Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: jih.uobaghdad.edu.iq

α-Sumudu Transformation Homotopy Perturbation Technique on Fractional Gas Dynamical Equation

Ali Moazzam
University of Agriculture Faisalab, Pakistan
alimoazzam7309723@gmail.com

Adnan Shokat
Comsats University, Islamabad, Pakistan,
Adnanshokat4118@gmail.com

Emad A. Kuffi
Al-Qadisiyah University, College of Engineering, Iraq.
emad.abbas@qu.edu.iq

Article history: Received 21 September 2022, Accepted 24 October 2022, Published in January 2023.

doi.org/10.30526/36.1.3029

Abstract

Transformation and many other substitution methods have been used to solve non-linear differential fractional equations. In this present work, the homotopy perturbation method to solve the non-linear differential fractional equation with the help of He's Polynomials is provided as the transformation plays an essential role in solving differential linear and non-linear equations. Here is the α-Sumudu technique to find the relevant results of the gas dynamics equation in fractional order. To calculate the non-linear fractional gas dynamical problem, a consumer method created on the new homotopy perturbation α-Sumudu transformation method (HPS_αTM) is suggested. In the Caputo type, the derivative is evaluated. α-Sumudu homotopy perturbation technique and He's polynomials are all incorporated in the HPSaTM. The availability of He's polynomials could be used to conveniently manage the non-linearity. The suggested approach shows that the strategy is simple to implement and provides results that can be compared to the results gained from any other transformation technique.

Keywords α-Sumudu, Mittag-Leffler, homotopy perturbation method, He's Polynomials.

1. Introduction

Fractional modeling is a collection of classification algorithms concerned with fractions having random order integrals and derivatives [1]. In [2], fractional derivatives have explored damping laws. Fractional calculus has been used in seemingly unrelated scientific and technological domains over the last ten years. Increasingly, difficulties in the mechanics of fluid [3], acoustics, electromagnetics, diffusion, processing of signals, and various physical phenomena are formulated using fractional differential equations [4]. Approximation approaches to fractional calculus perturbation describe a set of equations solved using different methods [5]. However, there are some drawbacks in perturbation approaches observed by [6] while solving the space-time diffusion equation of the fractional form. In [7] substitution method for solving the reaction-diffusion equation has been described. A recent overview of various new exponential approaches for searching isolated values of fractional nonlinear differential equations has been published [8]. He was the first to propose the homotopy perturbation method (HPM) [9]. Many researchers have looked at using the HPM to solve linear [10] and nonlinear problems in various science-technical sectors [11]. Method of Adomian decomposition [12] and variational iteration [13] have also been applied to learn about numerous physical problems. In the latest paper, [14] Appreciated consideration of homotopy perturbation Sumudu transform method (HPSTM) has been given to analyze results of partial differential nonlinear and linear equations. In [15], the iterative variational method is used to find the results for a system of linear and nonlinear differential equations with the help of He's Polynomials. Numerical solutions of the biological population model have been founded with the help of homotopy perturbation, and He's Polynomial by [16].In [17], the numerical of the Newell-Whitehead-Segal equation with the help of the variational iteration method and He's Polynomials. With the help of He's Polynomial and homotopy perturbation, the iterative variation method has been used to solve the KS equation [18]. The solution of the Black school option evaluation model [19], the KDV equation of fifth order in magneto-acoustic waves of numerical type [20], has been previously done by the mean of He's polynomials. [21] previously described the analytical and numeric solutions of vibration problems in nonlinear differential form. In [22-28], He's polynomial, homotopy perturbation methods, transformation of Sumudu, and Laplace type are used in solving nonlinear differential models.

In [29], HPSTM, which is a mixture of the Sumudu technique of homotopy perturbation and polynomials of He's typing, is applied. We assume the nonlinear gas dynamical fractional equation.

$$D_{\upsilon}^{\beta} V + \frac{1}{2}\left(V^2\right)_{\xi} + V\left(V-1\right) = 0 \tag{1.1}$$

