,)x\mu < 0$ and unstable if $\mu f_{x\mu}(0,0\,)x\mu > 0$. While $x_{*2} = \left(\frac{-6f_{x\mu}(x_0,\mu_0)}{f_{xxx}(x_0,\mu_0)}\right)^{\frac{1}{2}}$ is stable if $\mu f_{x\mu}(0,0\,)x\mu > 0$ and $\mu f_{x\mu}(0,0\,)x\mu < 0$. This is what we called a pitchfork bifurcation and the normal form of this kind of bifurcation is:
$\dot{x} = \frac{1}{\epsilon}(\,\mu \mp x^3)$. For pitchfork bifurcation to be occurred

- $\frac{\partial f}{\partial \mu}(0,0) = 0$
- $\frac{\partial^2 f}{\partial x^2}(0,0) = 0$
- $\frac{\partial^3 f}{\partial x^3}(0,0) = \frac{6}{\epsilon} \neq 0$
- $\frac{\partial^2 f}{\partial x \partial \mu}(0,0) = \frac{1}{\epsilon} \neq 0$

B. Bifurcation of higher dimensional on ODEs

Consider singularly perturbed ODEs

$$\dot{x} = f(x, y, \mu)$$
$$\epsilon\dot{x} = g(x, y, \mu), x \in R^n, y \in R^m, \mu \in R \qquad (12)$$

$Where\ 0 < \epsilon \ll 1$ is perturbed parameters, and μ is bifurcation parameter. Define the ODEs:

$$\dot{z} = F(x, y, \mu) = \begin{pmatrix} f(x, y, \mu) \\ \frac{1}{\epsilon} g(x, y, \mu) \end{pmatrix}$$

Where $\dot{z} = (\dot{x}, \dot{y})^T$. In other the bifurcation that occurred in above system we are going to apply Sotomayor Theorem on the system of singularly perturbed parameterized ODE without reduction.

Theorem (5): Assume that $F(x_0, y_0, \mu_0)$ and $A = DF(x_0, y_0, \mu_0) = J_{(x_0,y_0,\mu_0)}$ is $n \times n$ matrix has a simple zero eigenvalue($(\lambda = 0)$, (x_0, μ_0) is non hyperbolic critical point) with eigenvector V and A^T has eigenvector W corresponding $\lambda = 0$, suppose that A has k eigenvalues with negative real part and $(n - k - 1)$ eigenvalues with positive real part.

1-If the following conditions holds:

- $W^T \begin{pmatrix} f_\mu \\ \frac{1}{\epsilon} g_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} \neq 0$
- $W^T[D^2F(x_0, y_0, \mu_0)(V, V)] \neq 0$

Then the systems (12) undergoes saddle node bifurcation and (x_0, y_0, μ_0) is a bifurcation point.

2-If the following conditions hold:

- $W^T \begin{pmatrix} f_\mu \\ \frac{1}{\epsilon} g_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} \neq 0$

- $W^T \begin{pmatrix} f_{\mu x} & f_{\mu y} \\ \frac{1}{\epsilon} g_{\mu x} & \frac{1}{\epsilon} g_{\mu y} \end{pmatrix}_{(x_0,y_0,\mu_0)} V \neq 0$
- $W^T[D^2 f(x_0, y_0, \mu_0)(V,V)] \neq 0$

Then the systems (12) undergoes transcritical bifurcation and (x_0, y_0, μ_0) is a bifurcation point.

3- If the following conditions holds:

- $W^T \begin{pmatrix} f_\mu \\ \frac{1}{\epsilon} g_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} = 0$
- $W^T \begin{pmatrix} f_{\mu x} & f_{\mu y} \\ \frac{1}{\epsilon} g_{\mu x} & \frac{1}{\epsilon} g_{\mu y} \end{pmatrix}_{(x_0,y_0,\mu_0)} V \neq 0$
- $W^T[D^2 f(x_0, y_0, \mu_0)(V,V)] = 0$
- $W^T[D^3 f(x_0, y_0, \mu_0)(V,V,V)] \neq 0$

Then the systems (12) undergoes pitchfork bifurcation and (x_0, y_0, μ_0) is a bifurcation point.

