



## Study Sstrong Convergence and Acceleration of New Iteration Type Three – Step

**Zena Hussein Maibed**

Department of Mathematics , College of  
Education for Pure Sciences,Ibn Al –Haitham/  
University of Baghdad, Baghdad, Iraq.  
[mrs zena.hussein@yahoo.com](mailto:mrs_zena.hussein@yahoo.com)

**Alaa M. AL- Hameedwi**

Department of Mathematics , College of  
Education for Pure Sciences,Ibn Al –Haitham/  
University of Baghdad, Baghdad, Iraq.  
[Alaa.Mohsen1203a@ihcoedu.uobaghdad.edu.iq](mailto:Alaa.Mohsen1203a@ihcoedu.uobaghdad.edu.iq)

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### Abstract

I The paper aims to define a new iterative method called the three-step type in which Jungck resolvent CR-iteration and resolvent Jungck SP-iteration are discussed and study rate convergence and strong convergence in Banach space to reach the fixed point which is differentially solved of nonlinear equations. The studies also expanded around it to find the best solution for nonlinear operator equations in addition to the varying inequalities in Hilbert spaces and Banach spaces, as well as the use of these iterative methods to approximate the difference between algorithms and their images, where we examined the necessary conditions that guarantee the unity and existence of the solid point. Finally, the results show that resolvent CR-iteration is faster than resolvent Jungck SP-iteration using Jungck resolvent estimation.

**Keywords:** Jungck mapping, Rate of convergence, Jungck *CR* – iteration, Rsolvent Jungck *SP* – iteration.

### 1.Introduction and Preliminary

In the last three decades, a lot of literature has been published on the iterative approximation of fixed –points for certain classes of operators, using the methods of the Mann and Ishikawa algorithm. Fixed–point theorems were developed for single-valued or definite-value assignments of metric spaces and Banach. Among the topics of fixed–point theorem, the topic of fixed–point approximations for assignments is important because it is useful for proving the existence of fixed–point assignments. However, once one knows that there is a fixed– point for some map, then finding the value of that fixed point is not an easy task, which is the reason of why we use iterative operations to calculate it. Over time, many iterative processes have been developed. These



iterative processes have been used, to solve various types of differential equations in recent years. The well-known Bannack deflationary theorem uses the process of Picard's algorithm to approximate the fixed– point [1]. Some other well-known iterative processes are Agarwal [2], Noor [3], SP [4], Piccard S [5], and so on. Recently, Qing and Qhou [6] extended their results to a random period and the Ishikawa algorithm. They gave some control conditions for the convergence of the Ishikawa algorithm over a random period. In 2006, [7] presented that Mann's algorithm converges faster than Ishikawa's algorithm. Subsequently, many studies were conducted on this subject see [8, 9, 10].

This paper aims to propose new iterative schemes for a fixed point and proves the convergence and speediness using resolvent ZA-Jungck mapping

Now, let  $\mathcal{D}$  be a nonempty subset closed -convex of a Banach space  $N$ , we recall the following:

**Definition 1.1** [11]: Let  $\{v_n\}_{n=0}^\infty$  be a bounded sequence in  $N$  for all  $v \in N$  and  $r(v, \langle v_n \rangle) = \limsup_{n \rightarrow \infty} \|v_n - v\|$ . Then

1. the asymptotic radius of  $\langle v_n \rangle$  relative to  $\mathcal{D}$  is given by:  
 $r(\mathcal{D}, \langle v_n \rangle) = \inf \{r(v, \langle v_n \rangle) : v \in \mathcal{D}\}$ .

2. The asymptotic center  $A(\mathcal{D}, \langle v_n \rangle)$  of  $\langle v_n \rangle$  is defined as:

$A(\mathcal{D}, \langle v_n \rangle) = \{v \in \mathcal{D} : r(v, \langle v_n \rangle) = r(\mathcal{D}, \langle v_n \rangle)\}$ .

**Definition 1.2** [12]: Let  $\langle y_n \rangle, \langle o_n \rangle$  be a real sequence such that  $\langle y_n \rangle$  converges to  $y$ ,  $\langle o_n \rangle$  Converges to  $o$ . And  $E = \lim_{n \rightarrow \infty} \frac{|y_n - y|}{|o_n - o|}$

1. If  $E = 0 \rightarrow$  the sequence  $\langle y_n \rangle$  is converge to  $y$  Faster then  $\langle o_n \rangle$  converge to  $o$
2. if  $0 < E < \infty \rightarrow \langle Y_n \rangle$  and  $\langle O_n \rangle$  have the same rate of convergence.

**Lemma 1.3** [13]: Let  $\mathcal{B}$  be a uniformly convex - space and  $\langle u_n \rangle$  be any sequence such that  $0 < q \leq u_n < 1$  For some  $q \in \mathbb{R}$  and  $\forall n \geq 1$ . Let  $\langle v_n \rangle$  and  $\langle l_n \rangle$  be any two sequences of  $\mathcal{B}$  such that  $\limsup_{n \rightarrow \infty} \|v_n\| \leq c, \limsup_{n \rightarrow \infty} \|l_n\| \leq c$  and  $c = \limsup_{n \rightarrow \infty} \|u_n v_n + (1 - u_n) l_n\|$  for some  $c \geq 0$ .

