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RESEARCH ARTICLE

Posinormality of Operators Treated by Weyl's Theorem on Unbounded Hilbert Space

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ABSTRACT

Hamiltonians, momentum operators, and other quantum-mechanical perceptible take the form of self-adjoint operators when understood in quantized physical schemes. Unbounded and self-adjoint recognition are required in the situation of positive measurements. The selection of the proper Hilbert space(s) and the selection of the self-adjoint extension must be made in order for this to operate. In this effort, we define a new extension positive measure depending on the measurable field of nonzero positive self-adjoint operator in unbounded Hilbert space of analytic functions of complex variables. Consequently, we define an extension norm in the same space. We show several new properties of the suggested operator and its adjoin operator. These properties include the posinormality, inclusion property, isolating property and achieving the Weyl's and Browder's theorems.

Keywords: Unbounded operator, Browder's theorem, Weyl's theorem, Unbounded Hilbert space, Self-adjoint operator, Positive measurement, Posinormality

1. Introduction

Here the conception of an unbounded operator delivers a non-figurative background for allocating with differential operators, unbounded perceptible in quantum mechanics, and other circumstances. The Weyl's Theorem for bounded hermitian operators was established by Weyl [1]. Weyl's Theorem has since been expanded to encompass the class of bounded normal, hyponormal, and Toeplitz operators [2] as well as a number of other non-normal categories of bounded operators. The familiar Weyl's theorem is generalized in such a way. Furthermore, he established this modified version of the traditional Weyl's theorem for limited hyponormal operators in [3]. The classes of bounded operators have so far been the exclusive subject of this study. The works in this direction have recently been expanded to include the classes of unbounded posinormal operators and unbounded hyponormal operators [4].

For unbounded operators on different spaces such as the space of Banach with non-empty resolvent, the authors introduced the B-Fredholm theory in [5]. Weyl's Theorem for the category of paranormal operators in Banach spaces was established by Ramanujan [6], and it was further developed by Aiena and Guillen [7] to include the investigation of Weyl's. Theorems for the perturbation of paranormal operators by algebraic operators and unbounded compact operators defined on a Banach space are investigated by the authors [8, 9], including those by Browder and Weyl. The theory is demonstrated in the final section using examples involving isometrics, analytical Toeplitz operators, semi-shift operators, and weighted right shifts. Both generalized Weyl's theorem

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and generalized Browder's theorem are susceptible to failure for matrices with two-by-two operators. In this study, we also investigate the survival of generalized Weyl's, generalized Browder's, generalized a-Weyl's, and generalized a-theorems Browder's for 2D- upper triangular operator matrices on the Banach space [10]. The heritability of new, powerful forms of the Weyl-type theorem is given in [11–15]. As a practical example, the authors in [16] demonstrated a limit for the analytical continuation of the data concerning frequency. The inverse source problem's stability estimate is then derived. The high-frequency tail of the source function, whose value drops as the frequency's upper bound rises, and the Lipschitz-type data discrepancy make up the estimate.

In this project, we construct a novel extension positive measure that is dependent on the measurable field of a nonzero positive self-adjoint operator in an unbounded Hilbert space of analytic functions of complex variables. We define an extension norm in the same space as a result. The recommended operator and its adjoin operator have various new properties that we demonstrate. The Weyl's, a-Weyl's and Browder's theorems are met, as well as the inclusion and isolating properties and posinormality.

2. Preliminaries

In this section, we recall the following concepts (see [17-20]), which are used alter.

2.1. Concepts

Definition 2.1. Let $\mathbb{H}$ be a Hilbert space and let $\Delta_\Lambda \subset \mathbb{H}$ be a subset of $\mathbb{H}$ joining the operator Λ. Then the operator Λ, Δ_Λ is called bounded if

$$\|\Lambda\| = \sup_{\substack{\varphi\in\Delta,\varphi\neq0\\ \Lambda}} = \frac{\|\Lambda\varphi\|}{\|\varphi\|} = \sup_{\substack{\varphi\in\Delta\|\varphi\|=1\\ \Lambda}} \|\Lambda\varphi\| < \infty.$$

Otherwise, it is called an unbounded operator.