C. *Hopf Bifurcation*

Hopf bifurcation occurs when the system of non-linear equation is non-hyperbolic, and Hopf bifurcation cannot occur in one-dimension. a Hopf bifurcation typically occurs when real part of complex conjugate pair of purely imaginary eigenvalue. Hopf bifurcation signal or the annihilation of period orbits called limit cycle, a limit cycle is an isolated periodic solution of a nonlinear system $\dot{x} = f(x)$,here a periodic solution is a function $x(t)$ satisfying $\dot{x} = f(x)$ and having the property of $x(t+T) = x(t)$ for all t , where T is the period of the periodic solution. Therefore, a limit cycle is a closed curve in $n - dimensional$ space. There exist two types of Hopf bifurcation depending on the nature of the interaction with limit cycle.

. Subcritical HB: An unstable limit cycle, existing prior to the bifurcation, shrinks and eventually disappears as it coalesces with a stable equilibrium point at bifurcation, after the bifurcation the equilibrium point becomes unstable resulting in growing oscillation.

. Super critical HB: A stable limit cycle is generated at the bifurcation, and a stable equilibrium point becomes unstable with increasing amplitude oscillation, which are eventually attracted by the stable limit cycle.

Consider the system of singular perturbed ODE:

$$\begin{aligned} \dot{x} &= \mu x - y + p(x,y) \qquad (13) \\ \epsilon \dot{y} &= x + \mu y + q(x,y) \end{aligned}$$

Where

$$p(x,y) = \sum_{i+j\geq 2} a_{ij}\, x^i y^j = (a_{20}x^2 + a_{11}xy + a_{02}y^2) + (a_{30}x^3 + a_{21}x^2 y + a_{12}xy^2 + a_{03}y^3) + \cdots$$

$$q(x,y) = \sum_{i+j\geq 2} b_{ij}\, x^i y^j = (b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2 y + b_{12}xy^2 + b_{03}y^3) + \cdots$$

$$DF(0,0,\mu) = \begin{pmatrix} \mu & -1 \\ \frac{1}{\epsilon} & \frac{\mu}{\epsilon} \end{pmatrix}$$

Now for $\mu \neq 0, DF(0,0,\mu)$ has a pair of pure imaginary eigenvalues $\lambda_1 \simeq \frac{1}{\epsilon}(\mu + i), \lambda_2 \simeq \frac{1}{\epsilon}(\mu - i)$. So the origin is stable focus for $\mu < 0\,, 0 < \epsilon \leq 1$, unstable focus for $\mu < 0\,, 0 < \epsilon \leq 1$. For $\mu = 0\,, 0 < \epsilon \leq 1$, $DF(0,0,0)$ have two eigenvalues $\lambda_1 = \frac{1}{\sqrt{\epsilon}} i, \lambda_2 = \frac{-1}{\sqrt{\epsilon}} i$, in this case the origin is center. The bifurcation of limit cycle from the origin that occurs at bifurcation value $\mu = 0$ as the origin changes its stability. This change in structure stability as μ changes is referred to as a Hopf bifurcation. In order to determine which kind of Hopf bifurcation occurred at bifurcation point $\mu = 0$ we will use the idea of Lyapunov number σ. Where for $\mu = 0$ in system (13) we have:

$$\sigma = \frac{3\pi}{2}[3\left(a_{30} + \frac{1}{\epsilon} b_{03}\right) + \left(a_{12} + \frac{1}{\epsilon} b_{21}\right) - \frac{2}{\epsilon}(a_{20}b_{20} - a_{02}b_{02}) + a_{11}(a_{02} + a_{20}) - \frac{1}{\epsilon} b_{11}(b_{02} + b_{20})]$$

1- If $\sigma < 0$ then (0,0) is stable focus.

2- If $\sigma > 0$ then (0,0) is unstable focus.

In the first case $\sigma < 0$ the critical point generates a stable limit cycle as μ passes through the bifurcation value $\mu = 0$, then we have a supercritical Hopf bifurcation.

In the second case $\sigma > 0$ the critical point generates unstable limit cycle as μ passes through bifurcation point $\mu = 0$, then we have a subcritical Hopf bifurcation.