Then  $\lim_{n \rightarrow \infty} \|v_n - l_n\| = 0$ .

## 2.Main Results

In this part, we will present a new iteration and study the existence of fixed points and the convergence of these iterations

**Definition 2.1:** A resolvent Jungck  $\mathcal{CR}$  – iteration

$$Se_{n+1} = J_{rn}((1 - a_n)TSw_n + a_n J_{rn}TTw_n),$$

$$Sw_n = (1 - c_n)J_{rn}TSz_n + c_n TTz_n,$$

$Sz_n = (1 - b_n)J_{rn}TSe_n + b_n J_{rn} Te_n$ , where  $z_0 \in \mathcal{C}$ ,  $S, T$  commute mappings are on  $\mathcal{C}$  and  $\langle a_n \rangle, \langle b_n \rangle$  and  $\langle c_n \rangle$  are sequences in  $[0,1]$

**Definition 2.2:** A resolvent Jungck  $\mathcal{SP}$  – iteration

$$Su_{n+1} = (1 - a)J_{rn}TSx_n + a_n Tx_n,$$

$$Sx_n = J_{rn}((1 - c_n)Sv_n + c Tv_n),$$

$Sv_n = (1 - b_n)Su_n + b_n J_{rn} Tu_n$ , where  $u_0 \in \mathcal{C}$ ,  $\langle a_n \rangle, \langle b_n \rangle$  and  $\langle c_n \rangle$  are sequences in  $[0,1]$

**Definition 2.3** [14]: A self-mapping,  $T: \mathcal{D} \rightarrow \mathcal{D}$  is called a resolvent  $\mathcal{ZA}$ -Jungck mapping if

$$\|Tn - Th\| \leq \varepsilon \psi(a_1 \|Sn - Sh\| + a_2 \|Sn - J_{rn}h\|) + \min \left\{ \|J_{rn}n - Tn\|, \frac{\|Sn - J_{rn}n\| + \|J_{rn}h - Sh\|}{2}, \|Th - J_{rn}h\| \right\}$$

$\forall n, h \in \mathcal{C}, a_1, a_2, \varepsilon \in [0,1]$  and  $a_1 + a_2 \leq 1$

Now, we give new results.

**Lemma 2.4:** Let  $(N, \| \cdot \|)$  be a Banach - space, and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on  $\mathcal{D}$  with  $b \in [0,1]$ . Suppose that  $\langle Su_n \rangle$  be a resolvent Jungck  $\mathcal{SP}$  – iteration scheme in  $\mathcal{D}$ . There exists  $p \in F(s) \cap F(J) \cap A^{-1}(0) \neq \emptyset$ , then  $\lim_{n \rightarrow \infty} \|Su_n - t\|$  exists, for all  $n \in \mathbb{N}$ .

**Proof:** Since  $t$  be a fixed point of  $S, T$  and  $A^{-1}(0)$ . Then the following inequalities hold

$$\begin{aligned}
 \|Su_{n+1} - p\| &= \|(1 - a_n)J_{rn}TSx_n + a_nTx_n - p\| \\
 &\leq (1 - a_n)\|J_{rn}TSx_n - p\| + a_n\|Tx_n - p\| \\
 &\leq (1 - a_n)\|TSx_n - p\| + a_n\|Tx_n - p\| \\
 (1 - a_n)b\psi(a_1\|SSx_n - Sp\| + a_2\|SSx_n - J_{rn}p\|) &+ \min\left\{ \|J_{rn}Sx_n - \right. \\
 TSx_n\|, \frac{\|SSx_n - J_{rn}Sx_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \Big\} & \\
 + a_nb\psi(a_1\|Sx_n - Sp\| + a_2\|Sx_n - J_{rn}p\|) & \\
 + \min\left\{ \|J_{rn}x_n - Tx_n\|, \frac{\|Sx_n - J_{rn}x_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} & \\
 = (1 - a_n)b\psi\|Sx_n - p\| + a_nb\psi(\|Sx_n - p\|) & \\
 \leq (1 - a_n)b\|Sx_n - p\| + a_nb\|Sx_n - P\| & \\
 \leq ((1 - a_n) + a_nb)\|Sx_n - P\| & \\
 \leq b\|Sx_n - p\|. & \tag{2.1}
 \end{aligned}$$