Let $[\Lambda, \Delta_\Lambda]$ be unbounded operator on $\mathbb{H}$. The adjoint of $[\Lambda, \Delta_\Lambda]$ is the operator $[\Lambda^*, \Delta^*_\Lambda]$ satisfying the map

$$\Lambda^* : \varphi \in \Delta^*_\Lambda \to \psi, \quad (\varphi, \Lambda\phi) = (\psi, \phi), \forall\phi \in \Delta_\Lambda.$$

Note that the operator $[\Lambda^*, \Delta^*_\Lambda]$ is linear in $\mathbb{H}$.

A self-adjoint operator Λ is positive if $\langle\Lambda\chi, \chi\rangle \geq 0$ for every $\chi \in \Delta(\Lambda) \subseteq \mathbb{H}$.

A densely defined operator $[\Lambda, \Delta_\Lambda] \in \mathbb{H}$ with $\Delta(\Lambda) \subset \Delta(\Lambda^*)$ is called posinormal (positive-normal) if there occurs a positive operator, called interrupter Π, achieving $\Lambda\Lambda^* = \Lambda^*\Pi\Lambda$.

A densely close linear operator $[\Lambda, \Delta_\Lambda]$ with $\Delta\Lambda \subset \Delta\Lambda^*$ is called totally posinormal if $\Lambda - \lambda I$ is posinormal for every $\lambda \in \mathbb{C}$.

A closed linear operator $[\Lambda, \Delta\Lambda]$ is known as a semi-Fredholm if the range of the operator Λ denoting by $\Re(\Lambda)$ is closed and null of Λ and the dimension of the null space of Λ denoting by $\aleph(\Lambda) < \infty$.

Definition 2.2. Let $\sigma(\Lambda)$ be the spectrum of the operator Λ. Then $\sigma_A(\Lambda)$ indicates the approximate spectrum in $\sigma(\Lambda)$; $\sigma_W(\Lambda)$ presents the Weyl-spectrum (i.e. $\sigma(\Lambda - \lambda I)$ is not Weyl). The operator $[\Lambda, \Delta\Lambda]$ is said to be satisfied the property

if $\sigma_A(\Lambda)\backslash\sigma_W(\Lambda)$
if $\sigma(\Lambda)\backslash\sigma_W(\Lambda)$.

2.2. Unbounded Hilbert spaces

In this part, the unbounded Hilbert spaces $\mathbb{H}_m$, $m = 1, \ldots,$ of analytic functions

$$\Phi(z) = \Phi(z_1, \ldots, z_m),$$

where $z = (z_1, \ldots, z_m)$ is an element of the m-dimensional complex Euclidean space Ξ_m. The inner product of two analytic functions in $\Phi, \Phi' \in \mathbb{H}_m, m \to \infty$ is defined by the formula [21]

$$\langle \Phi, \Phi' \rangle = \int_{\Xi_m} \bar{\Phi}\Phi' d\mu_m(z),$$

where $z_n = x_n + iy_n$

$$d\mu_m(z) = \frac{\exp\left(-\sum_{n=1}^{m}\right)|z_n|^2 \prod_{n=1}^{m} dx_n dy_n}{\pi^m}$$

Then $\Phi \in \mathbb{H}_m$ if and only if

$$\langle \Phi, \Phi' \rangle = \int_{\Xi_m} \bar{\Phi}\Phi' d\mu_m(z) < \infty.$$