III. PERTURBED TAYLOR EXPANSION BIFURCATION OF SOLUTION IN DIFFERENTIAL ALGEBRAIC EQUATION:

We will study bifurcation of differential algebraic equation (DAE) when perturbed parameter ϵ=0, we work to convert DAEs to singularly perturbed ODEs, then by using implicit function theorem. we can apply bifurcation theory of singularly perturbed ODE on DAE. The three kinds of bifurcation saddle, pitchfork and transcritical bifurcation and singularity induced bifurcation (SIB) will be investigated to be occurred in such systems. We convert DAEs to ODEs by using PTE (perturbed Taylor expansion) [1]and discuss bifurcation of higher dimensional when ϵ=0. In section three we apply bifurcation theory on the differential algebraic equation (DAE) when μ is bifurcation parameter.

Consider the differential algebraic equation DAEs when ϵ equal to zero:

$$\dot{x} = f(x, y, \mu) \qquad (14)$$
$$0 = g(x, y, \mu)$$

Where $f: R^n \times R^m \times R \to R^n, g: R^n \times R^m \times R \to R^n$.

This studying the bifurcation of solution in parameterized DAE we will apply the method used for singular perturbed parameterized ODE. We have two ways that one could approach the bifurcation of solution of DAE. The first one is to use implicit function theorem, where gy(0,0)is non singular, which can be summarized as following:

- By implicit function theorem there exist a function $y(x): R^n \to R^m, y(0) = 0$ such that the DAE can be written as

$$\dot{x} = f(x, y(x), \mu) \qquad (15)$$

- Apply the bifurcation theory to the reduced ODE.

The second one is to use ϵ-embedding method with the following steps:

- Consider the DAEs with $0 < \epsilon \leq 1$and replacing the constraint equation $0 = g(x, y, \mu)$ by singular perturbed equation $\epsilon\dot{y} = g(x, y, \mu)$. to get the system:

$$\dot{x} = f(x, y, \mu) \qquad (16)$$
$$\epsilon\dot{y} = g(x, y, \mu)$$

- Apply bifurcation theory to the system.
- Set $\epsilon = 0$ to obtain a method for investigating bifurcation for the DAE system.

A. Bifurcation of solution of DAE by implicit function theorem ($g_y(0,0)$is non-singular)

Consider the parameterized DAEs and we assume here$g_y(0,0)$ is non-singular. Then by implicit function theorem there is a function $y(x): R^n \to R^m$ such that $g(x, y(x), \mu) = 0$ and the DAE will reduced to the ODE (15) :

$$\dot{x} = f(x, \mu)$$

So bifurcation theory can be applied on the equivalence reduced ODE (15) .

1) Saddle node bifurcation solution of DAE with index one:

The following theorem talk about the occurrence of saddle node bifurcation in DAE (14) under satisfying some conditions.

Theorem(6):consider the parameterized DAE such that $g_y(0,0)$ is nonsingular where (0,0) is non hyperbolic critical point, Assume that following conditions are satisfied:

2) 1-$f_\mu(0,0) \neq 0$

3) 2-$f_{xx}(0,0) \neq 0$

Then the DAE undergoes saddle node bifurcation and (0,0) is a bifurcation point.

Proof: we will follow the same processes given in the Theorem (2) for dimension one and Theorem (6) for higher dimension. So for one dimension, i.e., $x \in R, y \in R, \mu \in R$ for Theorem (1) applying Taylor expansion on $\dot{x} = f(x, y(x), \mu)$ yields:

$$\dot{x} \approx \frac{1}{\epsilon}\mu f_\mu(0,0) + \frac{x^2}{2\epsilon} f_{xx}(0,0). \qquad (17)$$

Analyzing we have

$$x_* = \pm(\frac{-2\mu f_\mu(0,0)}{f_{xx}(0,0)})^{\frac{1}{2}}$$

As critical points. The Jacobain at x_* is

$$Df(x,\mu)|_{x_*} = \pm\frac{1}{\epsilon}\left(\frac{-2f_\mu(0,0)}{f_{xx}(0,0)}\right)^{\frac{1}{2}} f_{xx}(0,0)$$

If $\frac{\mu f_\mu(0,0)}{f_{xx}(0,0)} < 0$ and $f_{xx}(0,0) < 0$ then x_* is stable and is unstable if $f_{xx}(0,0) > 0$. Which means that (0,0) is saddle node bifurcation point and the system $\dot{x} = f(x,\mu)$ undergoes a saddle node bifurcation when the conditions 1,2 given in the theorem satisfied. It follow that DAE undergoes saddle node bifurcation and (0,0) is saddle node bifurcation point.

