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Fuzzy Generalized Alpha Generalized Closed Sets

المجموعات المعممة ألفا المعممة المغلقة الضبابية

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Abstract:

In this paper we introduced a new class of fuzzy closed sets called fuzzy $g\alpha$g-closed sets and study their basic properties in fuzzy topological spaces. We also introduced fuzzy $g\alpha$g-continuous functions with some of its properties. Moreover, the investigation will include some of the properties of the fuzzy separation axioms such fuzzy $g\alpha g$-R_i-space and fuzzy $g\alpha g$-T_j-space (note that, the indexes i and j are natural numbers of the spaces R and T are from 0 to 1 and from 0 to 2 respectively).

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الخلاصة:

في هذا البحث قدمنا فئة جديدة من المجموعات المغلقة الضبابية تسمى بالمجموعات المعممة ألفا المعممة المغلقة الضبابية ودراسة خصائصها الأساسية في الفضاءات التبولوجية الضبابية. قدمنا أيضا الدوال المعممة ألفا المعممة المستمرة الضبابية مع بعض خصائصها. علاوة على ذلك، سيشمل تحقيق بعض خصائص بديهيات الفصل الضبابية مثل فضاء $g\alpha g$-R_i الضبابي و فضاء $g\alpha g$-T_j الضبابي (لاحظ أن المؤشرات i و j هي اعداد طبيعية من الفضاءات R و T هي من 0 إلى 1 ومن 0 إلى 2 على التوالي).

1. Introduction

In 1965 Zadeh studied the fuzzy sets (briefly F-sets) (see [5]) which plays such a role in the field of fuzzy topological spaces (or simply FTS). The fuzzy topological spaces investigated by Chang in 1968 (see [2]). A. S. Bin Shahna [1] defined fuzzy α-closed sets. In 1997, fuzzy generalized closed set (briefly Fg-CS) was introduced by G. Balasubramania and P. Sundaram [4]. S. Kalaiselvi and V. Seenivasan [12] introduced the concept of fuzzy gsg-closed sets in FTS. The purpose of this paper is to introduce the concept of fuzzy $g\alpha$g-closed sets and study their basic properties in FTS. We also introduce fuzzy $g\alpha$g-continuous functions by using fuzzy $g\alpha$g-closed sets and study some of their fundamental properties. Furthermore, the investigation will include some of the properties of the fuzzy separation axioms such fuzzy $g\alpha g$-R_i-space and fuzzy $g\alpha g$-T_j-space (here the indexes i and j are natural numbers of the spaces R and T are from 0 to 1 and from 0 to 2 respectively).

2. Preliminaries

Throughout this paper, $(X,\tau),(Y,\psi)$ and (Z,ρ) (or simply X,Y and Z) always mean FTS on which no separation axioms are assumed unless otherwise mentioned. A fuzzy point [3] with support $x \in X$ and value λ $(0 < \lambda \leq 1)$ at $x \in X$ will be denoted by x_λ, and for F-set $\mathcal{M}$, $x_\lambda \in \mathcal{M}$ iff $\lambda \leq \mathcal{M}(x)$. Two fuzzy points x_λ and y_μ are said to be distinct iff their supports are distinct. That is, by 0_X and 1_X we mean the constant F-sets taking the values 0 and 1 on X, respectively. For a F-set $\mathcal{M}$ in a FTS (X,τ), $cl(\mathcal{M})$, $int(\mathcal{M})$ and $\mathcal{M}^c = 1_X - \mathcal{M}$ denote the fuzzy closure of $\mathcal{M}$, the fuzzy interior of $\mathcal{M}$ and the fuzzy complement of $\mathcal{M}$ respectively.

Definition 2.1:[11] A fuzzy point in a set X with support x and membership value 1 is called crisp point, denoted by x_1. For any F-set $\mathcal{M}$ in X, we have $x_1 \in \mathcal{M}$ iff $\mathcal{M}(x) = 1$.

Proposition 2.2:[10] Let $\mathcal{M}, \mathcal{N}, \mathcal{O}$ be F-sets in a FTS (X, τ). Then $\mathcal{M}q(\mathcal{N} \vee \mathcal{O})$ iff $\mathcal{M}q\mathcal{N}$ or $\mathcal{M}q\mathcal{O}$.

Definition 2.3:[6] A fuzzy point $x_\lambda \in \mathcal{M}$ is called quasi-coincident (briefly q-coincident) with the F-set $\mathcal{M}$ is denoted by $x_\lambda q\mathcal{M}$ iff $\lambda + \mathcal{M}(x) > 1$. A F-set $\mathcal{M}$ in a FTS (X, τ) is called q-coincident with a F-set $\mathcal{N}$ which is denoted by $\mathcal{M}q\mathcal{N}$ iff there exists $x \in X$ such that $\mathcal{M}(x) + \mathcal{N}(x) > 1$. If the F-sets $\mathcal{M}$ and $\mathcal{N}$ in a FTS (X, τ) are not q-coincident then we write $\mathcal{M}\bar{q}\mathcal{N}$. Note that $\mathcal{M} \leq \mathcal{N} \Leftrightarrow \mathcal{M}\bar{q}(1_X - \mathcal{N})$.

Definition 2.4:[6] A F-set $\mathcal{M}$ in a FTS (X, τ) is called q-neighbourhood(briefly q-nhd)of a fuzzy point x_λ (resp. F-set $\mathcal{N}$)if there is a F-OS $\mathcal{A}$ in a FTS (X, τ) such that $x_\lambda q\mathcal{A} \leq \mathcal{M}$ (resp. $\mathcal{N}q\mathcal{A} \leq \mathcal{M}$).

Proposition 2.5:[4,7] Let $\mathcal{M}, \mathcal{N}$ be two F-sets in a FTS (X, τ). Then the following properties hold:
(i) $\mathcal{M}$ is a F-OS iff $\mathcal{M} = int(\mathcal{M})$.
(ii) $\mathcal{M}$ is a F-CS iff $\mathcal{M} = cl(\mathcal{M})$.
(iii) $int(\mathcal{M}) \leq \mathcal{M}$, $int(int(\mathcal{M})) = int(\mathcal{M})$.
(iv) $int(\mathcal{M}) \leq int(\mathcal{N})$, whenever $\mathcal{M} \leq \mathcal{N}$.
(v) $int(\mathcal{M} \wedge \mathcal{N}) = int(\mathcal{M}) \wedge int(\mathcal{N})$, $int(\mathcal{M} \vee \mathcal{N}) \geq int(\mathcal{M}) \vee int(\mathcal{N})$.
(vi) $\mathcal{M} \leq cl(\mathcal{M})$, $cl(cl(\mathcal{M})) = cl(\mathcal{M})$.
(vii) $cl(\mathcal{M}) \leq cl(\mathcal{N})$, whenever $\mathcal{M} \leq \mathcal{N}$.
(viii) $cl(\mathcal{M} \wedge \mathcal{N}) \leq cl(\mathcal{M}) \wedge cl(\mathcal{N})$, $cl(\mathcal{M} \vee \mathcal{N}) = cl(\mathcal{M}) \vee cl(\mathcal{N})$.

Lemma 2.6:[7] Let $\mathcal{M}$ be any F-set in a FTS (X, τ). Then the following properties hold:
(i) $cl(1_X - \mathcal{M}) = 1_X - int(\mathcal{M})$.
(ii) $int(1_X - \mathcal{M}) = 1_X - cl(\mathcal{M})$.

Definition 2.7:[2] Let X and Y be two non-empty sets, and $f: (X, \tau) \longrightarrow (Y, \psi)$ be a function. If $\mathcal{M}$ is a F-set of X and $\mathcal{N}$ is a F-set of Y, then:
(i) $f(\mathcal{M})$ is a F-set of Y, where

$$f(\mathcal{M}) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mathcal{M}(x), & \text{if } f^{-1}(y) \neq 0, \\ 0\,, & \text{otherwise} \end{cases}$$

for every $y \in Y$.
(ii) $f^{-1}(\mathcal{N})$ is a F-set of X, where $f^{-1}(\mathcal{N})(x) = \mathcal{N}(f(x))$ for each $x \in X$.
(iii) $f^{-1}(1_Y - \mathcal{N}) = 1_X - f^{-1}(\mathcal{N})$.

Theorem 2.8:[2] Let X and Y be two non-empty sets, and $f: (X, \tau) \longrightarrow (Y, \psi)$ be a function, then:
(i) $f^{-1}(\mathcal{N}^c) = (f^{-1}(\mathcal{N}))^c$, for any F-set $\mathcal{N}$ in Y.
(ii) $f(f^{-1}(\mathcal{N})) \leq \mathcal{N}$, for any F-set $\mathcal{N}$ in Y.
(iii) $\mathcal{M} \leq f^{-1}(f(\mathcal{M}))$, for any F-set $\mathcal{M}$ in X.

Definition 2.9:[1] A F-set $\mathcal{M}$ of a FTS (X, τ) is said to be a fuzzy α-open set (briefly Fα-OS) if $\mathcal{M} \leq int(cl(int(\mathcal{M})))$ and a fuzzy α-closed set (briefly Fα-CS) if $cl(int(cl(\mathcal{M}))) \leq \mathcal{M}$. The fuzzy α-closure of a F-set $\mathcal{M}$ of a FTS (X, τ) is the intersection of all Fα-CS that contain $\mathcal{M}$ and is denoted by $\alpha cl(\mathcal{M})$.

Definition 2.10:[4] A F-set $\mathcal{M}$ of a FTS (X, τ) is said to be a fuzzy g-closed set (briefly Fg-CS) if $cl(\mathcal{M}) \leq \mathcal{U}$ whenever $\mathcal{M} \leq \mathcal{U}$ and $\mathcal{U}$ is a F-OS in X. The complement of a Fg-CS in X is a Fg-OS in X.

Definition 2.11:[9] A F-set $\mathcal{M}$ of a FTS (X,τ) is said to be a fuzzy αg-closed set (briefly Fαg-CS) if $\alpha cl(\mathcal{M}) \leq \mathcal{U}$ whenever $\mathcal{M} \leq \mathcal{U}$ and $\mathcal{U}$ is a Fα-OS in X. The complement of a Fαg-CS in X is a Fαg-OS in X.

Definition 2.12:[8] A F-set $\mathcal{M}$ of a FTS (X,τ) is said to be a fuzzy gα-closed set (briefly Fgα-CS) if $\alpha cl(\mathcal{M}) \leq \mathcal{U}$ whenever $\mathcal{M} \leq \mathcal{U}$ and $\mathcal{U}$ is a F-OS in X. The complement of a Fgα-CS in X is a Fgα-OS in X.

Theorem 2.13:[1,4] In a FTS (X,τ), then the following statements hold and the converse of each statements are not true:
(i) Every F-OS (resp. F-CS) is a Fα-OS (resp. Fα-CS).
(ii) Every F-OS (resp. F-CS) is a Fg-OS (resp. Fg-CS).