$$\begin{aligned}
 \|Sx_n - P\| &= \|J_{rn}[(1 - c_n)Sv_n + c_nTv_n] - p\| \\
 &\leq \|(1 - c_n)Sv_n + c_nTv_n - p\| \\
 &\leq (1 - c_n)\|Sv_n - t\| + c_n\|Tv_n - p\| \\
 &\leq (1 - c_n)\|Sv_n - p\| + c_nb\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - J_{rn}p\|) \\
 + \min\left\{ \|J_{rn}v_n - Tv_n\|, \frac{\|Sv_n - J_{rn}v_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} & \\
 \leq (1 - c_n)\|Sv_n - p\| + c_nb\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|) & \\
 \leq (1 - c_n)\|Sv_n - p\| + c_nb\psi(\|Sv_n - p\|) & \\
 \leq (1 - c_n)\|Sv_n - p\| + c_nb\|Sv_n - p\| & \\
 = ((1 - c_n) + c_nb)\|Sv_n - p\|. & \tag{2.2}
 \end{aligned}$$

$$\begin{aligned}
 \|Sv_n - p\| &= \|(1 - b_n)Su_n + b_nJ_{rn}Tu_n - p\| \\
 &\leq (1 - b_n)\|Su_n - p\| + b_n\|Tu_n - p\| \\
 &\leq (1 - b_n)\|Su_n - P\| + b_n\|Tu_n - P\| \\
 &\leq (1 - b_n)\|Su_n - p\| + b_nb\psi(a_1\|Su - Sp\| + a_2\|Su_n - J_{rn}p\|) \\
 + \min\left\{ \|J_{rn}u_n - Tu_n\|, \frac{\|Su_n - J_{rn}u_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} & \\
 = (1 - b_n)\|Su_n - P\| + b_nb\psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|) & \\
 \leq (1 - b_n)\|Su_n - p\| + b_nb\psi(\|Su_n - p\|) & \\
 \leq (1 - b_n)\|Su_n - t\| + b_nb\|Su_n - p\| & \\
 = ((1 - b_n) + b_nb)\|Su_n - p\| & \tag{2.3}
 \end{aligned}$$

We substitute (2.3) in (2.2) as follows

$$\|Sx_n - p\| \leq ((1 - c_n) + c_nb)((1 - b_n) + b_nb)\|Su_n - p\| \tag{2.4}$$

We substitute (2.4) in (2.1) as follows

$$\begin{aligned}
 \|Su_{n+1} - p\| &\leq b(1 - c_n(1 - b)(1 - b_n(1 - b)))\|Su_n - p\| \\
 &\leq (1 - c_n(1 - b))\|Su_n - p\|
 \end{aligned}$$

$$\|Su_{n+1} - p\| \leq \|Su_n - p\|$$

$$\|Su_{n+1} - p\| \leq \|Su_n - p\| \leq \dots \dots \dots \leq \|Su_0 - p\|.$$

So,  $\{\|Su_n - p\|\}$  is decreasing and bounded, for each  $p \in f(S) \cap f(T) \cap A^{-1}(0)$ , this implies that  $\lim_{n \rightarrow \infty} \|Su_n - p\|$  exists.

**Theorem 2.2:** Let  $(N, \|\cdot\|)$  be a Banach-space, and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on. If  $\langle Su_n \rangle$  is resolvent Jungck  $SP$  – iteration scheme. Then the resolvent Jungck  $SP$  – iteration scheme converge-strongly to a unique common fixed- point of  $S, T$  and  $A^{-1}(0)$

**Proof:** Let  $P \in F(S) \cap F(T) \cap A^{-1}(0)$ . By Lemma. (2.1), we have

$$\begin{aligned} \|Su_{n+1} - p\| &\leq (1 - c_n(1 - b)) \|Su_n - p\| \\ \|Su_{n+1} - p\| &\leq \prod_{j=0}^n (1 - c_j(1 - b)) \|Su_0 - p\| \end{aligned} \tag{2.5}$$

$$\|Su_n - p\| \leq e^{-(1-b)\sum_{j=0}^{\infty} c_j} \|Su_0 - p\|$$

Since  $c_j \in [0, 1]$  and  $b \in [0, 1]$  and  $\sum_{j=0}^{\infty} c_j = \infty$

Then  $e^{-(1-b)\sum_{j=0}^{\infty} c_j} \rightarrow 0$  as  $n \rightarrow \infty$

from (2.5),  $\lim_{n \rightarrow \infty} \|Su_n - p\| = 0$ . Therefore,  $\langle Se_n \rangle$  converges strongly to  $t$

Now, we prove that the unique common fixed point of  $S, T$  and  $A^{-1}(0)$ . Let there exist another point  $p^* \in C$  such that  $Sp^* = J_{rn} p^* = Tp^* = p^*$

$$\begin{aligned} 0 \leq \|p - p^*\| &= \|Tp - Tp^*\| \\ &\leq b\psi(a_1\|Sp - Sp^*\| + a_2\|Sp - J_{rn} p^*\|) + \\ &\quad \min \left\{ \|J_{rn}p - Tp\|, \frac{\|Sp - J_{rn} p^*\| + \|J_{rn} p^* - Sp^*\|}{2}, \|Tp^* - J_{rn} p^*\| \right\} \\ &\leq b(a_1\|Sp - Sp^*\| + a_2\|Sp - J_{rn} p^*\|) \\ &\leq b\|Sp - Sp^*\| \\ &\quad p = p^* \end{aligned}$$

**Theorem 2.3:** Let  $\mathcal{N}$  be a uniformly convex Banach-space,  $\mathcal{D}$  be a nonempty convex-closed subset of  $\mathcal{N}$  and  $T$  be a resolvent  $\mathcal{Z}\mathcal{A}$ -jungck self-mapping on  $\mathcal{D}$ . If  $\langle Su_n \rangle$  the resolvent Jungck  $\mathcal{SP}$ -iteration scheme in  $\mathcal{D}$  then  $F(S) \cap F(T) \cap A^{-1}(0) \neq \emptyset$  if and only if  $\lim_{n \rightarrow \infty} \|Tu_n - J_{rn}Su_{n+1}\| = 0$ ..