2) Transcritical bifurcation in DAE with index one

Following Theorem (3) which is the bifurcation of solution in singular perturbed ODE, we formulate a similar theorem the occurrence of transcritical bifurcation in DAE.

Theorem(7):Consider the parameterized DAE such that $g_y(0,0)$ is non singular, and (0,0) is non-hyperbolic critical point. Assume that the following conditions are satisfied:

1-$f_\mu(0,0) = 0$

2-$f_{xx}(0,0) \neq 0$

3-$f_{x\mu}(0,0) \neq 0$

Then the DAEs (14) undergoes transcritical bifurcation and (0,0) is transcritical bifurcation point.

Proof: Assume that (0,0) is non-hyperbolic equilibrium point. Since $det g_y(0,0) \neq 0$ so by implicit function theorem there exist function $y(x), y(0) = 0$ such that $g(x, y(x), \mu) = 0$. The DAE (14) will be reduced to ODE $\dot{x} = f(x, y, \mu)$. Then from conditions 1,2,3 given in the theorem Taylor expansion gives:

$$\dot{x} = (f_{xx}(0,0)x^2 + 2f_{x\mu}(0,0)x\mu + f_{\mu\mu}(0,0)\,\mu^2) + O(|x|^3, |\mu|^3, \dots) \qquad (9)$$

then we have:

$$\dot{x} \approx f_{xx}(0,0)x^2 + 2f_{x\mu}(0,0)x\mu$$

The critical points are $x_{*1} = 0$, and $x_{*2} = \frac{-2\mu f_{x\mu}(0,0)}{f_{xx}(0,0)}$. And the Jacobian is:

$$Df(x,\mu) = x f_{xx}(0,0) + \mu f_{x\mu}(0,0).$$

Then we have

$$Df(x,\mu)|_{x_{*1}} = \mu f_{x\mu}(0,0).$$

So if $\mu f_{x\mu}(0,0) > 0$ then x_{*1} is unstable, and is stable if $\mu f_{x\mu}(0,0) < 0$. Also we have

$$Df(x,\mu)|_{x_{*1}} = \mu f_{x\mu}(0,0)$$

Then for $\mu > 0$ *we have* x_{*2} is stable, unstable if $\mu f_{x\mu}(0,0) > 0$, $\mu f_{x\mu}(0,0) < 0$ respectively. Then we conclude from above that this kind of bifurcation is transcritical bifurcation and (0,0) is transcritical bifurcation point.

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3) Pitchfork bifurcation in DAE with index one:

In pitchfork bifurcation of solution of DAE (14) with $g_y(0,0)$ is non singular, we will follow the same in the previous theorems.

Theorem(8): Consider the parameterized DAE (14) such that $g_y(0,0) \neq 0$ where (0,0) is non hyperbolic critical point. Suppose that the following conditions are satisfied:

1-$f_\mu(0,0) = 0$

2-$f_{xx}(0,0) \neq 0$

3-$f_{x\mu}(0,0) \neq 0$

4-$f_{xxx}(0,0) \neq 0$

Then the system (14) undergoes pitchfork bifurcation and (0,0) is a pitchfork bifurcation point.

Proof: Assume (0,0) non hyperbolic equilibrium point. Then with conditions above Taylor expansion becomes:

$$\dot{x} \approx \frac{1}{2}\left(2f_{x\mu}(0,0)x\mu + f_{\mu\mu}(0,0)\mu^2\right) + \frac{1}{6}\left(f_{xxx}(0,0)x^3 + f_{\mu\mu\mu}(0,0)\mu^3 + \cdots\right) \qquad (19)$$

By balancing the first term and third term we get:

$$f_{x\mu}(0,0)x\mu + \frac{1}{6}f_{xxx}(0,0)x^3 = 0$$

Then $x_{*1} = 0$ and $x_{*2} = (\frac{-6f_{x\mu}(0,0)}{f_{xxx}(0,0)})^{\frac{1}{2}}$ are critical points. And the Jacobian is:

$$Df(x,\mu) = f_{x\mu}(0,0)\mu + \frac{1}{2\epsilon}f_{xxx}(0,0)x^2 + \cdots$$

Then $x_{*1} = 0$ is stable if $\mu f_{x\mu}(0,0\,)x\mu < 0$ and unstable if $\mu f_{x\mu}(0,0\,)x\mu > 0$. While $x_{*2} = \left(\frac{-6f_{x\mu}(0,0)}{f_{xxx}(0,0)}\right)^{\frac{1}{2}}$ is stable if $\mu f_{x\mu}(0,0\,)x\mu > 0$ and $\mu f_{x\mu}(0,0\,)x\mu < 0$. This is what we called a pitchfork bifurcation point.