with the initial condition

$$V(\xi, 0) = e^{-\xi}. \tag{1.2}$$

Where fractional derivative order is β. The probability distribution $V\left(\xi, \upsilon\right)$ is demarcated as a probability density function, where υ is the period and ξ is the spatially component. In the Caputo meaning, the derivation is grasped. The ordering of the fractional differential equation is described by a factor in the average response formulation, which may be changed to get different replies. The fractionated gas dynamical equation is reduced to the classic gas dynamical equation when $\beta = 1$. The universal laws of preservation, such as mass conservation, momentum preservation, sustainable energy, and so on, are used to create the gas dynamical equations. The non-linear time-fractional of gas dynamics has been deliberate before by [30]. To address the non-linear time-fractional problem, we use HPSTM and Adomian decomposition method (ADM). The study aims to expand the HPSTM's applicability to provide analytical and approximation answers to the time-fractional gas dynamical problem. The HPSTM has the benefit of being able to combine two effective approaches for finding precise and numerical approximation models for non-linear equations. It generates solutions in terms of convergent series with readily computed constituents without any need for linearization, perturbation, or restricted conditions. It is important to note that, compared to conventional approaches, the HPSTM can reduce the quantity of computational effort while keeping a high accuracy output; the size reduction equates to an enhancement in the approach's efficiency.

2. Materials and Methods

Definition 1. [31] If $f(t)$ is defined on R^+, then the α-Sumudu transform is

$$F_\alpha(v) = S_\alpha[f(t)](v) = \int_0^\infty \frac{1}{v^{\frac{1}{\alpha}}} e^{-\frac{t}{v^{\frac{1}{\alpha}}}} f(t)dt; \quad v \in R. \tag{1.3}$$

Definition 2. [1] The integral operator of Riemann-Liouville fractional with order β>0 is given as

$$J^\beta f(\upsilon) = \frac{1}{\Gamma(\beta)} \int_0^\tau (\upsilon - \tau) f(\tau) d\tau \quad (\beta > 0), \tag{1.4}$$

$$J^0 f(\upsilon) = f(\upsilon). \tag{1.5}$$

It can also be expressed in gamma relationship as

$$J^\beta \upsilon^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1)} \upsilon^{\beta+\gamma}. \tag{1.6}$$

Definition 3. [32] In the Caputo interpretation, the fractional derivative of $f(t)$ is specified as

$$D_\upsilon^\beta f(\upsilon) = J^{m-\beta} D^n f(\upsilon) = \frac{1}{\Gamma(n-\beta)} \int_0^\upsilon (\upsilon - \tau)^{m-\beta-1} f^{(m)}(\tau) d\tau,$$

for $m \geq \beta > m-1$, m is natural number and $\upsilon > 0$.

For the relationship between the fractional integral of Riemann-Liouville and the fractional Caputo derivatives, we have:

$$J_\upsilon^\beta D_\upsilon^\beta f(\upsilon) = f(\upsilon) - \sum_{q=0}^{m-1} f^{(q)}(0) + \frac{\upsilon^q}{q!}. \tag{1.7}$$

Definition 4. [31] Caputo fractional derivative's α-Sumudu transformation is described as:

$$S_\alpha[D_\upsilon^\beta f(\upsilon)] = \frac{G(v)}{v^{\frac{\beta}{\alpha}}} - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{v^{\frac{\beta-k}{\alpha}}}. \tag{1.8}$$

Solution by Homotopy Perturbation of Caputo Fractional Differential Equations

General Idea of HPS$_\alpha$TM

Let us assume that the general non-homogeneous and non-linear partial fractional differential equation to demonstrate the general idea of this technique. We have

$$D_\upsilon^\beta V(\xi, \upsilon) + RV(\xi, \upsilon) + NV(\xi, \upsilon) = g(\xi, \upsilon), \tag{2.1}$$

$$V(\xi, 0) = f(\xi), \tag{2.2}$$

where R indicates the linearly difference operator, N denotes the generic non-linear differential controller and $D_\upsilon^\beta V(\xi, \upsilon)$ is the Caputo derivative of functions.

By performing the α-Sumudu transform on (2.1), we get

$$S_\alpha[D_\upsilon^\beta V(\xi, \upsilon) + S_\alpha(RV(\xi, \upsilon)) + S_\alpha(NV(\xi, \upsilon))] = S_\alpha[g(\xi, \upsilon)]. \tag{2.3}$$

Applying the rules of the α-Sumudu technique and after simplification, we get

$$S_\alpha[V(\xi,\upsilon)] = f(\xi) + v^{\frac{\beta}{\alpha}} S_\alpha[g(\xi,\upsilon)] - v^{\frac{\beta}{\alpha}} S_\alpha[RV(\xi,\upsilon) + NV(\xi,\upsilon)]. \tag{2.4}$$