Theorem 2.14:[8,9] In a FTS (X,τ), then the following statements hold and the converse of each statements are not true:
(i) Every F-OS (resp. F-CS) is a Fαg-OS (resp. Fαg-CS).
(ii) Every Fg-OS (resp. Fg-CS) is a Fgα-OS (resp. Fgα-CS).
(iii) Every Fα-OS (resp. Fα-CS) is a Fαg-OS (resp. Fαg-CS).
(iv) Every Fαg-OS (resp. Fαg-CS) is a Fgα-OS (resp. Fgα-CS).

Definition 2.15:[4] A FTS (X,τ) is said to be a fuzzy $T_{\frac{1}{2}}$-space (briefly F$T_{\frac{1}{2}}$-space) if every Fg-CS in it is a F-CS.

Definition 2.16: Let (X,τ) and (Y,ψ) be FTS. Then the function $f:(X,\tau) \longrightarrow (Y,\psi)$ is called:
(i) F-continuous [2] if $f^{-1}(\mathcal{V})$ is a F-OS (resp. F-CS) set in X, for each F-OS (resp. F-CS) $\mathcal{V}$ in Y.
(ii) Fα-continuous [1] if $f^{-1}(\mathcal{V})$ is a Fα-OS (resp. Fα-CS) in X, for each F-OS (resp. F-CS) $\mathcal{V}$ in Y.
(iii) Fg-continuous [4] if $f^{-1}(\mathcal{V})$ is a Fg-OS (resp. Fg-CS) in X, for each F-OS (resp. F-CS) $\mathcal{V}$ in Y.
(iv) Fαg-continuous [9] if $f^{-1}(\mathcal{V})$ is a Fαg-OS (resp. Fαg-CS) in X, for each F-OS (resp. F-CS) $\mathcal{V}$ in Y.
(v) Fgα-continuous [8] if $f^{-1}(\mathcal{V})$ is a Fgα-OS (resp. Fgα-CS) in X, for each F-OS (resp. F-CS) $\mathcal{V}$ in Y.

Theorem 2.17:[1,4] Let $f:(X,\tau) \longrightarrow (Y,\psi)$ be a function. Then the following statements hold and the converse of each statements are not true:
(i) Every F-continuous function is a Fα-continuous.
(ii) Every F-continuous function is a Fg-continuous.

Theorem 2.18:[8,9] Let $f:(X,\tau) \longrightarrow (Y,\psi)$ be a function. Then the following statements hold and the converse of each statements are not true:
(i) Every Fg-continuous function is a Fgα-continuous.
(ii) Every Fα-continuous function is a Fαg-continuous.
(iii) Every Fαg-continuous function is a Fgα-continuous.

3. Fuzzy gαg-Closed Sets

Definition 3.1: A F-set $\mathcal{M}$ of a FTS (X,τ) is said to be a fuzzy generalized αg-closed set (briefly Fgαg-CS) if $cl(\mathcal{M}) \leq \mathcal{U}$ whenever $\mathcal{M} \leq \mathcal{U}$ and $\mathcal{U}$ is a Fαg-OS in X. The family of all Fgαg-CS of a FTS (X,τ) is denoted by Fgαg-C(X).

Example 3.2: Let $X = \{a,b\}$ and the F-set $\mathcal{M}$ in X defined as follows: $\mathcal{M}(a) = 0.5, \mathcal{M}(b) = 0.5$. Let $\tau = \{0_X, \mathcal{M}, 1_X\}$ be a FTS. Then the F-sets $0_X, \mathcal{M}$ and 1_X are Fgαg-CS in X.

Definition 3.3: The intersection of all Fgαg-CS in a FTS (X,τ) containing $\mathcal{M}$ is called fuzzy gαg-closure of $\mathcal{M}$ and is denoted by gαg-$cl(\mathcal{M})$, gαg-$cl(\mathcal{M}) = \wedge\{\mathcal{N}: \mathcal{M} \leq \mathcal{N}, \mathcal{N}$ is a Fgαg-CS$\}$.

Theorem 3.4: In a FTS (X,τ), then the following statements are true:
(i) Every F-CS is a Fgαg-CS.
(ii) Every Fgαg-CS is a Fg-CS.
(iii) Every Fgαg-CS is a Fαg-CS.
(iv) Every Fgαg-CS is a Fgα-CS.

Proof: (i) Let $\mathcal{M}$ be a F-CS in a FTS (X,τ) and let $\mathcal{U}$ be any Fαg-OS containing $\mathcal{M}$. Then $cl(\mathcal{M}) = \mathcal{M} \leq \mathcal{U}$. Hence $\mathcal{M}$ is a Fgαg-CS.
(ii) Let $\mathcal{M}$ be a Fgαg-CS in a FTS (X,τ) and let $\mathcal{U}$ be any F-OS containing $\mathcal{M}$. By theorem (2.14) part (i), $\mathcal{U}$ is a Fαg-OS in X. Since $\mathcal{M}$ is a Fgαg-CS, we have $cl(\mathcal{M}) \leq \mathcal{U}$. Hence $\mathcal{M}$ is a Fg-CS.
(iii) Let $\mathcal{M}$ be a Fgαg-CS in a FTS (X,τ) and let $\mathcal{U}$ be any Fα-OS containing $\mathcal{M}$. By theorem (2.14) part (iii), $\mathcal{U}$ is a Fαg-OS in X. Since $\mathcal{M}$ is a Fgαg-CS, we have $\alpha cl(\mathcal{M}) \leq cl(\mathcal{M}) \leq \mathcal{U}$. Hence $\mathcal{M}$ is a Fαg-CS.
(iv) Let $\mathcal{M}$ be a Fgαg-CS in a FTS (X,τ) and let $\mathcal{U}$ be any F-OS containing $\mathcal{M}$. By theorem (2.14) part (i), $\mathcal{U}$ is a Fαg-OS in X. Since $\mathcal{M}$ is a Fgαg-CS, we have $\alpha cl(\mathcal{M}) \leq cl(\mathcal{M}) \leq \mathcal{U}$. Hence $\mathcal{M}$ is a Fgα-CS.

The converse of the above theorem need not be true as shown in the following examples.

Example 3.5: Let $X = \{x, y, z\}$ and the F-sets $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ from X to [0,1] be defined as: $\mathcal{M}(x) = 0.0$, $\mathcal{M}(y) = 0.0$, $\mathcal{M}(z) = 0.4$; $\mathcal{N}(x) = 0.9$, $\mathcal{N}(y) = 0.6$, $\mathcal{N}(z) = 0.0$; $\mathcal{O}(x) = 1.0$, $\mathcal{O}(y) = 0.7$, $\mathcal{O}(z) = 1.0$. Let $\tau = \{0_X, \mathcal{M}, \mathcal{N}, \mathcal{M} \vee \mathcal{N}, 1_X\}$ be a FTS. Then the F-set $\mathcal{O}$ is a Fgαg-CS but not F-CS in X.

Example 3.6: Let $X = \{x, y, z\}$ and the F-sets $\mathcal{M}$,$\mathcal{N}$,$\mathcal{O}$ and $\mathcal{P}$ from X to [0,1] be defined as: $\mathcal{M}(x) = 0.7$, $\mathcal{M}(y) = 0.3$, $\mathcal{M}(z) = 1.0$; $\mathcal{N}(x) = 0.7$, $\mathcal{N}(y) = 0.0$, $\mathcal{N}(z) = 0.0$; $\mathcal{O}(x) = 0.9$, $\mathcal{O}(y) = 0.2$, $\mathcal{O}(z) = 0.1$; $\mathcal{P}(x) = 0.2$, $\mathcal{P}(y) = 0.7$, $\mathcal{P}(z) = 0.2$. Let $\tau = \{0_X, \mathcal{M}, \mathcal{N}, 1_X\}$ be a FTS. Then the F-set $\mathcal{O}$ is a Fg-CS and hence Fgα-CS, but not Fgαg-CS in X. And the F-set $\mathcal{P}$ is a Fαg-CS but not Fgαg-CS in X.

Definition 3.7: A F-set $\mathcal{M}$ of a FTS (X,τ) is said to be a fuzzy generalized αg-open set (briefly Fgαg-open set) iff $1_X - \mathcal{M}$ is a Fgαg-CS. The family of all Fgαg-open sets of a FTS (X,τ) is denoted by Fgαg-O(X).

Example 3.8: By example (3.2). Then the F-sets $0_X, \mathcal{M}$ and 1_X are Fgαg-OS in X.

Definition 3.9: The union of all Fgαg-OS in a FTS (X,τ) contained in $\mathcal{M}$ is called fuzzy gαg-interior of $\mathcal{M}$ and is denoted by gαg-$int(\mathcal{M})$, gαg-$int(\mathcal{M}) = \vee\{\mathcal{N}: \mathcal{M} \geq \mathcal{N}, \mathcal{N}$ is a Fgαg-OS$\}$.

Proposition 3.10: Let $\mathcal{M}$ be any F-set in a FTS (X,τ). Then the following properties hold:
(i) gαg-$int(\mathcal{M}) = \mathcal{M}$ iff $\mathcal{M}$ is a Fgαg-OS.
(ii) gαg-$cl(\mathcal{M}) = \mathcal{M}$ iff $\mathcal{M}$ is a Fgαg-CS.
(iii) gαg-$int(\mathcal{M})$ is the largest Fgαg-OS contained in $\mathcal{M}$.
(iv) gαg-$cl(\mathcal{M})$ is the smallest Fgαg-CS containing $\mathcal{M}$.

Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 3.11: Let $\mathcal{M}$ be any F-set in a FTS (X,τ). Then the following properties hold:
(i) gαg-$int(1_X - \mathcal{M}) = 1_X - (g\alpha$g-$cl(\mathcal{M}))$,
(ii) gαg-$cl(1_X - \mathcal{M}) = 1_X - (g\alpha$g-$int(\mathcal{M}))$.

Proof: (i) By definition, gαg-$cl(\mathcal{M}) = \wedge\{\mathcal{N}: \mathcal{M} \leq \mathcal{N}, \mathcal{N}$ is a Fgαg-CS$\}$
$1_X - ($gαg-$cl(\mathcal{M})) = 1_X - \wedge\{\mathcal{N}: \mathcal{M} \leq \mathcal{N}, \mathcal{N}$ is a Fgαg-CS$\}$
$= \vee\{1_X - \mathcal{N}: \mathcal{M} \leq \mathcal{N}, \mathcal{N}$ is a Fgαg-CS$\}$
$= \vee\{\mathcal{U}: 1_X - \mathcal{M} \geq \mathcal{U}, \mathcal{U}$ is a Fgαg-OS$\}$

$= \text{g}\alpha\text{g-}int(1_X - \mathcal{M})$

(ii) The proof is similar to (i).

Theorem 3.12: Let (X, τ) be a FTS. If $\mathcal{M}$ is a F-OS, then it is a Fgαg-OS in X.

Proof: Let $\mathcal{M}$ be a F-OS in a FTS (X, τ), then $1_X - \mathcal{M}$ is a F-CS in X. By theorem (3.4) part (i), $1_X - \mathcal{M}$ is a Fgαg-CS. Hence $\mathcal{M}$ is a Fgαg-OS in X.

Theorem 3.13: Let (X, τ) be a FTS. If $\mathcal{M}$ is a Fgαg-OS, then it is a Fg-OS in X.