B. Bifurcation of solution in DAE with higher dimension by ϵ-embedding method:

We use ϵ -embedding method to investigate the bifurcation of solution in DAE (14). For this purpose, we apply the perturbation Taylor expansion (PTE) technique to convert a DAEs to a singularly perturbed ODEs (14). The method of investigation bifurcation of solution given is then easily performed to this ODE. Simulation results show that PTE technique satisfies the fast convergence requirement and preserves the bifurcation properties of the original DAE system. To convert a DAEs to singularly perturbed ODEs by perturbed Taylor expansion we have the following:

Perturbation of variable y around the algebraic constraint $g(x,y,\mu) = 0$ by a small positive scalar ϵ,

$$0 = g(x, y + \epsilon\dot{y}, \mu)$$

Apply Taylor expansion:

$$0 = g(x,y,\mu) + g_y(x,y,\mu)\epsilon\dot{y} + \mathrm{o}(\epsilon)$$

Ignoring higher order terms, we have

$$g_y(x,y,\mu)\epsilon\dot{y} = -g(x,y,\mu)$$

If g_y is nonsingular, then

$$\epsilon\dot{y} = -{g_y}^{-1}(x,y,\mu)g(x,y,\mu)$$

Therefor, the singularly perturbed ODEs obtained by PTE is as below

$$\begin{aligned} \dot{x} &= f(x,y,\mu,\epsilon) \qquad (20)\\ \epsilon\dot{y} &= {g_y}^{-1}(x,y,\mu)g(x,y,\mu) \end{aligned}$$

The singular perturbed ODE is preserved the topological properties of solution of the original DAE. So study the bifurcation of solution of ODE (20) gives us information about the bifurcation of solution of DAE (14). In particular, the two systems have the same critical solution that is the critical pints of the two systems is the solution of

$$\begin{aligned} f(x,y,\mu) &= 0 \qquad (21)\\ g(x,y,\mu) &= 0 \end{aligned}$$

Now the following results indicate study of bifurcation of solution in DAEs after transform it into singular perturbed ODE by perturbed Taylor expansion. This result is a reformulation of Sotomayor Theorem in direction of DAEs.

Theorem (9): Consider the parameter dependent DA (14) and the singular perturbed ODE (19) obtained from perturbed Taylor expansion. Assume that $g_y(x_0, y_0, \mu_0) \neq 0$ where (x_0, y_0, μ_0) is a non hyperbolic critical point. Assume that $A = J_{(x_0,y_0,\mu_0)}$ is $n \times n$ Jacobian matrix of ODE (20) has a simple zero eigenvalue $(\lambda) = 0$ with eigenvector V and A^Thas eigenvector Wcorresponding to $\lambda = 0$, Suppose that A has K eigenvalues with negative real part and $(n-k-1)$ eigenvalues with positive real part.

1- If the following conditions satisfied:

- $W^T \begin{pmatrix} f_\mu \\ \frac{-1}{\epsilon}(g_y^{-1} g)_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} \neq 0$
- $W^T D^2 \begin{pmatrix} f \\ \frac{-1}{\epsilon} g_y^{-1} g \end{pmatrix}_{(x_0,y_0,\mu_0)} (V,V) \neq 0$

then the DAE (14) undergoes saddle node bifurcation and (x_0, y_0, μ_0) is a saddle node bifurcation point.