Using the inverse of α-Sumudu on (2.4), we find

$$V(\xi,\upsilon) = g(\xi,\upsilon) - S_\alpha^{-1}[\upsilon^{\frac{\beta}{\alpha}} S_\alpha[RV(\xi,\upsilon) + NV(\xi,\upsilon)]], \tag{2.5}$$

where $g(\xi,\upsilon)$ is the source term. Now we apply the HPM:

$$V(\xi,\upsilon) = \sum_{n=0}^{\infty} p^n V_n(\xi,\upsilon) \tag{2.6}$$

and the non-linear part can be decomposed as:

$$NV(\xi,\upsilon) = \sum_{n=0}^{\infty} p^n H_n(V). \tag{2.7}$$

For little He's described polynomials $H_n(\xi,\upsilon)$ that is given below as

$$H_n(V_0, V_1, ..., V_n),$$

$$\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i V_i\right)\right]_{p=0}; \quad n = 0, 1, 2, ... \tag{2.8}$$

By putting the values of (2.6)-(2.7) into (2.5), we find

$$\sum_{n=0}^{\infty} p^n V_n(\xi,\upsilon) = g(\xi,\upsilon) - p\left(S_\alpha^{-1}\left[\upsilon^{\frac{\beta}{\alpha}} S_\alpha\left[R\sum_{n=0}^{\infty} p^n V_n(\xi,\upsilon) + \sum_{n=0}^{\infty} p^n H_n(V)\right]\right]\right). \tag{2.9}$$

It is also the He's polynomials-based connection of the α-Sumudu technique and the HPM. The preceding estimates are produced by analyzing the factors of like powers of p,

$$\left.\begin{aligned} p^0 &: V_0(\xi,\upsilon) = g(\xi,\upsilon) \\ p^1 &: V_1(\xi,\upsilon) = -S_\alpha^{-1}[\upsilon^{\frac{\beta}{\alpha}} S_\alpha[RV_0(\xi,\upsilon) + V_0(\xi,\upsilon)]] \\ p^2 &: V_2(\xi,\upsilon) = -S_\alpha^{-1}[\upsilon^{\frac{\beta}{\alpha}} S_\alpha[RV_1(\xi,\upsilon) + V_1(\xi,\upsilon)]] \\ p^3 &: V_3(\xi,\upsilon) = -S_\alpha^{-1}[\upsilon^{\frac{\beta}{\alpha}} S_\alpha[RV_2(\xi,\upsilon) + V_2(\xi,\upsilon)]] \end{aligned}\right\}. \tag{2.10}$$

The remaining elements of the $V_n(\xi,\upsilon)$ may be acquired in almost the same way, and the appropriate route can therefore be found completely. Ultimately, we use a curtailed series to estimate the analytic answer $V(\xi,\upsilon)$:

$$V(\xi,\upsilon) = \lim_{N\to\infty} \sum_{n=0}^{\infty} V_n(\xi,\upsilon). \tag{2.11}$$

The series mentioned in (2.11) generally converges fastly.

3. Results

Solution of Gas Dynamic Problem

Let us assume that the proceeding non-linear gas dynamic time-fractional equation:

$$D_{\upsilon}^{\beta} V + \frac{1}{2}\left(V^2\right)_{\xi} + V\left(V-1\right) = 0 \tag{3.1}$$

with given initial condition

$$V(\xi, 0) = e^{-\xi}. \tag{3.2}$$

By performing the α-Sumudu technique, and with the help of (3.2), equation (3.1) becomes

$$S_{\alpha}[V(\xi, \upsilon)] = e^{-\xi} - v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2}(V^2)_{\xi} + V(V-1)\right]. \tag{3.3}$$

By performing α-Sumudu inverse transform,

$$V(\xi, \upsilon) = e^{-\xi} - S_{\alpha}^{-1}\left[v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2}(V^2)_{\xi} + V(1-V)\right]\right]. \tag{3.4}$$

By HPM technique,

$$\sum_{n=0}^{\infty} p^n V_n(\xi, \upsilon) = e^{-\xi} - S_{\alpha}^{-1}\left[v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2}\left(\sum_{n=0}^{\infty} p^n H_n(V)\right) - \sum_{n=0}^{\infty} p^n V_n(\xi, \upsilon) + \left(\sum_{n=0}^{\infty} p^n H_n^*(V)\right)\right]\right]. \tag{3.5}$$