In this effort, we shall generalize the above definition using the measurable field κ_m of nonzero positive self-adjoint operators, as follows: The inner product of two analytic functions in $\Phi, \Phi' \in \mathbb{H}_m^{\kappa}$ is defined by the formula

$$\langle \Phi, \Phi' \rangle = \int_{\Xi_m} \bar{\Phi}\Phi' d\nu_m(z),$$

where $z_n = x_n + iy_n$ and

$$d\nu_m := \kappa_m \left(\frac{\exp\left(-\sum_{n=1}^{m}\right)|z_n|^2 \prod_{n=1}^{m} dx_n dy_n}{\pi^m} \right)$$

Then $\Phi \in \mathbb{H}_m^{\kappa}$ if and only if

$$\langle \Phi, \Phi \rangle = \int_{\Xi_m} \bar{\Phi}\Phi d\nu_m(z) < \infty.$$

It is clear that when $\kappa_m = 1$ for all m, we obtain the the above measure $d\mu_m(z)$. In this study, we shall investigate some important properties of the space $\mathbb{H}_m^{\kappa}$. Consequently, every function $\Phi \in \mathbb{H}_m^{\kappa}$ can be written by the formula

$$\Phi(z) = \sum_{\iota} \alpha[\iota]\nu[\iota](z),$$

where

$$\nu[\iota] = \prod_{n=1}^{m} \frac{(z_n^{\iota_n})}{\sqrt{\iota_n!}},$$

and $\iota = (\iota_1, \ldots, \iota_m)$ indicates an m-tuple of nonnegative integers. Moreover,

$$\|\Phi\|^2 = \sum_{\iota} |\alpha[\iota]|^2, \quad |z_n| \leq 1.$$

In general, we define the following norm:

$$\|\Phi\|^2 = \sum_{\iota} |\alpha[\iota]\nu[\iota]|^2.$$

3. Results

We present our results as follows:

Proposition 3.1. Let $\mathbb{H}_m^{\kappa}$ be the collections of Hilbert spaces of analytic functions of complex variables. Let the operator $[\Omega, \Delta\Omega]$ be defined as follows:

$$\Omega(z) = \Omega(\xi_1 z_1, \ldots, \xi_m z_m) = \sum_{\iota} \xi[\iota]\nu[\iota](z),$$

$$(z_m = x_m + iy_m, \xi_m, \in \mathbb{C})$$

where

$$\Delta\Omega = \left\{ (z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]|^2 < \infty \right\}.$$

Then the operator $[\Omega, \Delta\Omega]$ is an unbounded posinormal operator.

Define a set

$$h_0 := \{z = (z_1, \ldots, z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, |z_j| \leq 1, j = 1, \ldots, m\},$$

where $\{z\}_n$ presents the sequence generated by the element $(z_1, \ldots, z_m)$. That is there is at least one element of the following sequence which is different from zero element

$$\{z\}_1 = (z_1, \ldots, z_m)_1, \ldots, \{z\}_n = (z_1, \ldots, z_m)_n.$$

Thus, the set h_0 is dense in $\mathbb{H}_m^{\kappa}$. Consequently, we have $h_0 \subseteq \Delta\Omega$ and then $\Delta\Omega$ is dense in $\mathbb{H}_m^{\kappa}$. The adjoint operator of Ω is defined as follows $[\Omega^*(z), \Delta\Omega^*]$:

$$\Omega^*(z) = \left(\overline{\xi_1 z_1}, \ldots, \overline{\xi_m z_m}\right) = \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}(\bar{z}),$$

where

$$\Delta\Omega^* = \left\{ (z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\overline{\xi[\iota]}|^2 < \infty \right\}.$$

Obviously, we have

$$\begin{aligned}
\Delta\Omega &= \left\{ (z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]|^2 < \infty \right\} \\
&= \left\{ (z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\overline{\xi[\iota]}|^2 < \infty \right\} \\
&= \Delta\Omega^*.
\end{aligned}$$