Proof: Let $\mathcal{M}$ be a Fgαg-OS in a FTS (X, τ), then $1_X - \mathcal{M}$ is a Fgαg-CS in X. By theorem (3.4) part (ii), $1_X - \mathcal{M}$ is a Fg-CS. Hence $\mathcal{M}$ is a Fg-OS in X.

Lemma 3.14: Let (X, τ) be a FTS. If $\mathcal{M}$ is a Fgαg-OS, then it is a Fαg-OS (resp. Fgα-OS) in X.

Proof: Similar to above theorem.

Proposition 3.15: If $\mathcal{M}$ and $\mathcal{N}$ are Fgαg-CS in a FTS (X, τ), then $\mathcal{M} \vee \mathcal{N}$ is a Fgαg-CS.

Proof: Let $\mathcal{M}$ and $\mathcal{N}$ be Fgαg-CS in a FTS (X, τ) and let $\mathcal{U}$ be any Fαg-OS containing $\mathcal{M}$ and $\mathcal{N}$. Then $\mathcal{M} \vee \mathcal{N} \leq \mathcal{U}$. Then $\mathcal{M} \leq \mathcal{U}$ and $\mathcal{N} \leq \mathcal{U}$. Since $\mathcal{M}$ and $\mathcal{N}$ are Fgαg-CS, $cl(\mathcal{M}) \leq \mathcal{U}$ and $cl(\mathcal{N}) \leq \mathcal{U}$. Now, $cl(\mathcal{M} \vee \mathcal{N}) = cl(\mathcal{M}) \vee cl(\mathcal{N}) \leq \mathcal{U}$ and so $cl(\mathcal{M} \vee \mathcal{N}) \leq \mathcal{U}$. Hence $\mathcal{M} \vee \mathcal{N}$ is a Fgαg-CS.

Proposition 3.16: If $\mathcal{M}$ and $\mathcal{N}$ are Fgαg-OS in a FTS (X, τ), then $\mathcal{M} \wedge \mathcal{N}$ is a Fgαg-OS.

Proof: Let $\mathcal{M}$ and $\mathcal{N}$ be Fgαg-OS in a FTS (X, τ). Then $1_X - \mathcal{M}$ and $1_X - \mathcal{N}$ are Fgαg-CS. By proposition (3.15), $(1_X - \mathcal{M}) \vee (1_X - \mathcal{N})$ is a Fgαg-CS. Since $(1_X - \mathcal{M}) \vee (1_X - \mathcal{N}) = 1_X - (\mathcal{M} \wedge \mathcal{N})$. Hence $\mathcal{M} \wedge \mathcal{N}$ is a Fgαg-OS.

Proposition 3.17: If a F-set $\mathcal{M}$ is Fgαg-CS in a FTS (X, τ), then $cl(\mathcal{M}) - \mathcal{M}$ contains no non-empty F-CS in X.

Proof: Let $\mathcal{M}$ be a Fgαg-CS in a FTS (X, τ) and let $\mathcal{F}$ be any F-CS in X such that $\mathcal{F} \leq cl(\mathcal{M}) - \mathcal{M}$. Since $\mathcal{M}$ is a Fgαg-CS, we have $cl(\mathcal{M}) \leq 1_X - \mathcal{F}$. This implies $\mathcal{F} \leq 1_X - cl(\mathcal{M})$. Then $\mathcal{F} \leq cl(\mathcal{M}) \wedge (1_X - cl(\mathcal{M})) = 0_X$. Thus, $\mathcal{F} = 0_X$. Hence $cl(\mathcal{M}) - \mathcal{M}$ contains no non-empty F-CS in X.

Proposition 3.18: If a F-set $\mathcal{M}$ is Fgαg-CS in a FTS (X, τ), then $cl(\mathcal{M}) - \mathcal{M}$ contains no non-empty Fαg-CS in X.

Proof: Let $\mathcal{M}$ be a Fgαg-CS in a FTS (X, τ) and let $\mathcal{D}$ be any Fαg-CS in X such that $\mathcal{D} \leq cl(\mathcal{M}) - \mathcal{M}$. Since $\mathcal{M}$ is a Fgαg-CS, we have $cl(\mathcal{M}) \leq 1_X - \mathcal{D}$. This implies $\mathcal{D} \leq 1_X - cl(\mathcal{M})$. Then $\mathcal{D} \leq cl(\mathcal{M}) \wedge (1_X - cl(\mathcal{M})) = 0_X$. Thus, $\mathcal{D} = 0_X$. Hence $cl(\mathcal{M}) - \mathcal{M}$ contains no non-empty Fαg-CS in X.

Theorem 3.19: If $\mathcal{M}$ is a Fαg-OS and a Fgαg-CS in a FTS (X, τ), then $\mathcal{M}$ is a F-CS in X.

Proof: Suppose that $\mathcal{M}$ is a Fαg-OS and a Fgαg-CS in a FTS (X, τ), then $cl(\mathcal{M}) \leq \mathcal{M}$ and since $\mathcal{M} \leq cl(\mathcal{M})$. Thus, $cl(\mathcal{M}) = \mathcal{M}$. Hence $\mathcal{M}$ is a F-CS.

Theorem 3.20: If $\mathcal{M}$ is a Fgαg-CS in a FTS (X, τ) and $\mathcal{M} \leq \mathcal{N} \leq cl(\mathcal{M})$, then $\mathcal{N}$ is a Fgαg-CS in X.

Proof: Suppose that $\mathcal{M}$ is a Fgαg-CS in a FTS (X, τ). Let $\mathcal{U}$ be a Fαg-OS in X such that $\mathcal{N} \leq \mathcal{U}$. Then $\mathcal{M} \leq \mathcal{U}$. Since $\mathcal{M}$ is a Fgαg-CS, it follows that $cl(\mathcal{M}) \leq \mathcal{U}$. Now, $\mathcal{N} \leq cl(\mathcal{M})$ implies $cl(\mathcal{N}) \leq cl(cl(\mathcal{M})) = cl(\mathcal{M})$. Thus, $cl(\mathcal{N}) \leq \mathcal{U}$. Hence $\mathcal{N}$ is a Fgαg-CS.

Theorem 3.21: If $\mathcal{M}$ is a Fgαg-OS in a FTS (X, τ) and $int(\mathcal{M}) \leq \mathcal{N} \leq \mathcal{M}$, then $\mathcal{N}$ is a Fgαg-OS in X.

Proof: Suppose that $\mathcal{M}$ is a Fgαg-OS in a FTS (X,τ) and $int(\mathcal{M}) \le \mathcal{N} \le \mathcal{M}$. Then $1_X - \mathcal{M}$ is a Fgαg-CS and $1_X - \mathcal{M} \le 1_X - \mathcal{N} \le cl(1_X - \mathcal{M})$. Then $1_X - \mathcal{N}$ is a Fgαg-CS by theorem (3.20). Hence, $\mathcal{N}$ is a Fgαg-OS.

Theorem 3.22: A F-set $\mathcal{M}$ is Fgαg-OS iff $\mathcal{C} \le int(\mathcal{M})$ where $\mathcal{C}$ is a Fgαg-CS and $\mathcal{C} \le \mathcal{M}$.

Proof: Suppose that $\mathcal{C} \le int(\mathcal{M})$ where $\mathcal{C}$ is a Fgαg-CS and $\mathcal{C} \le \mathcal{M}$. Then $1_X - \mathcal{M} \le 1_X - \mathcal{C}$ and $1_X - \mathcal{C}$ is a Fαg-OS by lemma (3.14). Now, $cl(1_X - \mathcal{M}) = 1_X - int(\mathcal{M}) \le 1_X - \mathcal{C}$. Then $1_X - \mathcal{M}$ is a Fgαg-CS. Hence $\mathcal{M}$ is a Fgαg-OS.
Conversely, let $\mathcal{M}$ be a Fgαg-OS and $\mathcal{C}$ be a Fgαg-CS and $\mathcal{C} \le \mathcal{M}$. Then $1_X - \mathcal{M} \le 1_X - \mathcal{C}$. Since $1_X - \mathcal{M}$ is a Fgαg-CS and $1_X - \mathcal{C}$ is a Fαg-OS, we have $cl(1_X - \mathcal{M}) \le 1_X - \mathcal{C}$. Then $\mathcal{C} \le int(\mathcal{M})$.

Definition 3.23: A F-set $\mathcal{M}$ in a FTS (X,τ) is said to be a fuzzy gαg-neighbourhood (briefly Fgαg-nhd) of a fuzzy point x_λ if there exists a Fgαg-OS $\mathcal{N}$ such that $x_\lambda \in \mathcal{N} \le \mathcal{M}$. A Fg$\alpha$g-nhd $\mathcal{M}$ is said to be a Fgαg-open-nhd (resp. Fgαg-closed-nhd) iff $\mathcal{M}$ is a Fgαg-OS (resp. Fgαg-CS). A F-set $\mathcal{M}$ in a FTS (X,τ) is said to be a fuzzy gαg-q-neighbourhood (briefly Fgαg-q-nhd) of a fuzzy point x_λ (resp. F-set $\mathcal{N}$) if there exists a Fgαg-OS $\mathcal{A}$ in a FTS (X,τ) such that $x_\lambda q\mathcal{A} \le \mathcal{M}$ (resp. $\mathcal{N}q\mathcal{A} \le \mathcal{M}$).

Theorem 3.24: A F-set $\mathcal{M}$ of a FTS (X,τ) is Fgαg-CS iff $\mathcal{M}\bar{q}\mathcal{C} \Rightarrow cl(\mathcal{M})\bar{q}\mathcal{C}$, for every F$\alpha$g-CS $\mathcal{C}$ of X.

Proof: Necessity. Let $\mathcal{C}$ be a Fαg-CS and $\mathcal{M}\bar{q}\mathcal{C}$. Then $\mathcal{M} \le 1_X - \mathcal{C}$ and $1_X - \mathcal{C}$ is a Fαg-OS in X which implies that $cl(\mathcal{M}) \le 1_X - \mathcal{C}$ as $\mathcal{M}$ is a Fgαg-CS. Hence, $cl(\mathcal{M})\bar{q}\mathcal{C}$.
Sufficiency. Let $\mathcal{U}$ be a Fαg-OS of a FTS (X,τ) such that $\mathcal{M} \le \mathcal{U}$. Then $\mathcal{M}\bar{q}(1_X - \mathcal{U})$ and $1_X - \mathcal{U}$ is a Fαg-CS in X. By hypothesis, $cl(\mathcal{M})\bar{q}(1_X - \mathcal{U})$ implies $cl(\mathcal{M}) \le \mathcal{U}$. Hence, $\mathcal{M}$ is a Fgαg-CS in X.

Theorem 3.25: Let x_λ and $\mathcal{M}$ be a fuzzy point and a F-set respectively in a FTS (X,τ). Then $x_\lambda \in$ gαg-$cl(\mathcal{M})$ iff every Fgαg-q-nhd of x_λ is q-coincident with $\mathcal{M}$.