2- If the following conditions satisfied:

- $W^T \begin{pmatrix} f_\mu \\ \frac{-1}{\epsilon}(g_y^{-1} g)_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} = 0$
- $W^T \begin{pmatrix} f_{\mu x} & f_{\mu y} \\ \frac{-1}{\epsilon}(g_y^{-1} g)_{\mu x} & \frac{-1}{\epsilon}(g_y^{-1} g)_{\mu y} \end{pmatrix}_{(x_0,y_0,\mu_0)} V \neq 0$
- $W^T [D^2 \begin{pmatrix} f \\ \frac{-1}{\epsilon} g_y^{-1} g \end{pmatrix}_{(x_0,y_0,\mu_0)} (V,V)] \neq 0$

then the DAE (14) undergoes transcritical bifurcation and (x_0, y_0, μ_0) is a transcritical bifurcation point.

3-If the following conditions satisfied:

- $W^T \begin{pmatrix} f_\mu \\ \frac{-1}{\epsilon}(g_y^{-1} g)_\mu \end{pmatrix}_{(x_0,y_0,\mu_0)} = 0$
- $W^T \begin{pmatrix} f_{\mu x} & f_{\mu y} \\ \frac{-1}{\epsilon}(g_y^{-1} g)_{\mu x} & \frac{-1}{\epsilon}(g_y^{-1} g)_{\mu y} \end{pmatrix}_{(x_0,y_0,\mu_0)} V \neq 0$
- $W^T [D^2 \begin{pmatrix} f \\ \frac{-1}{\epsilon} g_y^{-1} g \end{pmatrix}_{(x_0,y_0,\mu_0)} (V,V)] = 0$
- $W^T [D^3 \begin{pmatrix} f \\ \frac{-1}{\epsilon} g_y^{-1} g \end{pmatrix}_{(x_0,y_0,\mu_0)} (V,V,V)] \neq 0$

then the DAE (14) undergoes pitchfork bifurcation and (x_0, y_0, μ_0) is a pitchfork bifurcation point.

Example 1: Consider the following singular perturbed ODEs:

$$\begin{aligned} \dot{x} &= x^2 - 5\mu x + 4\mu^2 \qquad (22) \\ \epsilon \dot{y} &= -y \end{aligned}$$

The Jacobian $J_{(0,0,0)} = DF_{(0,0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{\epsilon} \end{pmatrix}$. Then the eigenvectors for J and J^T are $W = (\frac{1}{\epsilon}, 0)^T$ and $V = (\frac{1}{\epsilon}, 0)^T$. Now application the conditions given in theorem above to investigate which kind of bifurcation occurred in given system:

1) $W^T F_\mu(0,0,0) = \left(\frac{1}{\epsilon}, 0\right) \begin{pmatrix} -5x + 8\mu \\ 0 \end{pmatrix}_{(0,0,0)} = 0$

2) $W^T\left[DF_\mu(0,0,0)V\right] = \left(\frac{1}{\epsilon}, 0\right) \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\epsilon} \\ 0 \end{pmatrix} = \frac{-5}{\epsilon^2} \neq 0$

3) $W^T\left[D^2F_\mu(0,0,0)V\right] = \left(\frac{1}{\epsilon}, 0\right) \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\epsilon} \\ 0 \\ \frac{1}{\epsilon} \\ 0 \end{pmatrix} = \frac{2}{\epsilon^2} \neq 0.$

Then we conclude that the system undergoes transcritica bifurcation. Now we try to show the bifurcation diagram and the phase portrait: We have $(4\mu, 0)$ and $(\mu, 0)$ are critical points for $\mu \neq 0$ and $(0, 0)$ is the only critical point for $\mu = 0$ .The Jacobian:

$$DF(x, \mu) = \begin{pmatrix} 2x - 5\mu & 0 \\ 0 & \frac{-1}{\epsilon} \end{pmatrix}.$$

Then

$$DF(4\mu, \mu) = \begin{pmatrix} 3\mu & 0 \\ 0 & \frac{-1}{\epsilon} \end{pmatrix}.$$

So $\lambda_1 = 3\mu$, $\lambda_2 = \frac{-1}{\epsilon}$ are eigenvalues.

And

$$DF(\mu, \mu) = \begin{pmatrix} -3\mu & 0 \\ 0 & \frac{-1}{\epsilon} \end{pmatrix},$$

so $\lambda_1 = -3\mu$, $\lambda_2 = \frac{-1}{\epsilon}$ are eigenvalues.

If $\mu < 0$, $0 < \epsilon \leq 1$ then $(4\mu, 0)$ is stable node and $(\mu, 0)$ is saddle.