Moreover, a computation yields

$$\begin{aligned}
\Omega(z) \cdot \Omega^*(z) &= \sum_{\iota} \xi[\iota]\nu[\iota](z) \cdot \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}(\bar{z}) \\
&= \sum_{\iota} \xi[\iota]\nu[\iota]\,(z_1, \ldots, z_m) \cdot \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\,(\overline{z_1}, \ldots, \overline{z_m}) \\
&= \left(\sum_{\iota} \xi[\iota]\nu[\iota] z_1, \ldots, \sum_{\iota} \xi[\iota]\nu[\iota] z_m \right) \cdot \left(\sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_1}, \ldots, \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_m} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{\iota} \xi[\iota]\nu[\iota] \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\right) z_1\overline{z_1} + \ldots + \left(\sum_{\iota} \xi[\iota]\nu[\iota] \overline{\sum_{\iota}} \overline{\xi[\iota]\nu[\iota]}\right) z_m\overline{z_m} \\
&= \left(\sum_{\iota} \overline{\xi[\iota]\nu[\iota]} \sum_{\iota} \xi[\iota]\nu[\iota]\right) \overline{z_1}z_1 + \ldots + \left(\sum_{\iota} \overline{\xi[\iota]\nu[\iota]} \sum_{\iota} \xi[\iota]\nu[\iota]\right) \overline{z_m}z_m \\
&= \left(\sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_1}, \ldots, \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_m}\right) \cdot \left(\sum_{\iota} \xi[\iota]\nu[\iota]z_1, \ldots, \sum_{\iota} \xi[\iota]\nu[\iota]z_m\right) \\
&= \left(\sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_1}, \ldots, \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}\overline{z_m}\right) \cdot 1_m \cdot \left(\sum_{\iota} \xi[\iota]\nu[\iota]z_1, \ldots, \sum_{\iota} \xi[\iota]\nu[\iota]z_m\right) \\
&= \Omega^*(z) \cdot 1_m \cdot \Omega(z),
\end{aligned}
$$

where $1_m = (\underbrace{1, \ldots, 1}_{m-times})$. Thus, the operator $[\Omega, \Delta\Omega]$ is an unbounded posinormal operator over the set h_0. This completes the proof.

Corollary 3.1. Let $\mathbb{H}_m^\kappa$ be the collections of Hilbert spaces of analytic functions of complex variables. Let the operator $[\Omega, \Delta\Omega]$ be defined as follows:

$$\Omega(z) = \Omega(\xi_1 z_1, \ldots, \xi_m z_m) = \sum_{\iota} \xi[\iota]\nu[\iota](z),$$

$$\left(z_m = x_m + iy_m, \xi_m \in \mathbb{C}\right)$$

where

$$\Delta\Omega = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]|^2 < \infty\right\}.$$

Then the null space of Ω denoting by $\aleph(\Omega)$ satisfies the inclusion property $\aleph(\Omega) \subset \aleph(\Omega^*)$.

In view of Proposition 3.1, we have that Ω is unbounded posinormal operator. Thus, in view of [17] – Theorem 2.2 (part one), we have the result.

Proposition 3.2. Let $\mathbb{H}_m^\kappa$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Omega, \Delta\Omega]$ be defined as follows:

$$\Omega(z) = \Omega(\xi_1 z_1, \ldots, \xi_m z_m) = \sum_{\iota} \xi[\iota]\nu[\iota](z),$$

$$\left(z_m = x_m + iy_m, \xi_m \in \mathbb{C}\right)$$

where

$$\Delta\Omega = \left\{(z_1, \ldots, z_m) \in \Xi_m : \frac{\left(\sum_{\iota} |\xi[\iota]|^2\right)}{2} < \infty\right\}.$$