**Proof:** Since  $F(S) \cap F(T) \cap A^{-1}(0) \neq \emptyset \Rightarrow \exists p \in F(S) \cap F(T) \cap A^{-1}(0) \Rightarrow p = F(S) = f(t) = A^{-1}(0)$ .

By Lemma (2.1) We have,  $\lim_{n \rightarrow \infty} \|Su_n - p\| = \delta$

$$\text{We get, } \limsup_{n \rightarrow \infty} \|J_{rn}Su_{n+1} - p\| \leq \delta \tag{2.6}$$

Now, To prove that  $\limsup_{n \rightarrow \infty} \|Tu_n - p\| \leq \delta$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Tu_n - p\| &\leq \limsup_{n \rightarrow \infty} b\psi(a_1\|Su_n - Sp\| + a_2\|Su_n - J_{rn}p\|) \\ &\quad + \min \left\{ \|J_{rn}u_n - Tu_n\|, \frac{\|Su_n - J_{rn}u_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &= \limsup_{n \rightarrow \infty} b\psi(a_1\|Su_n - Sp\| + a_2\|Su_n - J_{rn}p\|) \\ &\leq \limsup_{n \rightarrow \infty} \|Su_n - Sp\|. \\ &= \delta \end{aligned}$$

$$\text{We get, } \limsup_{n \rightarrow \infty} \|Tu_n - p\| \leq \delta \tag{2.7}$$

Now by Lemma (2.1) we have

$$\begin{aligned} \delta &= \limsup_{n \rightarrow \infty} \|Su_{n+1} - p\| = \limsup_{n \rightarrow \infty} \|(1 - a_n)J_{rn}TSx_n + a_nTx_n - p\| \\ &\leq \limsup_{n \rightarrow \infty} [(1 - a_n)\|J_{rn}TSx_n - p\| + a_n\|Tx_n - p\|] \\ &\leq \limsup_{n \rightarrow \infty} [(1 - a_n)\|TSx_n - p\| + a_n\|Tx_n - p\|] \\ &\leq \limsup_{n \rightarrow \infty} [(1 - a_n)b\psi(a_1\|SSx_n - Sp\| + a_2\|SSx_n - J_{rn}p\|) + \\ &\quad \min \left\{ \|J_{rn}Sx_n - TSx_n\|, \frac{\|SSx_n - J_{rn}Sx_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\quad + a_nb\psi(a_1\|Sx_n - Sp\| + a_2\|Sx_n - J_{rn}p\|) \\ &\quad + \min \left\{ \|J_{rn}x_n - Tx_n\|, \frac{\|Sx_n - J_{rn}x_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\}] \\ &\leq \limsup_{n \rightarrow \infty} [(1 - a_n)b\psi\|Sx_n - p\| + a_nb\psi(\|Sx_n - p\|)] \\ &\leq \limsup_{n \rightarrow \infty} [(1 - a_n)b\|Sx_n - p\| + a_nb\|Sx_n - p\|] \\ &= \limsup_{n \rightarrow \infty} [(1 - a_n) + a_nb]\|Sx_n - p\| \end{aligned}$$

$$\leq \limsup_{n \rightarrow \infty} \|Sx_n - p\| . \tag{2.8}$$

$$\begin{aligned} \|Sx_n - P\| &= \|J_{rn}[(1 - c_n)Sv_n + c_nTv_n] - p\| \\ &\leq \|(1 - c_n)Sv_n + c_nTv_n - p\| \\ &\leq (1 - c_n)\|Sv_n - t\| + c_n\|Tv_n - p\| \\ &\leq (1 - c_n)\|Sv_n - p\| + c_n b\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - J_{rn}p\|) \\ &+ \min \left\{ \|J_{rn}v_n - Tv_n\|, \frac{\|Sv_n - J_{rn}v_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\leq (1 - c_n)\|Sv_n - p\| + c_n b\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|) \\ &\leq (1 - c_n)\|Sv_n - p\| + c_n b\psi(\|Sv_n - p\|) \\ &\leq (1 - c_n)\|Sv_n - p\| + c_n b\|Sv_n - p\| \\ &= ((1 - c_n) + c_n b)\|Sv_n - p\| \end{aligned} \tag{2.9}$$