In the above equation, $H_n(V)$ and $H_n^*(V)$ are the pre-described He's polynomials denoting the non-linear part. So, polynomials are illustrated by

$$\sum_{n=0}^{\infty} p^n H_n(V) = \frac{1}{2}(V^2)_{\xi}. \tag{3.6}$$

The first limited constituents of He's polynomials are given by

$$H_0(V) = (V_0^2)_{\xi}, \quad H_1(V) = (V_0 V_1)_{\xi}, \quad H_2(V) = (V_1^2 + V_0 V_2)_{\xi} \tag{3.7}$$

and for $H_n^*(V)$, we use

$$\sum_{n=0}^{\infty} p^n H_n^*(V) = V^2,$$

$$H_0^*(V) = V_0^2, \quad H_1^*(V) = 2V_0 V_1, \quad H_2^*(V) = V_1^2 + 2V_0 V. \tag{3.8}$$

Equating the factors of powers of p, we get

$$p^0 : V_0(\xi, \upsilon) = e^{\xi},$$

$$p^1 : V_1(\xi, \upsilon) = S_{\alpha}^{-1}\left[v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2} H_0(V) - V_0 + H_0(V)\right]\right] = e^{-\xi} \frac{\upsilon^{\beta}}{\Gamma(\beta+1)},$$

$$p^2 : V_2(\xi, \upsilon) = S_{\alpha}^{-1}\left[v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2} H_1(V) - V_1 + H_1(V)\right]\right] = e^{-\xi} \frac{\upsilon^{2\beta}}{\Gamma(2\beta+1)},$$

$$p^3 : V_3(\xi, \upsilon) = S_{\alpha}^{-1}\left[v^{\frac{\beta}{\alpha}} S_{\alpha}\left[\frac{1}{2} H_2(V) - V_2 + H_2(V)\right]\right] = e^{-\xi} \frac{\upsilon^{3\beta}}{\Gamma(3\beta+1)}. \tag{3.9}$$

Consequently, the solution series is

$$V(\xi, \upsilon) = e^{-\xi}\left[1 + \frac{\upsilon^{\beta}}{\Gamma(\beta+1)} + \frac{\upsilon^{2\beta}}{\Gamma(2\beta+1)} + \frac{\upsilon^{3\beta}}{\Gamma(3\beta+1)}\right]. \tag{3.10}$$

Setting $\beta = 1$ in equation (3.10), we repeat the solution of the equation as follows:

$$V(\xi, \upsilon) = e^{-\xi}\left[1 + \upsilon + \frac{\upsilon^2}{2!} + \frac{\upsilon^3}{3!} + \cdots\right]. \tag{3.11}$$

This result is comparable to the precise result in the form:

$$V(\xi, \upsilon) = e^{\upsilon-\xi}. \tag{3.12}$$

Equation (3.12) provides the exact solution of gas dynamic fractional equation.

5. Conclusion

Throughout this study, The non-linear time fractional gaseous dynamical problem is effectively solved using the homotopy perturbation α-Sumudu transformation method (HPS$_\alpha$TM). As a result, this approach is highly productive and helpful in addressing many types of fractional differential non-linear and linear equations that arise in various science and technology sectors. The HPSaTM, on the other hand, offers significant advantages. The HPSaTM is a great modification of current approaches that might have a broad spectrum of uses.

References

1.Podlubny, I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications. *Elsevier* **1998**.

2.Beyer, H.; Kempfle, S. Definition of physically consistent damping laws with fractional derivatives. *ZAMM - Journal of Applied Mathematics and Mechanics* **1995**, *75(8),* 623-635.

3.He, J.H. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *computer methods in applied mechanics and engineering* **1998**, *167(1),* 57-68.

4.Hilfer, R. Fractional time evolution. *Applications of Fractional Calculus in Physics* **2000**, *87*, 87-130.

5.He, J.H. Some applications of nonlinear fractional differential equations and their approximations. *Bulletin* of *Science*, *Technology* and *Society* **1999a**, *15(2),* 86-90.

6.Mainardi, F.; Luchko Y.; Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *Fractional Calculus and Applied Analysis* **2001**, *4(2),* 153–192.

7.Rida, S.Z.; El-sayed A.M.A.; Arafa, A.M.A. On the solutions of time-fractional reaction-diffusion equations. Communications in nonlinear science and numerical simulation **2010**, *15(12),* 3847-3854.

8.He, J.H. Asymptotic methods for solitary solutions and compactions. *Abstract and Applied Analysis* **2012**, *2012(1),* 1-30.

9.He, J.H. Homotopy perturbation technique. *computer methods in applied mechanics and engineering* ,**1999b**, *178(3)*, 257-262.

10.He, J.H. Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics* and *Computation* ,**2003**, *135(1),* 73-79.