Proof: We prove by contradiction. Let $x_\lambda \in$ gαg-$cl(\mathcal{M})$. Suppose there exists a Fgαg-q-nhd $\mathcal{A}$ of x_λ such that $\mathcal{A}\bar{q}\mathcal{M}$. Since $\mathcal{A}$ is a Fgαg-q-nhd of x_λ, there exists a Fgαg-OS $\mathcal{B}$ in X such that $x_\lambda q\mathcal{B} \le \mathcal{A}$ whish gives that $\mathcal{B}\bar{q}\mathcal{M}$ and hence $\mathcal{M} \le 1_X - \mathcal{B}$. Then g$\alpha$g-$cl(\mathcal{M}) \le 1_X - \mathcal{B}$, as $1_X - \mathcal{B}$ is a Fgαg-CS. Since $x_\lambda \notin 1_X - \mathcal{B}$, we have $x_\lambda \notin$ gαg-$cl(\mathcal{M})$, a contradiction. Thus every Fgαg-q-nhd of x_λ is q-coincident with $\mathcal{M}$.
Conversely, suppose $x_\lambda \notin$ gαg-$cl(\mathcal{M})$. Then there exists a Fgαg-CS $\mathcal{N}$ such that $\mathcal{M} \le \mathcal{N}$ and $x_\lambda \notin \mathcal{N}$. Then we have $x_\lambda q(1_X - \mathcal{N})$ and $\mathcal{M}\bar{q}(1_X - \mathcal{N})$, a contradiction. Hence $x_\lambda \in$ gαg-$cl(\mathcal{M})$.

Proposition 3.26: Let $\mathcal{M}$ and $\mathcal{N}$ be two F-sets in a FTS (X,τ). Then the following properties hold:
(i) gαg-$cl(0_X) = 0_X$, gαg-$cl(1_X) = 1_X$.
(ii) gαg-$cl(\mathcal{M})$ is a Fgαg-CS in X.
(iii) gαg-$cl(\mathcal{M}) \le$ gαg-$cl(\mathcal{N})$ when $\mathcal{M} \le \mathcal{N}$.
(iv) $\mathcal{A}q\mathcal{M}$ iff $\mathcal{A}q$gαg-$cl(\mathcal{M})$, when $\mathcal{A}$ is a Fgαg-OS in X.
(v) gαg-$cl(\mathcal{M}) =$ gαg-$cl($gαg-$cl(\mathcal{M}))$.
(vi) gαg-$cl(\mathcal{M} \wedge \mathcal{N}) \le$ gαg-$cl(\mathcal{M}) \wedge$ gαg-$cl(\mathcal{N})$.
(vii) gαg-$cl(\mathcal{M} \vee \mathcal{N}) =$ gαg-$cl(\mathcal{M}) \vee$ gαg-$cl(\mathcal{N})$.

Proof: (i) and (ii) are obvious.
(iii) Suppose that $x_\lambda \notin$ gαg-$cl(\mathcal{N})$. By theorem (3.25), there is a Fgαg-q-nhd $\mathcal{B}$ of a fuzzy point x_λ such that $\mathcal{B}\bar{q}\mathcal{N}$, so there is a Fg$\alpha$g-OS $\mathcal{A}$ such that $x_\lambda q\mathcal{A} \le \mathcal{B}$ and $\mathcal{A}\bar{q}\mathcal{N}$. Since $\mathcal{M} \le \mathcal{N}$, then $\mathcal{A}\bar{q}\mathcal{M}$. Hence $x_\lambda \notin$ gαg-$cl(\mathcal{M})$ by theorem (3.25). This shows that gαg-$cl(\mathcal{M}) \le$ gαg-$cl(\mathcal{N})$.
(iv) Let $\mathcal{A}$ be a Fgαg-OS in X. Suppose that $\mathcal{A}\bar{q}\mathcal{M}$, then $\mathcal{M} \le 1_X - \mathcal{A}$. Since $1_X - \mathcal{A}$ is a Fgαg-CS and by a part (iii), gαg-$cl(\mathcal{M}) \le$ gαg-$cl(1_X - \mathcal{A}) = 1_X - \mathcal{A}$. Hence, $\mathcal{A}\bar{q}$gαg-$cl(\mathcal{M})$.

Conversely, suppose that $\mathcal{A}\bar{q}$gαg-$cl(\mathcal{M})$. Then gαg-$cl(\mathcal{M}) \leq 1_X - \mathcal{A}$. Since $\mathcal{M} \leq$ gαg-$cl(\mathcal{M})$, we have $\mathcal{M} \leq 1_X - \mathcal{A}$. Hence $\mathcal{A}\bar{q}\mathcal{M}$. Thus $\mathcal{A}q\mathcal{M}$ if and only if $\mathcal{A}q$gαg-$cl(\mathcal{M})$.

(v) Since gαg-$cl(\mathcal{M}) \leq$ gαg-$cl($gαg-$cl(\mathcal{M}))$. We prove that gαg-$cl($gαg-$cl(\mathcal{M})) \leq$ gαg-$cl(\mathcal{M})$. Suppose that $x_\lambda \notin$ gαg-$cl(\mathcal{M})$. Then by theorem (3.25), there exists a Fgαg-q-nhd $\mathcal{B}$ of a fuzzy point x_λ such that $\mathcal{B}\bar{q}\mathcal{M}$ and so there is a Fgαg-OS $\mathcal{A}$ in X such that $x_\lambda q\mathcal{A} \leq \mathcal{B}$ and $\mathcal{A}\bar{q}\mathcal{M}$. By a part (iv), $\mathcal{A}\bar{q}$gαg-$cl(\mathcal{M})$. Then by theorem (3.25), $x_\lambda \notin$ gαg-$cl($gαg-$cl(\mathcal{M}))$. Thus gαg-$cl($gαg-$cl(\mathcal{M})) \leq$ gαg-$cl(\mathcal{M})$. Hence gαg-$cl(\mathcal{M}) =$ gαg-$cl($gαg-$cl(\mathcal{M}))$.

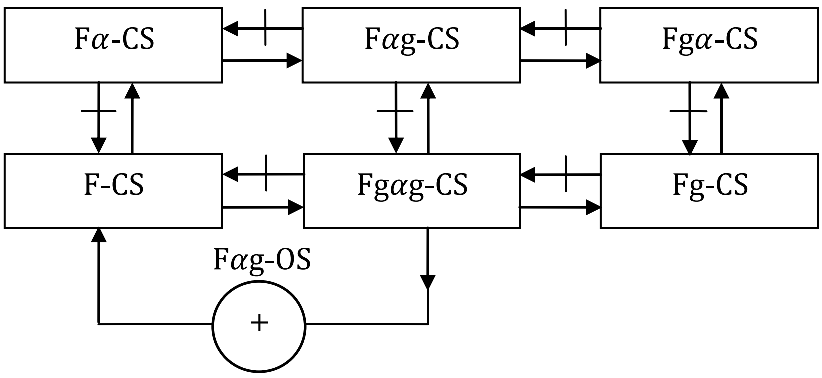
(vi) Since $\mathcal{M} \wedge \mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \wedge \mathcal{N} \leq \mathcal{N}$. Then g$\alpha$g-$cl(\mathcal{M} \wedge \mathcal{N}) \leq$ gαg-$cl(\mathcal{M})$ and gαg-$cl(\mathcal{M} \wedge \mathcal{N}) \leq$ gαg-$cl(\mathcal{N})$ by a part (iii). Hence, gαg-$cl(\mathcal{M} \wedge \mathcal{N}) \leq$ gαg-$cl(\mathcal{M}) \wedge$ gαg-$cl(\mathcal{N})$.

(vii) Since $\mathcal{M} \leq \mathcal{M} \vee \mathcal{N}$ and $\mathcal{N} \leq \mathcal{M} \vee \mathcal{N}$. By a part (iii), we have gαg-$cl(\mathcal{M}) \leq$ gαg-$cl(\mathcal{M} \vee \mathcal{N})$ and gαg-$cl(\mathcal{N}) \leq$ gαg-$cl(\mathcal{M} \vee \mathcal{N})$. Then g$\alpha$g-$cl(\mathcal{M}) \vee$ gαg-$cl(\mathcal{N}) \leq$ gαg-$cl(\mathcal{M} \vee \mathcal{N})$.
Conversely, let $x_\lambda \in$ gαg-$cl(\mathcal{M} \vee \mathcal{N})$. Then by theorem (3.25), there exists a Fgαg-q-nhd $\mathcal{A}$ of a fuzzy point x_λ such that $\mathcal{A}q(\mathcal{M} \vee \mathcal{N})$. By proposition (2.2), either $\mathcal{A}q\mathcal{M}$ or $\mathcal{A}q\mathcal{N}$. Then by theorem (3.25), $x_\lambda \in$ gαg-$cl(\mathcal{M})$ or $x_\lambda \in$ gαg-$cl(\mathcal{N})$. That is $x_\lambda \in$ gαg-$cl(\mathcal{M}) \vee$ gαg-$cl(\mathcal{N})$. Then gαg-$cl(\mathcal{M} \vee \mathcal{N}) \leq$ gαg-$cl(\mathcal{M}) \vee$ gαg-$cl(\mathcal{N})$. Hence, gαg-$cl(\mathcal{M} \vee \mathcal{N}) =$ gαg-$cl(\mathcal{M}) \vee$ gαg-$cl(\mathcal{N})$.

Proposition 3.27: Let $\mathcal{M}$ and $\mathcal{N}$ be two F-sets in a FTS (X, τ). Then the following properties hold:
(i) gαg-$int(0_X) = 0_X$, gαg-$int(1_X) = 1_X$.
(ii) gαg-$int(\mathcal{M})$ is a Fgαg-OS in X.
(iii) gαg-$int(\mathcal{M}) \leq$ gαg-$int(\mathcal{N})$ when $\mathcal{M} \leq \mathcal{N}$.
(iv) gαg-$int(\mathcal{M}) =$ gαg-$int($gαg-$int(\mathcal{M}))$.
(v) gαg-$int(\mathcal{M} \wedge \mathcal{N}) =$ gαg-$int(\mathcal{M}) \wedge$ gαg-$int(\mathcal{N})$.
(vi) gαg-$int(\mathcal{M} \vee \mathcal{N}) \geq$ gαg-$int(\mathcal{M}) \vee$ gαg-$int(\mathcal{N})$.

Proof: Obvious.

Remark 3.28: The following diagram shows the relations among the different types of weakly F-CS that were studied in this section:



4. Fuzzy gαg-Continuous Functions

Definition 4.1: A function $f: (X, \tau) \longrightarrow (Y, \psi)$ is said to be a fuzzy gαg-continuous (briefly Fgαg-continuous) if $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X for every F-CS $\mathcal{V}$ in Y.

Proposition 4.2: Let (X, τ) and (Y, ψ) be FTS, and $f: (X, \tau) \longrightarrow (Y, \psi)$ be a function. Then f is a Fgαg-continuous function iff $f^{-1}(\mathcal{V})$ is a Fgαg-OS in X, for every F-OS $\mathcal{V}$ in Y.