If Ω satisfies

$\rho(\Omega) \neq \emptyset$ (the resolvent set of an operator of Ω)
$\sigma(\Omega - \lambda I) = \{0\}$ (the spectrum of the operator $\Omega - \lambda I$) on some invariant subspace

then

$$\aleph(\Omega - \lambda I) \subseteq \aleph(\Omega - \lambda I)^*$$

Since the domain $\Delta\Omega$ is dense in $\mathbb{H}_m^\kappa$ (Proposition [pro-1]) and, consequently, the adjoint operator $[\Omega^*, \Delta\Omega^*]$, is uniquely determined, then $[\Omega^*, \Delta\Omega^*]$ is a closed operator (see [22, 23]). Furthermore, $\Delta\Omega \subset \Delta\Omega^*$ then $[\Omega, \Delta\Omega]$

is closed and totally posinormal in $\mathbb{H}_m^\kappa$. Now to prove that $[\Omega, \Delta\Omega]$ is closed and totally posinormal in $\mathbb{H}_m^\kappa$, we aim to prove that $[\Omega - \lambda I, \Delta(\Omega - \lambda I)]$ is posinormal in $\mathbb{H}_m^\kappa$. Following the same steps in Proposition 3.1, we

$$\begin{aligned}(\Omega - \lambda I)(z)\cdot\left(\Omega^* - \lambda^* I\right)(z) &= \sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])(z)\cdot\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}(\bar{z})\\
&= \sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])(z_1,\dots,z_m)\cdot\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\left(\overline{z_1},\dots,\overline{z_m}\right)\\
&= \left(\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_1,\dots,\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_m\right)\cdot\\
&\quad\left(\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_1},\dots,\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_m}\right)\\
&= \left(\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\right)z_1\overline{z_1}\\
&\quad+\dots+\left(\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\right)z_m\overline{z_m}\end{aligned}$$

have

$$\begin{aligned}&= \left(\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])\right)\overline{z_1}z_1\\
&\quad+\dots+\left(\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])\right)\overline{z_m}z_m\\
&= \left(\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_1},\dots,\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_m}\right)\cdot\\
&\quad\left(\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_1,\dots,\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_m\right)\\
&= \left(\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_1},\dots,\sum_\iota \overline{(\xi[\iota]\nu[\iota] - \lambda[\iota])}\overline{z_m}\right)\cdot 1_m.\\
&\quad\left(\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_1,\dots,\sum_\iota (\xi[\iota]\nu[\iota] - \lambda[\iota])z_m\right)\\
&= \left(\Omega^* - \lambda^* I\right)(z)\cdot 1_m\cdot(\Omega - \lambda I)(z),\end{aligned}$$

where $1_m = (\underbrace{1,\dots,1}_{m-times})$. Thus, the operator $[\Omega, \Delta\Omega]$ is an unbounded posinormal operator over the set h_0.

Clearly, the operator $[\Omega, \Delta\Omega]$ is linear, because

$$\begin{aligned}\Omega(az + bw) &= \sum_\iota \xi[\iota]\nu[\iota](az + bw)\\
&= a\sum_\iota \xi[\iota]\nu[\iota](z) + b\sum_\iota \xi[\iota]\nu[\iota](w)\\
&= a\Omega(z) + b\Omega(w)\end{aligned}$$

Hence, the operator $[\Omega, \Delta\Omega]$ is a closed linear operator in the space $\mathbb{H}_m^\kappa$.

Thus, in view of [17] – Theorem 2.2 (part two), we have the result.

Corollary 3.2. Let the assumptions of Proposition 3.2 be valid. Then the set

$$\wp(\Omega - \lambda I) := \inf\left\{k : \aleph\left(\Omega^{\kappa}\right) = \aleph\left(\Omega^{1+\kappa}\right)\right\} \leq 1.$$

Moreover, λ is an isolated point of $\sigma(\Omega)$ if and only if λ is a simple pole of $\rho(\Omega)$.

A direct application of Theorem 2.1 in [17], we have the result. And the second part comes from [17] – Lemma 2.1.