11.Javidi, M.; Raji, M.A. Combination of Laplace transform and homotopy perturbation method to solve the parabolic partial differential equations. *Communications in Fractional Calculus* ,**2012**, *3(1)*, 10-19.

12.Ganji, D.D. A semi-analytical technique for non-linear settling particle equation of motion. *Journal of Hydro-environment Research* ,**2012**, *6(4)*, 323-327.

13.Duan, J.S.; Rach, R.; Baleanu, D.; Wazwaz, A.M. A review of the Adomian decomposition method and its applications to fractional differential equations. *Communications in Fractional Calculus* ,**2012**, *3(2),* 73-99.

14.Singh, J.; Kumar,D.; Sushila, D. Homotopy perturbation Sumudu transform method for nonlinear equations. *Advances in Theoretical* and *Applied Mechanics* ,**2011**, *4(4)*, 165-175.

15.Nadeem, M.; Yao, S.W. Solving system of partial differential equations using variational iteration method with He's polynomials. *Journal of Mathematics and Computer Science* ,**2019**, *19(3)*, 203-11.

16. Zellal, M.; Belghaba, K. An accurate algorithm for solving biological population model by the variational iteration method using He's polynomials. *Arab Journal of Basic and Applied Sciences* **2018,** *25(3)*, 142-9.

17.Nadeem,M.; Yao, S.W.; Parveen, N. Solution of Newell-Whitehead-Segel equation by variational iteration method with He's polynomials. *Journal of Mathematics and Computer Science* ,**2020**, *20(1)*, 21-9.

18.Ahmad, J. The variational iteration method on KS equation using he's polynomials. *Journal of Science and Arts* ,**2017**, *17(3)*, 443-8.

19.Edeki, S.O.; Ugbebor, O.O.; Owoloko, E.A. Construction of analytical solutions to the Black-Scholes option valuation model by means of He's polynomials technique. *Iniaeng transactions on engineering sciences: Special Issue for the International Association of Engineers Conferences* **2016**, 45-53.

20.Goswami, A.; Singh, J.; Kumar, D. Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves. *Ain Shams Engineering Journal* ,**2018**, *9(4)*, 2265-73.

21.Nadeem M.; Li, F. He–Laplace method for nonlinear vibration systems and nonlinear wave equations. *Journal of Low Frequency Noise, Vibration and Active Control* ,**2019**, *38(3-4)*, 1060-74.

22.Pandey, P.; Kumar, S.; Gómez-Aguilar JF, Baleanu D. An efficient technique for solving the space-time fractional reaction-diffusion equation in porous media. *Chinese Journal of Physics* **2020**, *68*, 483-92.

23.Nadeem, M.; He, J.H. He–Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics. *Journal of Mathematical Chemistry* ,**2021**, *59(5)*, 1234-45.

24.Ahmad, J,; Mushtaq, M. Decomposition method using He's polynomials for Kdv Boussinesq and coupled Kdv Boussinesq equations. *New Trends in Mathematical Sciences* ,**2017,** *5(4)*, 58-64.

25.Li, F.; Nadeem, M. He–Laplace method for nonlinear vibration in shallow water waves. *Journal of Low Frequency Noise, Vibration and Active Control* ,**2019**, *38, 3-4*, 1305-13.

26.Mishra, H.K.; Tripathi, R. Homotopy Perturbation Method of Delay Differential Equation Using He's Polynomial with Laplace Transform. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences* ,**2020**, *90(2)*, 289-98.

27.Goswami, A.; Singh, J.; Kumar, D. Numerical computation of fractional Kersten-Krasil'shchik coupled KdV-mKdV system occurring in multi-component plasmas. *AIMS Mathematics* ,**2020**, *5(3)*, 2346-69.

28.Chu, Y.M.; Ali, S.N.; Agarwal. P.; Dong, C. J. Analysis of fractional multi-dimensional Navier–Stokes equation. *Advances in Difference Equations,* **2021,** *1*, 1-8.

29.Ghorbani, A. Beyond Adomian polynomials: He polynomials. *Chaos Solitons Fractals* , **2009**, *39(3)*, 1486-1492.

30.Das S.; Kumar, R. Approximate analytical solutions of fractional gas dynamic equations. *Applied Mathematics* and *Computation,* **2011**, *217(24)*, 9905-9915.

31.Nanware, JA.; Patil, NG.; On Properties of α-Sumudu Transform and Applications. *Punjab University Journal of Mathematics* ,**2021,** *53(9)*, 606-620.
32.Caputo, M. Elasticità e Dissipazione. *Zanichelli*, Bologna, **1969.**