$$\begin{aligned} \|Sv_n - p\| &= \|(1 - b_n)Su_n + b_nJ_{rn}Tu_n - p\| \\ &\leq (1 - b_n)\|Su_n - p\| + b_n\|Tu_n - p\| \\ &\leq (1 - b_n)\|Su_n - P\| + b_n\|Tu_n - P\| \\ &\leq (1 - b_n)\|Su_n - p\| + b_n b\psi(a_1\|Su - Sp\| + a_2\|Su_n - J_{rn}p\|) \\ &+ \min \left\{ \|J_{rn}u_n - Tu_n\|, \frac{\|Su_n - J_{rn}u_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &= (1 - b_n)\|Su_n - P\| + b_n b\psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|) \\ &\leq (1 - b_n)\|Su_n - p\| + b_n b\psi(\|Su_n - p\|) \\ &\leq (1 - b_n)\|Su_n - t\| + b_n b\|Su_n - p\| \\ &= ((1 - b_n) + b_n b)\|Su_n - p\| \end{aligned} \tag{2.10}$$

We substitute (2.10) in (2.9) as follows:

$$\|Sx_n - p\| \leq ((1 - c_n) + c_n b)((1 - b_n) + b_n b)\|Su_n - p\| \tag{2.11}$$

We substitute (2.11) in (2.8) as follows:

$$\begin{aligned} \|Su_{n+1} - p\| &\leq \limsup_{n \rightarrow \infty} b(1 - c_n(1 - b)(1 - b_n(1 - b)))\|Su_n - p\| \\ &\leq \limsup_{n \rightarrow \infty} (1 - c_n(1 - b))\|Su_n - p\|. \end{aligned} \tag{2.12}$$

$$\limsup_{n \rightarrow \infty} \|Se_{n+1} - t\| \leq \limsup_{n \rightarrow \infty} \|Se_n - t\| = \delta$$

By (2.6), (2.7), (2.12) and use Lemma (1.3) we get

$$\lim_{n \rightarrow \infty} \|Te_n - J_{rn}Se_{n+1}\| = 0$$

Now, to prove that  $f(T) \neq \emptyset$

$$\text{Let } p \in A(\mathcal{D}, \langle J_{rn}Su_{n+1} \rangle) \Rightarrow r(\mathcal{D}, \langle Su_n \rangle) = r(\mathcal{T}p, \langle J_{rn}Su_{n+1} \rangle)$$

$$\text{Now, } r(\mathcal{T}p, \langle J_{rn}Su_{n+1} \rangle) = \limsup_{n \rightarrow \infty} \|J_{rn}Su_{n+1} - \mathcal{T}p\| \leq \limsup_{n \rightarrow \infty} [\|J_{rn}Su_{n+1} - \mathcal{T}O_n\| + \|Tu_n - \mathcal{T}p\|]$$

$$\begin{aligned} &\leq \limsup_{n \rightarrow \infty} [b\psi(a_1\|Su_n - p\| + a_2\|Su_n - p\|) \\ &+ \min \left\{ \|J_{rn}u_n - Tu_n\|, \frac{\|Su_n - J_{rn}u_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\}] \\ &= \limsup_{n \rightarrow \infty} b\psi(a_1\|Su_n - p\| + a_2\|Su_n - p\|) \\ &\leq \limsup_{n \rightarrow \infty} \|Su_n - p\| = r(p, \langle Su_n \rangle) = r(\mathcal{D}, \langle Su_n \rangle) \\ &= r(\mathcal{D}, \langle Su_n \rangle) = r(\mathcal{T}p, \langle J_{rn}Su_{n+1} \rangle) \end{aligned}$$

$\mathcal{T}p \in A(\mathcal{D}, \langle Su_n \rangle)$   $\mathcal{D}$  a uniformly convex  $\Rightarrow A(\mathcal{D}, \langle Su_n \rangle)$  is singleton

$$\Rightarrow p = \mathcal{T}p \Rightarrow p \in f(T) \Rightarrow f(T) \neq \emptyset$$

In the same way, we get,  $\rho \in f(T)$  &  $\rho \in A^{-1}(0)$

$$\Rightarrow f(T) \cap f(S) \cap A^{-1}(0) \neq \emptyset$$

**Lemma 2.7:** Let  $(N, \| \cdot \|)$  be a Banach-space, and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on  $\mathcal{D}$  with  $b \in [0,1)$ . Suppose that  $\langle Se_n \rangle$  be a resolvent Jungck  $\mathcal{CR}$  – iteration scheme in  $\mathcal{D}$ . There exists  $p \in F(S) \cap F(T) \cap A^{-1}(0) \neq \emptyset$ , then  $\lim_{n \rightarrow \infty} \|Se_n - p\|$  exists, for all  $n \in \mathbb{N}$ .