Proof: Let $\mathcal{V}$ be a F-OS in Y. Then $1_Y - \mathcal{V}$ is a F-CS in Y, so $f^{-1}(1_Y - \mathcal{V}) = 1_X - f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. Thus, $f^{-1}(\mathcal{V})$ is a Fgαg-OS in X. The proof of the converse is obvious.

Theorem 4.3: Every Fgαg-continuous function is a Fαg-continuous.

Proof: Let $f:(X,\tau) \longrightarrow (Y,\psi)$ be a Fgαg-continuous function and let $\mathcal{V}$ be a F-CS in Y. Since f is a Fgαg-continuous, $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. By theorem (3.4) part (iii), $f^{-1}(\mathcal{V})$ is a Fαg-CS in X. Thus, f is a Fαg-continuous.

Theorem 4.4: Every Fgαg-continuous function is a Fgα-continuous.

Proof: Let $f:(X,\tau) \longrightarrow (Y,\psi)$ be a Fgαg-continuous function and let $\mathcal{V}$ be a F-CS in Y. Since f is a Fgαg-continuous, $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. By theorem (3.4) part (iv), $f^{-1}(\mathcal{V})$ is a Fgα-CS in X. Thus, f is a Fgα-continuous.

The converse of the above theorems need not be true as shown in the following example.

Example 4.5: Let $X = \{x, y\}$, $Y = \{u, v\}$. F-set $\mathcal{M}$ is defined as: $\mathcal{M}(x) = 0.4$, $\mathcal{M}(y) = 0.6$.
Let $\tau = \{0_X, \mathcal{M}, 1_X\}$ and $\psi = \{0_Y, 1_Y\}$ be FTS. Then the function $f:(X,\tau) \longrightarrow (Y,\psi)$ defined by $f(x) = u$, $f(y) = v$ is a Fαg-continuous and hence Fgα-continuous but not Fgαg-continuous.

Theorem 4.6: If $f:(X,\tau) \longrightarrow (Y,\psi)$ is a Fgαg-continuous function then for each fuzzy point x_λ of X and $\mathcal{N} \in \psi$ such that $f(x_\lambda) \in \mathcal{N}$, there exists a Fg$\alpha$g-OS $\mathcal{M}$ of X such that $x_\lambda \in \mathcal{M}$ and $f(\mathcal{M}) \leq \mathcal{N}$.

Proof: Let x_λ be a fuzzy point of X and $\mathcal{N} \in \psi$ such that $f(x_\lambda) \in \mathcal{N}$. Take $\mathcal{M} = f^{-1}(\mathcal{N})$. Since $1_Y - \mathcal{N}$ is a F-CS in Y and f is a Fgαg-continuous function, we have $f^{-1}(1_Y - \mathcal{N}) = 1_X - f^{-1}(\mathcal{N})$ is a Fgαg-CS in X. This gives $\mathcal{M} = f^{-1}(\mathcal{N})$ is a Fgαg-OS in X and $x_\lambda \in \mathcal{M}$ and $f(\mathcal{M}) = f(f^{-1}(\mathcal{N})) \leq \mathcal{N}$.

Theorem 4.7: If $f:(X,\tau) \longrightarrow (Y,\psi)$ is a Fgαg-continuous function then for each fuzzy point x_λ of X and $\mathcal{N} \in \psi$ such that $f(x_\lambda)q\mathcal{N}$, there exists a Fgαg-OS $\mathcal{M}$ of X such that $x_\lambda q\mathcal{M}$ and $f(\mathcal{M}) \leq \mathcal{N}$.

Proof: Let x_λ be a fuzzy point of X and $\mathcal{N} \in \psi$ such that $f(x_\lambda)q\mathcal{N}$. Take $\mathcal{M} = f^{-1}(\mathcal{N})$. By above theorem (4.6), $\mathcal{M}$ is a Fgαg-OS in X and $x_\lambda q\mathcal{M}$ and $f(\mathcal{M}) = f(f^{-1}(\mathcal{N})) \leq \mathcal{N}$.

Definition 4.8: A function $f:(X,\tau) \longrightarrow (Y,\psi)$ is said to be a fuzzy gαg-irresolute (briefly Fgαg-irresolute) if $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X for every Fgαg-CS $\mathcal{V}$ in Y.

Proposition 4.9: Let (X,τ) and (Y,ψ) be FTS, and $f:(X,\tau) \longrightarrow (Y,\psi)$ be a function. Then f is a Fgαg-irresolute function iff $f^{-1}(\mathcal{V})$ is a Fgαg-OS in X, for every Fgαg-OS $\mathcal{V}$ in Y.

Proof: Let $\mathcal{V}$ be a Fgαg-OS in Y. Then $1_Y - \mathcal{V}$ is a Fgαg-CS in Y, so $f^{-1}(1_Y - \mathcal{V}) = 1_X - f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. Thus, $f^{-1}(\mathcal{V})$ is a Fgαg-OS in X. The proof of the converse is obvious.

Theorem 4.10: Every Fgαg-irresolute function is a Fgαg-continuous.

Proof: Let $f:(X,\tau) \longrightarrow (Y,\psi)$ be a Fgαg-irresolute function and let $\mathcal{V}$ be a F-CS in Y, by theorem (3.4) part (i), then $\mathcal{V}$ is a Fgαg-CS in Y. Since f is a Fgαg-irresolute, then $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. Thus, f is a Fgαg-continuous.

The following example shows that the converse of the above theorem not be true.

Example 4.11: Let $X = \{x, y, z\}$, $Y = \{u, v, w\}$. F-sets $\mathcal{M}$ and $\mathcal{N}$ are defined as follows:
$\mathcal{M}(x) = 0.7$, $\mathcal{M}(y) = 0.2$, $\mathcal{M}(z) = 0.1$; $\mathcal{N}(u) = 0.1$, $\mathcal{N}(v) = 0.7$, $\mathcal{N}(w) = 0.2$.
Let $\tau = \{0_X, \mathcal{M}, 1_X\}$ and $\psi = \{0_Y, \mathcal{N}, 1_Y\}$ be FTS. Then the function $f:(X,\tau) \longrightarrow (Y,\psi)$ defined by $f(x) = v$, $f(y) = w$, $f(z) = u$ is a Fgαg-continuous and it is not a Fgαg-irresolute.

Definition 4.12: A FTS (X,τ) is said to be a fuzzy $T_{g\alpha g}$-space (briefly F$T_{g\alpha g}$-space) if every Fgαg-CS in it is a F-CS.

Proposition 4.13: Every F$T_{\frac{1}{2}}$-space is a F$T_{g\alpha g}$-space.

Proof: Let (X,τ) be a F$T_{\frac{1}{2}}$-space and let $\mathcal{M}$ be a Fgαg-CS in X. Then $\mathcal{M}$ is a Fg-CS, by theorem (3.4) part (ii). Since (X,τ) is a F$T_{\frac{1}{2}}$-space, then $\mathcal{M}$ is a F-CS in X. Hence (X,τ) is a F$T_{g\alpha g}$-space.

The following example shows that the converse of the above proposition not be true.

Example 4.14: Let $X = \{x,y,z\}$ and the F-sets $\mathcal{M}$ and $\mathcal{N}$ from X to [0,1] be defined as:
$\mathcal{M}(x) = 0.7$, $\mathcal{M}(y) = 0.3$, $\mathcal{M}(z) = 1.0$; $\mathcal{N}(x) = 0.7$, $\mathcal{N}(y) = 0.0$, $\mathcal{N}(z) = 0.0$.
Let $\tau = \{0_X, \mathcal{M}, \mathcal{N}, 1_X\}$ be a FTS. Then (X,τ) is a F$T_{g\alpha g}$-space but not F$T_{\frac{1}{2}}$-space.

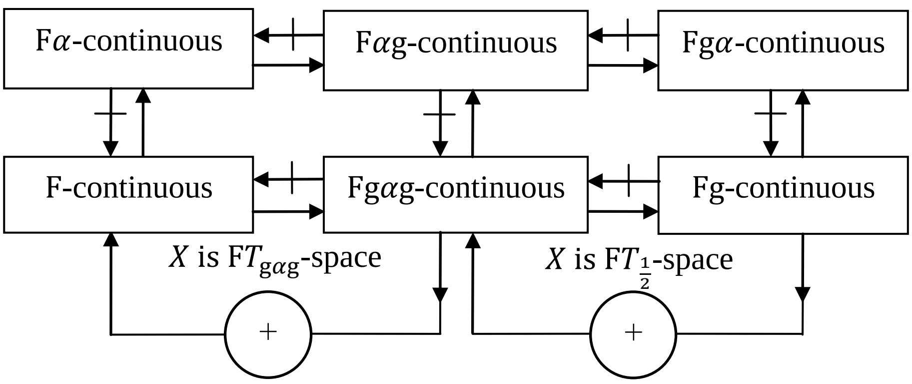
Theorem 4.15: If $f_1: (X,\tau) \longrightarrow (Y,\psi)$ is a Fgαg-continuous function and $f_2: (Y,\psi) \longrightarrow (Z,\rho)$ is a Fg-continuous function and (Y,ψ) is a F$T_{\frac{1}{2}}$-space. Then $f_2 \circ f_1: (X,\tau) \longrightarrow (Z,\rho)$ is a Fgαg-continuous function.

Proof: Let $\mathcal{W}$ be a F-CS in Z. Since f_2 is a Fg-continuous function and (Y,ψ) is a F$T_{\frac{1}{2}}$-space, ${f_2}^{-1}(\mathcal{W})$ is a F-CS in Y. Since f_1 is a Fgαg-continuous function, ${f_1}^{-1}({f_2}^{-1}(\mathcal{W}))$ is a Fgαg-CS in X. Thus, $f_2 \circ f_1$ is a Fgαg-continuous.

Theorem 4.16: Let (X,τ) and (Y,ψ) be FTS, and $f: (X,\tau) \longrightarrow (Y,\psi)$ be a function:
(i) If (X,τ) is a F$T_{\frac{1}{2}}$-space then f is a Fg-continuous iff it is a Fgαg-continuous.
(ii) If (X,τ) is a F$T_{g\alpha g}$-space then f is a F-continuous iff it is a Fgαg-continuous.

Proof: (i) Let $\mathcal{V}$ be any F-CS in Y. Since f is a Fg-continuous, $f^{-1}(\mathcal{V})$ is a Fg-CS in X. By (X,τ) is a F$T_{\frac{1}{2}}$-space, which implies, $f^{-1}(\mathcal{V})$ is a F-CS. By theorem (3.4) part (i), $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. Hence f is a Fgαg-continuous.
Conversely, suppose that f is a Fgαg-continuous. Let $\mathcal{V}$ be any F-CS in Y. Then $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. By theorem (3.4) part (ii), $f^{-1}(\mathcal{V})$ is a Fg-CS in X. Hence f is a Fg-continuous.
(ii) Let $\mathcal{V}$ be any F-CS in Y. Since f is a F-continuous, $f^{-1}(\mathcal{V})$ is a F-CS in X. By theorem (3.4) part (i), $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. Hence f is a Fgαg-continuous.
Conversely, suppose that f is a Fgαg-continuous. Let $\mathcal{V}$ be any F-CS in Y. Then $f^{-1}(\mathcal{V})$ is a Fgαg-CS in X. By (X,τ) is a F$T_{g\alpha g}$-space, which implies $f^{-1}(\mathcal{V})$ is a F-CS in X. Hence f is a F-continuous.