The single-valued extension possession (SVEP) of closed linear operators is significant in Fredholm's theory. In this effort, we are primarily interested in the SVEP at a point, a localized variant of SVEP first proposed by J. Finch [24], and we refer it to the finiteness of the descent of a closed linear operator. Therefore, we have the next outcome:

Corollary 3.3. Let the assumptions of Proposition 3.2 be valid. Then the operators $[\Omega, \Delta\Omega]$ and $[\Omega^{*}, \Delta\Omega^{*}]$ achieve Weyl's Theorem; the operator $[\Omega^{*}, \Delta\Omega^{*}]$ achieve a – Weyl's Theorem;
the operator Ω achieves a – Weyl's Theorem, whenever Ω^{*} has SVEP.
the operators $[\Omega, \Delta\Omega]$ and $[\Omega^{*}, \Delta\Omega^{*}]$ achieve Browder's Theorem.
the operator $[\Omega^{*}, \Delta\Omega^{*}]$ achieves a – Browder's Theorem.
the $[\Omega, \Delta\Omega]$ achieves a – Browder's Theorem, whenever $[\Omega^{*}, \Delta\Omega^{*}]$ has SVEP.
A direct application of [17] – Theorem 3.1 implies the first three items. The fourth item occurs by using [17] – Theorem 3.2. Whereas, the last two items exist by utilizing [17] – Theorem 3.3.

Corollary 3.4. Let the assumptions of Proposition 3.2 be held. If $[\Omega^{*}, \Delta\Omega^{*}]$ has SVEP then the operator $[\Omega, \Delta\Omega]$ satisfies the properties (A) and (W).

In virtue of [17] – Theorem 4.1, we have the requested result.

Proposition 3.3. Let $\mathbb{H}^{\kappa}_{m}$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Theta, \Delta\Theta]$ be defined as follows:

$$\Theta(z) = \Theta\left(\xi_1 z_1, \ldots, \xi_m z_m\right) = \sum_{\iota} \xi[\iota]\nu[\iota](z),$$

$$\left(z_m = x_m + iy_m \xi_m \in \mathbb{C}\right)$$

where

$$\Delta\Theta = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]\nu[\iota]|^2 < \infty\right\}$$

Then the operator $[\Theta, \Delta\Theta]$ is an unbounded posinormal operator.

Define a set

$$h_1 := \left\{z = (z_1, \ldots, z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, z \in \mathbb{C}^m\right\},$$

where $\{z\}_n$ presents the sequence generated by the element $(z_1, \ldots, z_m)$. That is there is at least one element of the following sequence which is different from zero element

$$\{z\}_1 = (z_1, \ldots, z_m)_1, \ldots, \{z\}_n = (z_1, \ldots, z_m)_n.$$

Thus, the set h_0 is dense in $\mathbb{H}^{\kappa}_{m}$. Consequently, we have $h_1 \subseteq \Delta\Theta$ and then $\Delta\Theta$ is dense in $\mathbb{H}^{\kappa}_{m}$. The adjoint operator of Θ is defined by $[\Theta^{*}(z), \Delta\Theta^{*}]$:

$$\Theta^{*}(z) = \left(\overline{\xi_1 z_1}, \ldots, \overline{\xi_m z_m}\right) = \sum_{\iota} \overline{\xi[\iota]\nu[\iota]}(\bar{z}),$$

where

$$\Delta\Theta^* = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\overline{\xi[\iota]\nu[\iota]}|^2 < \infty\right\}.$$

Clearly, we get

$$\begin{aligned}\Delta\Theta &= \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]\nu[\iota]|^2 < \infty\right\}\\ &= \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\overline{\xi[\iota]\nu[\iota]}|^2 < \infty\right\}\\ &= \Delta\Theta^*.\end{aligned}$$

Moreover, a computation yields

$$\Theta(z) \cdot \Theta^*(z) = \Theta^*(z) \cdot 1_m \cdot \Theta(z),$$

where $1_m = (\underbrace{1, \ldots, 1}_{m-times})$. Thus, the operator $[\Theta, \Delta\Theta]$ is an unbounded posinormal operator over the set h_1. This completes the proof.