**Proof:** As the same proof of lemma 2.1

**Theorem 2.8:** Let  $(N, \| \cdot \|)$  be a Banach-space, and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on  $\mathcal{D}$ . If  $\langle Se_n \rangle$  is resolvent Jungck  $\mathcal{CR}$  – iteration scheme. Then the resolvent Jungck  $\mathcal{CR}$  – iteration scheme converges-strongly to a unique common fixed– point of  $S, T$  and  $A^{-1}$

**Proof:** As the same proof of Theorem 2.2

**Theorem 2.9:** Let  $N$  be a uniformly convex Banach-space,  $\mathcal{D}$  be a nonempty convex-closed subset of  $N$  and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on  $\mathcal{D}$ . If  $\langle Se_n \rangle$  the resolvent Jungck  $\mathcal{CR}$  – Iteration scheme in  $\mathcal{D}$  then  $F(S) \cap F(T) \cap A^{-1}(0) \neq \emptyset$  if and only if  $\lim_{n \rightarrow \infty} \|Ge_n - J_{rn}Se_{n+1}\| = 0$

**Proof:** the same proof Theorem 2.3

**Theorem 2.10:** Let  $(N, \| \cdot \|)$  be a normed–space, and  $T$  be a resolvent  $\mathcal{ZA}$ -jungck self-mapping on  $\mathcal{D}$  if  $P \in f(s) \cap f(T) \cap A^{-1}(0) \neq \emptyset$  where  $0 < r \leq a_n, b_n, c_n < 1$  then the resolvent Jungck  $\mathcal{CR}$  – iteration scheme is faster the resolvent Jungck  $\mathcal{SP}$  – iteration scheme

**Proof:** Let  $p \in f(S) \cap f(T) \cap A^{-1}(0)$

The resolvent Jungck  $\mathcal{CR}$  – iteration scheme, we have

$$\begin{aligned} \|se_{n+1} - p\| &= \|J_{rn}((1 - a_n)TSw_n + a_n J_{rn}TTw_n) - p\| \\ &\leq \|(1 - a_n)TSw_n + a_n TTw_n - p\| \\ &\leq (1 - a_n)\|TSw_n - p\| + a_n\|TTw_n - p\| \\ &\leq (1 - a_n)b\psi(a_1\|SSw_n - Sp\| + a_2\|SSw_n - J_{rn}p\|) \\ &+ \min \left\{ \|J_{rn}Sw_n - TSw_n\|, \frac{\|SSw_n - J_{rn}Sw_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\quad + a_nb\psi(a_1\|STw_n - Sp\| + a_2\|STw_n - J_{rn}p\|) + \\ &\min \left\{ \|J_{rn}Tw_n - TTw_n\|, \frac{\|Sw_n - J_{rn}m_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\leq (1 - a_n)b\psi\|Sw_n - p\| + a_nb\|TSw_n - p\| \\ &\leq (1 - a_n)b\|Sw_n - p\| + a_nb\psi(a_1\|SSw_n - Sp\| + a_2\|SSw_n - J_{rn}p\|) \\ &+ \min \left\{ \|J_{rn}Sw_n - GSw_n\|, \frac{\|SSw_n - J_{rn}Sw_n\| + \|J_{rn}p - Sp\|}{2}, \|Gp - J_{rn}p\| \right\} \\ &\leq (1 - a_n)b\|Sw_n - p\| + a_nb^2\|Sw_n - p\| \tag{2.13} \\ &\leq b(1 - a_n(1 - b))\|Sw_n - p\| \end{aligned}$$

Now, to find  $\|Sw_n - p\|$ .

$$\begin{aligned} \|Sw_n - P\| &= \|(1 - c_n)J_{rn}TSz_n + c_n TTw_n - p\| \\ &\leq (1 - c_n)\|TSz_n - p\| + c_n\|TTw_n - p\| \\ &\leq (1 - c_n)\|TSz_n - p\| + c_n\|TTz_n - p\| \\ &\leq (1 - c_n)b\psi(a_1\|SSz_n - Sp\| + a_2\|SSz_n - J_{rn}p\|) + \\ &\min \left\{ \|J_{rn}Sz_n - TSz_n\|, \frac{\|SSz_n - J_{rn}Sz_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\quad + c_nb\psi(a_1\|STz_n - Sp\| + a_2\|STz_n - J_{rn}p\|) + \\ &\min \left\{ \|J_{rn}Tz_n - TTw_n\|, \frac{\|STz_n - J_{rn}Tz_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\leq (1 - c_n)b\|Sz_n - p\| + c_nb(\|TSz_n - p\|) \\ &\leq (1 - c_n)b\|Sz_n - p\| + c_nb\psi(a_1\|SSz_n - Sp\| + a_2\|SSz_n - J_{rn}p\|) \\ &+ \min \left\{ \|J_{rn}z_n - Gz_n\|, \frac{\|Sz_n - J_{rn}z_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\ &\leq (1 - c_n)b\|Sz_n - p\| + c_nb^2\|Sz_n - p\| \\ &= b(1 - c_n(1 - b))\|Sz_n - p\|. \tag{2.14} \end{aligned}$$