Remark 4.17: The following diagram shows the relations among the different types of weakly F-continuous functions that were studied in this section:



5. Fuzzy gαg-R_i-Spaces, $i = 0, 1$

Definition 5.1: The intersection of all Fgαg-open subset of a FTS (X, τ) containing $\mathcal{A}$ is called the fuzzy gαg-kernel of $\mathcal{A}$ (briefly gαg-$ker(\mathcal{A})$), this means gαg-$ker(\mathcal{A}) = \wedge \{\mathcal{M}: \mathcal{M} \in$ Fgαg-O(X) and $\mathcal{A} \leq \mathcal{M}\}$.

Definition 5.2: Let x_λ be a fuzzy point of a FTS (X, τ). The fuzzy gαg-kernel of x_λ, denoted by gαg-$ker(\{x_\lambda\})$ is defined to be the F-set gαg-$ker(\{x_\lambda\}) = \wedge \{\mathcal{M}: \mathcal{M} \in$ Fgαg-O(X) and $x_\lambda \in \mathcal{M}\}$.

Definition 5.3: In a FTS (X, τ), a F-set $\mathcal{A}$ is said to be weakly ultra fuzzy gαg-separated from $\mathcal{B}$ if there exists a Fgαg-OS $\mathcal{M}$ such that $\mathcal{M} \wedge \mathcal{B} = 0_X$ or $\mathcal{A} \wedge$ gαg-$cl(\mathcal{B}) = 0_X$.

By definition (5.3), we have the following: For every two distinct fuzzy points x_λ and y_μ of a FTS (X, τ),
(i) gαg-$cl(\{x_\lambda\}) = \{y_\mu : \{y_\mu\}$ is not weakly ultra fuzzy gαg-separated from $\{x_\lambda\}\}$.
(ii) gαg-$ker(\{x_\lambda\}) = \{y_\mu : \{x_\lambda\}$ is not weakly ultra fuzzy gαg-separated from $\{y_\mu\}\}$.

Lemma 5.4: Let (X, τ) be a FTS, then $y_\mu \in$ gαg-$ker(\{x_\lambda\})$ iff $x_\lambda \in$ gαg-$cl(\{y_\mu\})$ for each $x \neq y \in X$.

Proof: Suppose that $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$. Then there exists a Fgαg-OS $\mathcal{U}$ containing x_λ such that $y_\mu \notin \mathcal{U}$. Therefore, we have $x_\lambda \notin$ gαg-$cl(\{y_\mu\})$. The converse part can be proved in a similar way.

Definition 5.5: A FTS (X, τ) is said to be fuzzy gαg-R_0-space (Fgαg-R_0-space, for short) if for each Fgαg-OS $\mathcal{U}$ and $x_\lambda \in \mathcal{U}$, then g$\alpha$g-$cl(\{x_\lambda\}) \leq \mathcal{U}$.

Definition 5.6: A FTS (X, τ) is said to be fuzzy gαg-R_1-space (Fgαg-R_1-space, for short) if for each two distinct fuzzy points x_λ and y_μ of X with gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$, there exist disjoint Fgαg-OS $\mathcal{U}, \mathcal{V}$ such that gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$ and gαg-$cl(\{y_\mu\}) \leq \mathcal{V}$.

Theorem 5.7: Let (X, τ) be a FTS. Then (X, τ) is a Fgαg-R_0-space iff gαg-$cl(\{x_\lambda\}) =$ gαg-$ker(\{x_\lambda\})$, for each $x \in X$.

Proof: Let (X, τ) be a Fgαg-R_0-space. If gαg-$cl(\{x_\lambda\}) \neq$ gαg-$ker(\{x_\lambda\})$, for each $x \in X$, then there exist another fuzzy point $y \neq x$ such that $y_\mu \in$ gαg-$cl(\{x_\lambda\})$ and $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ this means there exist an $\mathcal{U}_{x_\lambda}$ Fgαg-OS, $y_\mu \notin \mathcal{U}_{x_\lambda}$ implies gαg-$cl(\{x_\lambda\}) \nleq \mathcal{U}_{x_\lambda}$ this contradiction. Thus gαg-$cl(\{x_\lambda\}) =$ gαg-$ker(\{x_\lambda\})$.
Conversely, let gαg-$cl(\{x_\lambda\}) =$ gαg-$ker(\{x_\lambda\})$, for each Fgαg-OS $\mathcal{U}, x_\lambda \in \mathcal{U}$, then g$\alpha$g-$ker(\{x_\lambda\}) =$ gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$ [by definition (5.1)]. Hence by definition (5.5), (X, τ) is a Fgαg-R_0-space.

Theorem 5.8: A FTS (X, τ) is an Fgαg-R_0-space iff for each $\mathcal{M}$ Fgαg-CS and $x_\lambda \in \mathcal{M}$, then g$\alpha$g-$ker(\{x_\lambda\}) \leq \mathcal{M}$.

Proof: Let for each $\mathcal{M}$ Fgαg-CS and $x_\lambda \in \mathcal{M}$, then g$\alpha$g-$ker(\{x_\lambda\}) \leq \mathcal{M}$ and let $\mathcal{U}$ be a Fgαg-OS, $x_\lambda \in \mathcal{U}$ then for each $y_\mu \notin \mathcal{U}$ implies $y_\mu \in \mathcal{U}^c$ is a Fgαg-CS implies gαg-$ker(\{y_\mu\}) \leq \mathcal{U}^c$[by assumption]. Therefore $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ implies $y_\mu \notin$ gαg-$cl(\{x_\lambda\})$ [by lemma (5.4)]. So gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$. Thus (X, τ) is a Fgαg-R_0-space.
Conversely, let (X, τ) be a Fgαg-R_0-space and $\mathcal{M}$ be a Fgαg-CS and $x_\lambda \in \mathcal{M}$. Then for each $y_\mu \notin \mathcal{M}$ implies $y_\mu \in \mathcal{M}^c$ is a Fgαg-OS, then gαg-$cl(\{y_\mu\}) \leq \mathcal{M}^c$[since (X, τ) is a Fgαg-R_0-space], so gαg-$ker(\{x_\lambda\}) =$ gαg-$cl(\{x_\lambda\})$. Thus gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}$.

Corollary 5.9: A FTS (X,τ) is Fgαg-R_0-space iff for each $\mathcal{U}$ Fgαg-OS and $x_\lambda \in \mathcal{U}$, then g$\alpha$g-$cl$(g$\alpha$g-$ker(\{x_\lambda\})) \leq \mathcal{U}$.

Proof: Clearly.

Theorem 5.10: Every Fgαg-R_1-space is a Fgαg-R_0-space.

Proof: Let (X,τ) be a Fgαg-R_1-space and let $\mathcal{U}$ be a Fgαg-OS, $x_\lambda \in \mathcal{U}$, then for each $y_\mu \notin \mathcal{U}$ implies $y_\mu \in \mathcal{U}^c$ is a Fgαg-CS and gαg-$cl(\{y_\mu\}) \leq \mathcal{U}^c$ implies gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$. Hence by definition (5.6), gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$. Thus (X,τ) is a Fgαg-R_0-space.

Theorem 5.11: A FTS (X,τ) is Fgαg-R_1-space iff for each $x \neq y \in X$ with gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$, then there exist Fgαg-CS $\mathcal{M}_1$, $\mathcal{M}_2$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}_1$, g$\alpha$g-$ker(\{x_\lambda\}) \wedge \mathcal{M}_2 = 0_X$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{M}_2$, g$\alpha$g-$ker(\{y_\mu\}) \wedge \mathcal{M}_1 = 0_X$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = 1_X$.

Proof: Let (X,τ) be a Fgαg-R_1-space. Then for each $x \neq y \in X$ with gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$. Since every Fgαg-R_1-space is a Fgαg-R_0-space [by theorem (5.10)], and by theorem (5.7), gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$, then there exist Fgαg-OS $\mathcal{U}_1$,$\mathcal{U}_2$ such that gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}_1$ and gαg-$cl(\{y_\mu\}) \leq \mathcal{U}_2$ and $\mathcal{U}_1 \wedge \mathcal{U}_2 = 0_X$ [since (X,τ) is a Fgαg-R_1-space], then $\mathcal{U}_1^c$ and $\mathcal{U}_2^c$ are Fgαg-CS such that $\mathcal{U}_1^c \vee \mathcal{U}_2^c = 1_X$. Put $\mathcal{M}_1 = \mathcal{U}_1^c$ and $\mathcal{M}_2 = \mathcal{U}_2^c$. Thus $x_\lambda \in \mathcal{U}_1 \leq \mathcal{M}_2$ and $y_\mu \in \mathcal{U}_2 \leq \mathcal{M}_1$ so that gαg-$ker(\{x_\lambda\}) \leq \mathcal{U}_1 \leq \mathcal{M}_2$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{U}_2 \leq \mathcal{M}_1$.
Conversely, let for each $x \neq y \in X$ with gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$, there exist Fgαg-CS $\mathcal{M}_1$, $\mathcal{M}_2$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}_1$, g$\alpha$g-$ker(\{x_\lambda\}) \wedge \mathcal{M}_2 = 0_X$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{M}_2$, g$\alpha$g-$ker(\{y_\mu\}) \wedge \mathcal{M}_1 = 0_X$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = 1_X$, then $\mathcal{M}_1^c$ and $\mathcal{M}_2^c$ are Fgαg-OS such that $\mathcal{M}_1^c \wedge \mathcal{M}_2^c = 0_X$. Put $\mathcal{M}_1^c = \mathcal{U}_2$ and $\mathcal{M}_2^c = \mathcal{U}_1$. Thus, g$\alpha$g-$ker(\{x_\lambda\}) \leq \mathcal{U}_1$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{U}_2$ and $\mathcal{U}_1 \wedge \mathcal{U}_2 = 0_X$, so that $x_\lambda \in \mathcal{U}_1$ and $y_\mu \in \mathcal{U}_2$ implies $x_\lambda \notin$ gαg-$cl(\{y_\mu\})$ and $y_\mu \notin$ gαg-$cl(\{x_\lambda\})$, then gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}_1$ and gαg-$cl(\{y_\mu\}) \leq \mathcal{U}_2$. Thus, (X,τ) is a Fgαg-R_1-space.

Corollary 5.12: A FTS (X,τ) is Fgαg-R_1-space iff for each $x \neq y \in X$ with gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$ there exist disjoint Fgαg-OS $\mathcal{U},\mathcal{V}$ such that gαg-cl(gαg-$ker(\{x_\lambda\})) \leq \mathcal{U}$ and gαg-cl(gαg-$ker(\{y_\mu\})) \leq \mathcal{V}$.