Corollary 3.5. Let $\mathbb{H}_m^{\kappa}$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Theta, \Delta\Theta]$ be defined as follows:

$$\Theta(z) = \Theta\left(\xi_1 z_1, \ldots, \xi_m z_m\right) = \sum_{\iota} \xi[\iota]\nu[\iota](z)$$
$$\left(z_m = x_m + iy_m, \xi_m \in \mathbb{C}\right)$$

where

$$\Delta\Theta = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_{\iota} |\xi[\iota]\nu[\iota]|^2 < \infty\right\}.$$

Then the null space of Θ denoting by $\aleph(\Theta)$ satisfies the inclusion property $\aleph(\Theta) \subset \aleph(\Theta^*)$.

In view of Proposition 3.3, we have that Θ is unbounded posinormal operator. Thus, in view of [17] – Theorem 2.2, we have the result.

Proposition 3.4. Let $\mathbb{H}_m^{\kappa}$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Theta, \Delta\Theta]$ be defined as follows:

$$\Theta(z) = \Theta\left(\xi_1 z_1, \ldots, \xi_m z_m\right) = \sum_{\iota} \xi[\iota]\nu[\iota](z)$$
$$\left(z_m = x_m + iy_m, \xi_m \in \mathbb{C}\right)$$

where

$$\Delta\Theta = \left\{(z_1, \ldots, z_m) \in \Xi_m : \frac{\left(\sum_{\iota} |\xi[\iota]\nu[\iota]|^2\right)}{2} < \infty\right\}.$$

If Θ satisfies

$\rho(\Theta) \neq \emptyset$ (the resolvent set of an operator of Θ)
$\sigma(\Theta - \lambda I) = \{0\}$ (the spectrum of the operator $\Theta - \lambda I$) on some invariant subspace

then

$\aleph(\theta - \lambda I) \subseteq \aleph(\theta - \lambda I)^*$

Since the domain $\Delta\Theta$ is dense in $\mathbb{H}_m^\kappa$ (Proposition 3.3) and, consequently, the adjoint operator $[\Theta^*, \Delta\Theta^*]$, is uniquely determined, then $[\Theta^*, \Delta\Theta^*]$ is a closed operator (see [22, 23]). In addition, a computation implies that $\Delta\Theta \subset \Delta\theta^*$ then $[\Theta, \Delta\Theta]$ is closed and totally posinormal operator in $\mathbb{H}_m^\kappa$. Clearly, the operator $[\Theta, \Delta\Theta]$ is linear, because

$$\begin{aligned}\Theta(az + bw) &= \sum_\iota \xi[\iota]\nu[\iota](az + bw)\\ &= a\sum_\iota \xi[\iota]\nu[\iota](z) + b\sum_\iota \xi[\iota]\nu[\iota](w)\\ &= a\Theta(z) + b\Theta(w).\end{aligned}$$

Hence, the operator $[\Theta, \Delta\Theta]$ is a closed linear operator in the space $\mathbb{H}_m^\kappa$.

Thus, in view of [17] – Theorem 2.2 (part two), we have the result.

Corollary 3.6. Let the assumptions of Proposition 3.4 be valid. Then the set

$$\wp(\Theta - \lambda I) := \inf\left\{k : \aleph\left(\Theta^k\right) = \aleph\left(\Theta^{1+k}\right)\right\} \le 1.$$

Moreover, λ is an isolated point of $\sigma(\Theta)$ if and only if λ is a simple pole of $\rho(\Theta)$.

A direct application of Theorem 2.1 – [17] for the first part and Lemma 2.1 – [17] for the second part.