$$\begin{aligned}
 \|S z_n - p\| &= \|(1 - b_n)J_{rn}TSe_n + b_nJ_{rn}Te_n p\| \\
 &\leq (1 - b_n)\|J_{rn}TSe_n - p\| + b_n\|J_{rn}Te_n - p\| \\
 &\leq (1 - b_n)\|TSe_n - p\| + b_n\|Te_n - p\| \\
 &\leq (1 - b)\mathfrak{b}\psi(a_1\|SSe_n - Sp\| + a_2\|SSe_n - J_{rn}p\|) + \min\left\{\|J_{rn}Se_n - \right. \\
 &TSe_n\|, \frac{\|SSe_n - J_{rn}Se_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\|\left.\right\} \\
 &\quad + b_n\mathfrak{b}\psi(a_1\|Se_n - Sp\| + a_2\|Se_n - J_{rn}p\|) \\
 &+ \min\left\{\|J_{rn}e_n - Te_n\|, \frac{\|Se_n - J_{rn}e_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\|\right\} \\
 &\leq (1 - b_n)\mathfrak{b}\|Se_n - p\| + b_n\mathfrak{b}(\|Se_n - p\|) \\
 &\leq \mathfrak{b}\|Se_n - p\| \\
 &\leq 1\|Se_n - p\|
 \end{aligned} \tag{2.15}$$

We substitute (2.15) in (2.14) as follows

$$\|Sw_n - p\| \leq \mathfrak{b}^2(1 - c_n(1 - \mathfrak{b}))\|Se_n - t\| \tag{2.16}$$

We substitute (2.16) in (2.13) as follows

$$\begin{aligned}
 \|Se_{n+1} - p\| &\leq \mathfrak{b}^2(1 - a_n(1 - \mathfrak{b}))(1 - c_n(1 - \mathfrak{b}))\|Se_n - t\| \\
 &\leq \mathfrak{b}^2(1 - r(1 - \mathfrak{b}))^2\|Se_n - t\|
 \end{aligned}$$

$$\|Se_{n+1} - t\| \leq \mathfrak{b}^{2n}(1 - r(1 - \mathfrak{b}))^{2n}\|Se_0 - t\|$$

The resolvent Jungck  $SP$  - Iteration scheme, we have

$$\begin{aligned}
 \|Su_{n+1} - p\| &= \|(1 - a_n)J_{rn}TSx_n + a_nTx_n - p\| \\
 &\leq (1 - a_n)\|J_{rn}TSx_n - p\| + a_n\|Tx_n - p\| \\
 &\leq (1 - a_n)\|TSx_n - p\| + a_n\|Tx_n - p\| \\
 &(1 - a_n)\mathfrak{b}\psi(a_1\|SSx_n - SP\| + a_2\|SSx_n - J_{rn}p\|) + \min\left\{\|J_{rn}Sx_n - \right. \\
 &TSx_n\|, \frac{\|SSx_n - J_{rn}Sx_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\|\left.\right\} \\
 &\quad + a_n\mathfrak{b}\psi(a_1\|Sx_n - Sp\| + a_2\|Sx_n - J_{rn}p\|) \\
 &+ \min\left\{\|J_{rn}x_n - Tx_n\|, \frac{\|Sx_n - J_{rn}x_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\|\right\} \\
 &= (1 - a_n)\mathfrak{b}\psi\|Sx_n - p\| + a_n\mathfrak{b}\psi(\|Sx_n - p\|) \\
 &\leq (1 - a_n)\mathfrak{b}\|Sx_n - p\| + a_n\mathfrak{b}\|Sx_n - P\| \\
 &\leq ((1 - a_n) + a_n\mathfrak{b})\|Sx_n - P\| \\
 &\leq \mathfrak{b}\|Sx_n - p\|
 \end{aligned} \tag{2.17}$$

$$\begin{aligned}
 \|Sx_n - P\| &= \|J_{rn}[(1 - c_n)Sv_n + c_nTv_n] - p\| \\
 &\leq \|(1 - c_n)Sv_n + c_nTv_n - p\| \\
 &\leq (1 - c_n)\|Sv_n - t\| + c_n\|Tv_n - p\| \\
 &\leq (1 - c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - J_{rn}p\|) \\
 &+ \min\left\{\|J_{rn}v_n - Tv_n\|, \frac{\|Sv_n - J_{rn}v_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\|\right\} \\
 &\leq (1 - c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|) \\
 &\leq (1 - c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(\|Sv_n - p\|) \\
 &\leq (1 - c_n)\|Sv_n - p\| + c_n\mathfrak{b}\|Sv_n - p\| \\
 &= ((1 - c_n) + c_n\mathfrak{b})\|Sv_n - p\|
 \end{aligned} \tag{2.18}$$

$$\begin{aligned}
 \|Sv_n - p\| &= \|(1 - b_n)Su_n + b_nJ_{rn}Tu_n - p\| \\
 &\leq (1 - b_n)\|Su_n - p\| + b_n\|Tu_n - p\| \\
 &\leq (1 - b_n)\|Su_n - P\| + b_n\|Tu_n - P\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq (1 - b_n)\|Su_n - p\| + b_n b \psi(a_1\|Su - Sp\| + a_2\|Su_n - J_{rn}p\|) \\
 &+ \min \left\{ \|J_{rn}u_n - Tu_n\|, \frac{\|Su_n - J_{rn}u_n\| + \|J_{rn}p - Sp\|}{2}, \|Tp - J_{rn}p\| \right\} \\
 &= (1 - b_n)\|Su_n - P\| + b_n b \psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|) \\
 &\leq (1 - b_n)\|Su_n - p\| + b_n b \psi(\|Su_n - p\|) \\
 &\leq (1 - b_n)\|Su_n - t\| + b_n b \|Su_n - p\| \\
 &= ((1 - b_n) + b_n b) \|Su_n - p\| \tag{2.19}
 \end{aligned}$$