Proof: Let (X,τ) be a Fgαg-R_1-space and let $x \neq y \in X$ with gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$, then there exist disjoint Fgαg-OS $\mathcal{U},\mathcal{V}$ such that gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$ and gαg-$cl(\{y_\mu\}) \leq \mathcal{V}$. Also (X,τ) is a Fgαg-R_0-space [by theorem (5.10)] implies for each $x \in X$, then gαg-$cl(\{x_\lambda\}) =$ gαg-$ker(\{x_\lambda\})$ [by theorem (5.7)], but gαg-$cl(\{x_\lambda\}) =$ gαg-cl(gαg-$cl(\{x_\lambda\})) =$ gαg-cl(gαg-$ker(\{x_\lambda\}))$. Thus gαg-cl(gαg-$ker(\{x_\lambda\})) \leq \mathcal{U}$ and gαg-cl(gαg-$ker(\{y_\mu\})) \leq \mathcal{V}$.
Conversely, let for each $x \neq y \in X$ with gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$ there exist disjoint Fgαg-OS $\mathcal{U},\mathcal{V}$ such that gαg-cl(gαg-$ker(\{x_\lambda\})) \leq \mathcal{U}$ and gαg-cl(gαg-$ker(\{y_\mu\})) \leq \mathcal{V}$. Since $\{x_\lambda\} \leq$ gαg-$ker(\{x_\lambda\})$, then gαg-$cl(\{x_\lambda\}) \leq$ gαg-cl(gαg-$ker(\{x_\lambda\}))$ for each $x \in X$. So we get gαg-$cl(\{x_\lambda\}) \leq \mathcal{U}$ and gαg-$cl(\{y_\mu\}) \leq \mathcal{V}$. Thus, (X,τ) is a Fgαg-R_1-space.

6. Fuzzy gαg-T_j-Spaces, $j = 0, 1, 2$

Definition 6.1: Let (X,τ) be a FTS. Then X is said to be:
(i) fuzzy gαg-T_0-space (Fgαg-T_0-space, for short) iff for each pair of distinct fuzzy points in X, there exists a Fgαg-OS in X containing one and not the other.
(ii) fuzzy gαg-T_1-space (Fgαg-T_1-space, for short) iff for each pair of distinct fuzzy points x_λ and y_μ of X, there exist Fgαg-OS $\mathcal{M},\mathcal{N}$ containing x_λ and y_μ respectively such that $y_\mu \notin \mathcal{M}$ and $x_\lambda \notin \mathcal{N}$.
(iii) fuzzy gαg-T_2-space (Fgαg-T_2-space, for short) iff for each pair of distinct fuzzy points x_λ and y_μ of X, there exist disjoint Fgαg-OS $\mathcal{M},\mathcal{N}$ in X such that $x_\lambda \in \mathcal{M}$ and $y_\mu \in \mathcal{N}$.

Example 6.2: Let $X = \{x, y\}$ and $\tau = \{0_X, x_1, 1_X\}$ be a FTS on X. Then x_1 is a crisp point in X and (X, τ) is a Fgαg-T_0-space.

Example 6.3: Let $X = \{a, b\}$ and $\tau = \{0_X, a_1, b_1, 1_X\}$ be a FTS on X. Then a_1, b_1 are crisp points in X and (X, τ) is a Fgαg-T_1-space and Fgαg-T_2-space.

Remark 6.4: Every Fgαg-T_k-space is a Fgαg-T_{k-1}-space, $k = 1,2$.

Proof: Clearly.

Theorem 6.5: A FTS (X, τ) is Fgαg-T_0-space iff either $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ or $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$, for each $x \neq y \in X$.

Proof: Let (X, τ) be a Fgαg-T_0-space then for each $x \neq y \in X$, there exists a Fgαg-OS $\mathcal{M}$ such that $x_\lambda \in \mathcal{M}, y_\mu \notin \mathcal{M}$ or $x_\lambda \notin \mathcal{M}, y_\mu \in \mathcal{M}$. Thus either $x_\lambda \in \mathcal{M}$, $y_\mu \notin \mathcal{M}$ implies $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ or $x_\lambda \notin \mathcal{M}, y_\mu \in \mathcal{M}$ implies $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$.
Conversely, let either $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ or $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$, for each $x \neq y \in X$. Then there exists a Fgαg-OS $\mathcal{M}$ such that $x_\lambda \in \mathcal{M}, y_\mu \notin \mathcal{M}$ or $x_\lambda \notin \mathcal{M}, y_\mu \in \mathcal{M}$. Thus (X, τ) is a Fgαg-T_0-space.

Theorem 6.6: A FTS (X, τ) is Fgαg-T_0-space iff either gαg-$ker(\{x_\lambda\})$ is weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ or gαg-$ker(\{y_\mu\})$ is weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$ for each $x \neq y \in X$.

Proof: Let (X, τ) be a Fgαg-T_0-space then for each $x \neq y \in X$, there exists a Fgαg-OS $\mathcal{M}$ such that $x_\lambda \in \mathcal{M}, y_\mu \notin \mathcal{M}$ or $x_\lambda \notin \mathcal{M}, y_\mu \in \mathcal{M}$. Now if $x_\lambda \in \mathcal{M}, y_\mu \notin \mathcal{M}$ implies gαg-$ker(\{x_\lambda\})$ is weakly ultra fuzzy gαg-separated from $\{y_\mu\}$. Or if $x_\lambda \notin \mathcal{M}$, $y_\mu \in \mathcal{M}$ implies gαg-$ker(\{y_\mu\})$ is weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$.
Conversely, let either gαg-$ker(\{x_\lambda\})$ be weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ or gαg-$ker(\{y_\mu\})$ be weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$. Then there exists a Fgαg-OS $\mathcal{M}$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}$ and $y_\mu \notin \mathcal{M}$ or gαg-$ker(\{y_\mu\}) \leq \mathcal{M}$, $x_\lambda \notin \mathcal{M}$ implies $x_\lambda \in \mathcal{M}$, $y_\mu \notin \mathcal{M}$ or $x_\lambda \notin \mathcal{M}, y_\mu \in \mathcal{M}$. Thus, (X, τ) is a Fgαg-T_0-space.

Theorem 6.7: A FTS (X, τ) is Fgαg-T_1-space iff for each $x \neq y \in X$, gαg-$ker(\{x_\lambda\})$ is weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ and gαg-$ker(\{y_\mu\})$ is weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$.

Proof: Let (X, τ) be a Fgαg-T_1-space, then for each $x \neq y \in X$, there exist Fgαg-OS $\mathcal{U}, \mathcal{V}$ such that $x_\lambda \in \mathcal{U}, y_\mu \notin \mathcal{U}$ and $x_\lambda \notin \mathcal{V}, y_\mu \in \mathcal{V}$. Implies g$\alpha$g-$ker(\{x_\lambda\})$ is weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ and gαg-$ker(\{y_\mu\})$ is weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$.
Conversely, let gαg-$ker(\{x_\lambda\})$ be weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ and gαg-$ker(\{y_\mu\})$ be weakly ultra fuzzy gαg-separated from $\{x_\lambda\}$. Then there exist Fgαg-OS $\mathcal{U}, \mathcal{V}$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{U}, y_\mu \notin \mathcal{U}$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{V}, x_\lambda \notin \mathcal{V}$ implies $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ and $x_\lambda \notin \mathcal{V}$, $y_\mu \in \mathcal{V}$. Thus, (X, τ) is a Fgαg-T_1-space.

Theorem 6.8: A FTS (X, τ) is Fgαg-T_1-space iff for each $x \in X$, gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$.

Proof: Let (X, τ) be a Fgαg-T_1-space and let gαg-$ker(\{x_\lambda\}) \neq \{x_\lambda\}$. Then g$\alpha$g-$ker(\{x_\lambda\})$ contains another fuzzy point distinct from x_λ say y_μ. So $y_\mu \in$ gαg-$ker(\{x_\lambda\})$ implies gαg-$ker(\{x_\lambda\})$ is not weakly ultra fuzzy gαg-separated from $\{y_\mu\}$. Hence by theorem (6.7), (X, τ) is not a Fgαg-T_1-space this is contradiction. Thus gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$.
Conversely, let gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$, for each $x \in X$ and let (X, τ) be not a Fgαg-T_1-space. Then by theorem (6.7), gαg-$ker(\{x_\lambda\})$ is not weakly ultra fuzzy gαg-separated from $\{y_\mu\}$ for some $x \neq y \in X$, this means that for every Fgαg-OS $\mathcal{M}$ contains gαg-$ker(\{x_\lambda\})$ then $y_\mu \in \mathcal{M}$ implies

$y_\mu \in \wedge\{\mathcal{M} \in \text{Fg}\alpha\text{g-O}(X): x_\lambda \in \mathcal{M}\}$ implies $y_\mu \in g\alpha g\text{-}ker(\{x_\lambda\})$, this is contradiction. Thus, (X,τ) is a Fgαg-T_1-space.

Theorem 6.9: A FTS (X,τ) is Fgαg-T_1-space iff for each $x \neq y \in X$, $y_\mu \notin g\alpha g\text{-}ker(\{x_\lambda\})$ and $x_\lambda \notin g\alpha g\text{-}ker(\{y_\mu\})$.

Proof: Let (X,τ) be a Fgαg-T_1-space then for each $x \neq y \in X$, there exist Fgαg-OS $\mathcal{U},\mathcal{V}$ such that $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ and $y_\mu \in \mathcal{V}$, $x_\lambda \notin \mathcal{V}$. Implies $y_\mu \notin g\alpha g\text{-}ker(\{x_\lambda\})$ and $x_\lambda \notin g\alpha g\text{-}ker(\{y_\mu\})$.
Conversely, let $y_\mu \notin g\alpha g\text{-}ker(\{x_\lambda\})$ and $x_\lambda \notin g\alpha g\text{-}ker(\{y_\mu\})$, for each $x \neq y \in X$. Then there exist Fgαg-OS $\mathcal{U},\mathcal{V}$ such that $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ and $y_\mu \in \mathcal{V}$, $x_\lambda \notin \mathcal{V}$. Thus, (X,τ) is a Fgαg-T_1-space.

Theorem 6.10: A FTS (X,τ) is Fgαg-T_1-space iff for each $x \neq y \in X$ implies $g\alpha g\text{-}ker(\{x_\lambda\}) \wedge g\alpha g\text{-}ker(\{y_\mu\}) = 0_X$.

Proof: Let (X,τ) be a Fgαg-T_1-space. Then $g\alpha g\text{-}ker(\{x_\lambda\}) = \{x_\lambda\}$ and $g\alpha g\text{-}ker(\{y_\mu\}) = \{y_\mu\}$ [by theorem (6.8)]. Thus, $g\alpha g\text{-}ker(\{x_\lambda\}) \wedge g\alpha g\text{-}ker(\{y_\mu\}) = 0_X$.
Conversely, let for each $x \neq y \in X$ implies $g\alpha g\text{-}ker(\{x_\lambda\}) \wedge g\alpha g\text{-}ker(\{y_\mu\}) = 0_X$ and let (X,τ) be not Fgαg-T_1-space, then for each $x \neq y \in X$ implies $y_\mu \in g\alpha g\text{-}ker(\{x_\lambda\})$ or $x_\lambda \in g\alpha g\text{-}ker(\{y_\mu\})$ [by theorem (6.9)], then $g\alpha g\text{-}ker(\{x_\lambda\}) \wedge g\alpha g\text{-}ker(\{y_\mu\}) \neq 0_X$ this is contradiction. Thus, (X,τ) is a Fgαg-T_1-space.