Corollary 3.7. Let the assumptions of Proposition 3.4 be valid. Then
The operators $[\Theta, \Delta\Theta]$ and $[\Theta^*, \Delta\Theta^*]$ achieve Weyl's Theorem;
The operator $[\Theta^*, \Delta\Theta^*]$ achieve a–Weyl's Theorem;
If Θ^* has SVEP, then Θ achieves a–Weyl's Theorem.
The operators $[\Theta, \Delta\Theta]$ and $[\Theta^*, \Delta\Theta^*]$ achieve Browder's Theorem.
The operator $[\Theta^*, \Delta\Theta^*]$ achieves a–Browder's Theorem.
If $[\Theta^*, \Delta\Theta^*]$ has SVEP, then $[\Theta, \Delta\Theta]$ achieves a–Browder's Theorem.
Similar proof of Corollary 3.3.

Corollary 3.8. Let the assumptions of Proposition 3.4 be held. If $[\Theta^*, \Delta\Theta^*]$ has SVEP then the operator $[\Theta, \Delta\Theta]$ satisfies the properties (A) and (W).

In virtue of [17] – Theorem 4.1, we have the requested result.

Example 3.1. Let $\mathbb{H}_m^\kappa$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Omega, \Delta\Omega]_\lambda$ be defined with state with eigenvalue $\lambda \in \mathbb{C}$, as follows:

$$\begin{aligned}&\Omega(z) = \Omega\left(\lambda_1 z_1, \ldots, \lambda_m z_m\right) = \sum_\iota \lambda[\iota](z),\\ &\qquad \left(z_m = x_m + iy_m, \lambda_m \in \mathbb{C}\right)\end{aligned}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$

$$\Delta\Omega = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_\iota |\lambda[\iota]|^2 < \infty\right\}.$$

Then the operator $[\Omega, \Delta\Omega]_\lambda$ is an unbounded posinormal operator.

Define a set

$$E_0 := \left\{z = (z_1, \ldots, z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, \left|z_j\right| \le \frac{1}{1-q}, q > 1, j = 1, \ldots, m\right\}.$$

It is clear that $E_0 \subset h_0$, where

$$|z| \leq \frac{1}{1-q} < 1, \quad q > 1.$$

Thus, in view of Proposition 3.1, the operator $[\Omega, \Delta\Omega]_\lambda$ is an unbounded posinormal operator, where the adjoint operator of $[\Omega, \Delta\Omega]_\lambda$ is defined as follows $[\Omega^*(z), \Delta\Omega^*]_\lambda$:

$$\Omega^*(z) = \left(\overline{\xi_1 z_1}, \ldots, \overline{\xi_m z_m}\right) = \sum_\iota \overline{\lambda[\iota]}(\bar{z}),$$

where

$$\Delta\Omega^* = \left\{(z_1, \ldots, z_m) \in \Xi_m : \sum_\iota |\overline{\lambda[\iota]}|^2 < \infty\right\}.$$

4. Conclusion

The above investigation was an illustration of the application of the most important property posinormality acting on the Hilbert space of analytic functions in the complex plane, with an extension norm. We have received different possess of the suggested operator acting on the Hilbert space of analytic functions, such as the unboundedness (see Proposition 3.1) and inclusion (Corollary 3.1). Proposition 3.2 showed under some extra conditions based on the resolvent and spectrum of the operator $[\Omega, \Delta\Omega]$ (similarly, for $[\Theta, \Delta\Theta]$) imply the inclusion property of the spectrum operator $[\Omega\text{-}\lambda I, \Delta(\Omega\text{-}\lambda I)]$. We proved that λ is an isolated point of $\sigma(\Omega)$ if and only if it is a simple pole in $\rho(\Omega)$ (see Corollary [3.4]). Finally, we presented the main conditions on the operator $[\Omega, \Delta\Omega]$ to satisfy Weyl's, a-Weyl's and Browder's theorems (see [25]). For future work, one can proceed to deliver other properties of the operator $[\Omega, \Delta\Omega]$ (unbounded linear operator) such as the A-property. Or suggest another Hilbert space of holomorphic functions, meromorphic functions and special functions.

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Conflicts of interest

The author declares no conflict of interest.

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