We substitute (2.19) in (2.18) as follows

$$\|Sx_n - p\| \leq ((1 - c_n) + c_n b)((1 - b_n) + b_n b) \|Su_n - p\| \tag{2.20}$$

We substitute (2.20) in (2.17) as follows

$$\begin{aligned}
 \|Su_{n+1} - p\| &\leq b(1 - c_n(1 - b)(1 - b_n(1 - b))) \|Su_n - p\| \\
 &\leq (1 - c_n(1 - b)) \|Su_n - p\| \\
 \|Su_{n+1} - t\| &\leq b^n(1 - r(1 - b))^{2n} \|Se_0 - t\|
 \end{aligned}$$

$$\frac{\|Se_{n+1} - t\|}{\|Su_{n+1} - t\|} \leq \frac{b^{2n}(1 - r(1 - b))^{2n} \|Se_0 - t\|}{b^n(1 - r(1 - b))^{2n} \|Su_0 - t\|} \rightarrow 0 \quad n \rightarrow \infty$$

### 3. Conclusion

In this section, we will provide some conclusions. New iterations were performed and the convergence and acceleration of these iterations to the common fixed point were demonstrated. We also install the resolvent Jungck CR – iteration is faster than resolvent Jungck SP – iteration.

### References

1. Ali ,J. ; Ali ,F., Approximation of Common Fixed Points and the Solution of Image Recovery Problem *J. Phys.: Conf. Ser.* **2019** ,74, 4, 0–22, 2019.
2. Bahuguna,D. ; Sharma, A., On the Convergence of a New Iterative Algorithm of Three Infinite Families of Generalized Nonexpansive Multi-Valued Mappings *Proc. Indian Acad. Sci. Math.* **2018** Sci. 128, 4, 1–15.
3. Garodia, Chanchal, ; Izhar Uddin., A New Fixed Point Algorithm for Finding the Solution of a Delay Differential Equation. *AIMS Mathematics* ,**2020**,3182-3200, 5.4
- 4.Saad Hussein ; Zena H. Maibed ,Iterative Methods for Approximation of Fixed Points Via Like Contraction Mapping, *J. Phys.: Conf. Ser.* **2021**,1818 012152.
5. Daengsaen,j ; Khemphet Hindawi Abstr.On the Rate of Convergence of P-Iteration, SP-Iteration, and D-Iteration Methods for Continuous Nondecreasing Functions on Closed Intervals, *Appl. Anal.* **2018**, 6.
6. Ullah ,K. ; Muhammad, New Three-Step Iteration Process and Fixed Point Approximation in Banach Spaces *Linear Topol. Algebra.* **2018**, 07, 02, 87–100,.
7. Zena. H. Maibed and Saad. S. Hussein, Some Theorems of Fixed Point Approximations by Iteration Processes, *J. Phys.* **2021**, 1818 01215
8. Zena,H .Maibed ; Ali Q Thajil, Zenali Iteration Method for Approximating Fixed Point of A 'ZA - Quasi Contractive Mappings" *Haitham Journal for Pure and Applied Science Oct* **2021**.
- 9.Zena H Maibed, Contractive Mappings Having Mixed Finite Monotone Property in Generalized Metric Spaces *Haitham Journal for Pure and Applied Science.1.* **2019**, 32 (1) .
10. Singh S. L.; Bhatnagar Charu ; Mishra S. N. Stability of Jungck-type iterative procedures, *Int. J. Math. Sci.*, **2015**, 19, 3035-3043.
- 11.Garodia ; Izhar, A New Fixed Point Algorithm for Finding the Solution of a Delay Differential Equation, *AIMS Math.*, **2020** , 3182–3200.



- 12.Karakaya, V., Atalan, ; Y., Doğan, K. Convergence Analysis for a New Faster Iteration Method *İstanbul Ticaret Üniversitesi Fen Bilim* . **2016**,15.30, 35–53.
- 13.Jamil, Z. Z. ; M. B. Abed, On A Modified SP-Iterative Scheme for approximating Fixed Point of A Contraction Mapping, *Iraqi Journal of Scienceno* .**2015**,1 56, 4B, 3230-3239,June.
- 14.Zena H. M ; Alaa M. AOn the Convergence of New Iteration Schemes by Resolvent ZA-Jungck mapping, *Journal of Interdisciplinary Mathematics* to appear, **2022**.