Theorem 6.11: A FTS (X,τ) is Fgαg-T_1-space iff (X,τ) is Fgαg-T_0-space and Fgαg-R_0-space.

Proof: Let (X,τ) be a Fgαg-T_1-space and let $x_\lambda \in \mathcal{U}$ be a Fgαg-OS, then for each $x \neq y \in X$, $g\alpha g\text{-}ker(\{x_\lambda\}) \wedge g\alpha g\text{-}ker(\{y_\mu\}) = 0_X$ [by theorem (6.10)] implies $x_\lambda \notin g\alpha g\text{-}ker(\{y_\mu\})$ and $y_\mu \notin g\alpha g\text{-}ker(\{x_\lambda\})$, this means $g\alpha g\text{-}cl(\{x_\lambda\}) = \{x_\lambda\}$, hence $g\alpha g\text{-}cl(\{x_\lambda\}) \leq \mathcal{U}$. Thus, (X,τ) is a Fgαg-R_0-space.
Conversely, let (X,τ) be a Fgαg-T_0-space and Fgαg-R_0-space, then for each $x \neq y \in X$ there exists a Fgαg-OS $\mathcal{U}$ such that $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ or $x_\lambda \notin \mathcal{U}$, $y_\mu \in \mathcal{U}$. Say $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ since (X,τ) is a Fgαg-R_0-space, then $g\alpha g\text{-}cl(\{x_\lambda\}) \leq \mathcal{U}$, this means there exists a Fgαg-OS $\mathcal{V}$ such that $y_\mu \in \mathcal{V}$, $x_\lambda \notin \mathcal{V}$. Thus, (X,τ) is a Fgαg-T_1-space.

Theorem 6.12: A FTS (X,τ) is Fgαg-T_2-space iff
(i) (X,τ) is a Fgαg-T_0-space and Fgαg-R_1-space.
(ii) (X,τ) is a Fgαg-T_1-space and Fgαg-R_1-space.

Proof: (i) Let (X,τ) be a Fgαg-T_2-space, then it is a Fgαg-T_0-space. Now since (X,τ) is a Fgαg-T_2-space, then for each $x \neq y \in X$, there exist disjoint Fgαg-OS $\mathcal{U},\mathcal{V}$ such that $x_\lambda \in \mathcal{U}$ and $y_\mu \in \mathcal{V}$ implies $x_\lambda \notin g\alpha g\text{-}cl(\{y_\mu\})$ and $y_\mu \notin g\alpha g\text{-}cl(\{x_\lambda\})$, therefore $g\alpha g\text{-}cl(\{x_\lambda\}) = \{x_\lambda\} \leq \mathcal{U}$ and $g\alpha g\text{-}cl(\{y_\mu\}) = \{y_\mu\} \leq \mathcal{V}$. Thus, (X,τ) is a Fgαg-R_1-space.
Conversely, let (X,τ) be a Fgαg-T_0-space and Fgαg-R_1-space, then for each $x \neq y \in X$, there exists a Fgαg-OS $\mathcal{U}$ such that $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ or $y_\mu \in \mathcal{U}$, $x_\lambda \notin \mathcal{U}$, implies $g\alpha g\text{-}cl(\{x_\lambda\}) \neq g\alpha g\text{-}cl(\{y_\mu\})$, since (X,τ) is a Fgαg-R_1-space [by assumption], then there exist disjoint Fgαg-OS $\mathcal{M},\mathcal{N}$ such that $x_\lambda \in \mathcal{M}$ and $y_\mu \in \mathcal{N}$. Thus, (X,τ) is a Fgαg-T_2-space.
(ii) By the same way of part (i) a Fgαg-T_2-space is Fgαg-T_1-space and Fgαg-R_1-space.
Conversely, let (X,τ) be a Fgαg-T_1-space and Fgαg-R_1-space, then for each $x \neq y \in X$, there exist Fgαg-OS $\mathcal{U},\mathcal{V}$ such that $x_\lambda \in \mathcal{U}$, $y_\mu \notin \mathcal{U}$ and $y_\mu \in \mathcal{V}$, $x_\lambda \notin \mathcal{V}$ implies $g\alpha g\text{-}cl(\{x_\lambda\}) \neq g\alpha g\text{-}cl(\{y_\mu\})$, since (X,τ) is a Fgαg-R_1-space, then there exist disjoint Fgαg-OS $\mathcal{M},\mathcal{N}$ such that $x_\lambda \in \mathcal{M}$ and $y_\mu \in \mathcal{N}$. Thus, (X,τ) is a Fgαg-T_2-space.

Corollary 6.13: A Fgαg-T_0-space is Fgαg-T_2-space iff for each $x \neq y \in X$ with gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$, then there exist Fgαg-CS $\mathcal{M}_1$,$\mathcal{M}_2$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}_1$, g$\alpha$g-$ker(\{x_\lambda\}) \wedge \mathcal{M}_2 = 0_X$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{M}_2$, g$\alpha$g-$ker(\{y_\mu\}) \wedge \mathcal{M}_1 = 0_X$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = 1_X$.

Proof: By theorem (5.11) and theorem (6.12).

Corollary 6.14: A Fgαg-T_1-space is Fgαg-T_2-space iff one of the following conditions holds:
(i) for each $x \neq y \in X$ with gαg-$cl(\{x_\lambda\}) \neq$ gαg-$cl(\{y_\mu\})$, then there exist Fgαg-OS $\mathcal{U}, \mathcal{V}$ such that gαg-$cl($gαg-$ker(\{x_\lambda\})) \leq \mathcal{U}$ and gαg-$cl($gαg-$ker(\{y_\mu\})) \leq \mathcal{V}$.
(ii) for each $x \neq y \in X$ with gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$, then there exist Fgαg-CS $\mathcal{M}_1$,$\mathcal{M}_2$ such that gαg-$ker(\{x_\lambda\}) \leq \mathcal{M}_1$, g$\alpha$g-$ker(\{x_\lambda\}) \wedge \mathcal{M}_2 = 0_X$ and gαg-$ker(\{y_\mu\}) \leq \mathcal{M}_2$, g$\alpha$g-$ker(\{y_\mu\}) \wedge \mathcal{M}_1 = 0_X$ and $\mathcal{M}_1 \vee \mathcal{M}_2 = 1_X$.

Proof: (i) By corollary (5.12) and theorem (6.12).
(ii) By theorem (5.11) and theorem (6.12).

Theorem 6.15: A Fgαg-R_1-space is Fgαg-T_2-space iff one of the following conditions holds:
(i) for each $x \in X$, gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$.
(ii) for each $x \neq y \in X$, gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$ implies gαg-$ker(\{x_\lambda\}) \wedge$ gαg-$ker(\{y_\mu\}) = 0_X$.
(iii) for each $x \neq y \in X$, either $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ or $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$.
(iv) for each $x \neq y \in X$ then $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ and $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$.

Proof: (i) Let (X, τ) be a Fgαg-T_2-space. Then (X, τ) is a Fgαg-T_1-space and Fgαg-R_1-space [by theorem (6.12)]. Hence by theorem (6.8), gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$ for each $x \in X$.
Conversely, let for each $x \in X$, gαg-$ker(\{x_\lambda\}) = \{x_\lambda\}$, then by theorem (6.8), (X, τ) is a Fgαg-T_1-space. Also (X, τ) is a Fgαg-R_1-space by assumption. Hence by theorem (6.12), (X, τ) is a Fgαg-T_2-space.
(ii) Let (X, τ) be a Fgαg-T_2-space. Then (X, τ) is a Fgαg-T_1-space [by remark (6.4)]. Hence by theorem (6.10), gαg-$ker(\{x_\lambda\}) \wedge$ gαg-$ker(\{y_\mu\}) = 0_X$ for each $x \neq y \in X$.
Conversely, assume that for each $x \neq y \in X$, gαg-$ker(\{x_\lambda\}) \neq$ gαg-$ker(\{y_\mu\})$ implies gαg-$ker(\{x_\lambda\}) \wedge$ gαg-$ker(\{y_\mu\}) = 0_X$. So by theorem (6.10), (X, τ) is a Fgαg-T_1-space, also (X, τ) is a Fgαg-R_1-space by assumption. Hence by theorem (6.12), (X, τ) is a Fgαg-T_2-space.
(iii) Let (X, τ) be a Fgαg-T_2-space. Then (X, τ) is a Fgαg-T_0-space [by remark (6.4)]. Hence by theorem (6.5), either $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ or $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ for each $x \neq y \in X$.
Conversely, assume that for each $x \neq y \in X$, either $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ or $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$ for each $x \neq y \in X$. So by theorem (6.5), (X, τ) is a Fgαg-T_0-space, also (X, τ) is a Fgαg-R_1-space by assumption. Thus (X, τ) is a Fgαg-T_2-space [by theorem (6.12)].
(iv) Let (X, τ) be a Fgαg-T_2-space. Then (X, τ) is a Fgαg-T_1-space and Fgαg-R_1-space [by theorem (6.12)]. Hence by theorem (6.9), $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ and $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$.
Conversely, let for each $x \neq y \in X$ then $x_\lambda \notin$ gαg-$ker(\{y_\mu\})$ and $y_\mu \notin$ gαg-$ker(\{x_\lambda\})$. Then by theorem (6.9), (X, τ) is a Fgαg-T_1-space. Also (X, τ) is a Fgαg-R_1-space by assumption. Hence by theorem (6.12), (X, τ) is a Fgαg-T_2-space.

Remark 6.16: Each fuzzy gαg-separation axiom is defined as the conjunction of two weaker fuzzy axioms: Fgαg-T_k-space = Fgαg-R_{k-1}-space and Fgαg-T_{k-1}-space = Fgαg-R_{k-1}-space and Fgαg-T_0-space, $k = 1,2$.

Remark 6.17: The relation between fuzzy gαg-separation axioms can be representing as a matrix. Therefore, the element a_{ij} refers to this relation. As the following matrix representation shows:

and	Fgαg-T_0	Fgαg-T_1	Fgαg-T_2	Fgαg-R_0	Fgαg-R_1
Fgαg-T_0	Fgαg-T_0	Fgαg-T_1	Fgαg-T_2	Fgαg-T_1	Fgαg-T_2
Fgαg-T_1	Fgαg-T_1	Fgαg-T_1	Fgαg-T_2	Fgαg-T_1	Fgαg-T_2
Fgαg-T_2	Fgαg-T_2	Fgαg-T_2	Fgαg-T_2	Fgαg-T_2	Fgαg-T_2
Fgαg-R_0	Fgαg-T_1	Fgαg-T_1	Fgαg-T_2	Fgαg-R_0	Fgαg-R_1
Fgαg-R_1	Fgαg-T_2	Fgαg-T_2	Fgαg-T_2	Fgαg-R_1	Fgαg-R_1

Matrix Representation
The relation between fuzzy gαg-separation axioms

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