And if $\mu > 0$, $0 < \epsilon \leq 1$ then $(4\mu, 0)$ is saddle and $(\mu, 0)$ is stable node.

The following example is application of above theorem in higher dimension DAE system.

C. Singularity-Induced Bifurcation (g_y singular)

A special type of bifurcation called singularity-induced bifurcation(SIB) exists in DAE system:

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \qquad (23) \\ 0 &= g(x, y, \mu) \end{aligned}$$

Singularity-induced bifurcation (SIB),occurs where g_y is singular and one eigenvalue is going to infinity at both sides of the singular point with opposite sign.

So the implicit function theorem, when Jacobian matrix g_y is singular cannot be applied, that is mean $det g_y = 0$ cannot be solved for the m dependent algebraic variables y. So the system cannot be dependent on the singular surface. Therefore. In other words, an(SIB) is defined by both singularities and equilibrium conditions, i.e., the following conditions are necessary conditions for SIB to be occurred in DAEs:

$$f(x,y,\mu)=0 \qquad (24)$$
$$g(x,y,\mu)=0$$
$$det g_y(x,y,\mu)=0$$

In DAE system

$$J_{(0,0)}=[f_x-f_y g_y{}^{-1} g_x]_{(0,0)}$$

having a zero eigenvalue then we have saddle-node bifurcation. Or when it has unbounded eigenvalue going from minus infinity to plus infinity, in this case SIB is occurred in DAEs. Consequently, the definition of SIB point is given as follows:

Definition 1 (Singularity induced bifurcation point)(SIB): A point (x_*,y_*,μ_*) on the singular surface

$$S=\{(x,y,\mu)\in R^n\times R^m\times R^r : g(x,y,\mu)=0, det g_y(x,y,\mu)=0\} \qquad (25)$$

satisfying $f(x_*,y_*,\mu_*)=0$ is called an SIB point DAEs

The singularly perturbed ODE:

$$\dot{x}=f(x,y,\mu,\epsilon) \qquad (26)$$
$$\epsilon\dot{y}=g_y{}^{-1}(x,y,\mu)g(x,y,\mu)$$

that obtained by perturbed Taylor expansion, preserves the bifurcation properties of the original DAE system. So by PTE method the DAEs (23) convert to Singular perturbed ODEs (26) for which the equilibrium points are in the correspondence by conditions SIB. On other hands

$$eig J_r \subseteq eig J_u$$

Where

$$J_r=f_x-f_y g_y{}^{-1}$$

and

$$J_u=\begin{pmatrix} f_x & f_y \\ (g_y^{-1}g)_x & (g_y^{-1}g)_y \end{pmatrix}$$

Following is an example for investigating SIB in DAEs:

Example 2: Consider the DAE given by:

$\dot{x}=-x+y+\mu$

$0=x^2+y^2$

We have

$$J_r(0,0)=-1$$

Where (0,0) is the equilibrium point, so $eig J_r=-1$. The corresponding singular perturbed ODE Obtained by PTE method is given by:

$\dot{x}=-x+y+\mu$

$$\epsilon\dot{y}=-(2y)^{-1}(x^2+y^2)$$

The unreduced Jacobian matrix J_u:

$$J_u=\begin{pmatrix} -1 & 1 \\ -xy^{-1} & x^2(2)^{-1}-2^{-1} \end{pmatrix}$$

$$J_u(0,0,0)=\begin{pmatrix} -1 & 1 \\ 0 & (-2\epsilon)^{-1} \end{pmatrix}$$

then $eigJ_u$ are $\lambda_1 = -1, \lambda_2 = (-2\epsilon)^{-1}$. Obviously as $\epsilon \to 0, \lambda \to \infty$. Then we conclude that the system undergoes SIB and $\mu = 0$ is bifurcation value.

In this paper we have shown That The differential algebraic equation (DAEs) can studied through the singularly perturbed system by perturbed Taylor expansion reduction method. That is the classical theorem of bifurcation that can be apply
to singularly perturbed (
ODEs). The known kind of bifurcation, fold (saddle node), trans critical, pitchfork and Hopf bifurcation have been investigated for (ODEs) in this case when $0<\epsilon\ll1$.
Also the case when ϵ=0 was careful dealt with to show that bifurcation occur in (DAEs) by using perturbed Taylor expansion method and Implicit Function Theorem.

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