# مجموعات الغاية المنتظمة وفضاء–G كارتان


**المستخلص**

أن الهدف الرئيسي من هذا البحث هو تقديم نوع جديد من فضاءات – G سمي فضاء –G كارتان المنتظم وكذلك نوع جديد من مجموعات الغاية أسميناه مجموعات الغاية المنتظمة $J^r(x)$ و $\Lambda^r(x)$ وأعطينا خصائص وبعض مكافئات تلك المفاهيم ثم بينا العلاقة بين فضاء –G كارتان المنتظم و بين المجموعتين $J^r(x)$ و $\Lambda^r(x)$ .

# *Regular Limit Sets and Cartan G – space*

Presented by

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## Abstract

The main goal of this work is to create a general type of $G$ – space , namely, regular Cartan $G$ – space and a new type of limit sets , namely, regular limit sets $\Lambda^r(x)$ , $J^r(x)$ and, give some properties and some equivalent statement of these concept also we explain the relationship among the definitions regular Cartan $G$ – space and $\Lambda^r(x)$, $J^r(x)$.

## Introduction

One of the very important concepts in topological groups is the concept of group actions and there are several types of these actions. This paper studies an important class of actions namely, regular Cartan $G$ – space.

Let $B$ be a subset of a topological space $(X,T)$. We denote the closure of $B$ and the interior of $B$ by $\overline{B}$ and $B^o$ , respectively .The subset $B$ of $(X, T)$ is called regular open (r – open) if $B = \overline{B}^{\circ}$. The complement of regular open set is defined to be a regular closed (r – closed) if $B = \overline{B^o}$ .The family of all r – open sets in $(X,T)$ forms a base of a smaller topology $T^r$ on $X$ ,called the semi – regularization of $T$, [2],in section one, we introduce some definitions, remarks, propositions, theorems which are needed in the next sections. In section two, we define the sets $\Lambda^r(x)$, $J^r(x)$ and prove its properties, also we give some equivalent statement of $\Lambda^r(x)$, $J^r(x)$ .In section three, we defines regular thin sets and regular Cartan $G$ – space and give some propositions and theorems which related with this concepts and shown the relationship among the regular Cartan $G$ – space and the sets $\Lambda^r(x)$ and $J^r(x)$.

# 1. Preliminaries

**1.1 Definition** [2]:A subset $B$ of $(X, T)$ is called regular open (r – open) if $B = \overline{B}^{\circ}$. The complement of regular open set is defined to be a regular closed (r – closed) if $B = \overline{B^{o}}$.The family of all r – open sets in $(X,T)$ forms a base of a smaller topology $T^{r}$ on $X$ ,called the semi – regularization of $T$

**1.2 Definition[2]:** A subset $B$ of a space $X$ is called regular neighborhood (r – neighborhood) of $x \in X$ if there is an r - open subset $O$ of $X$ such that $x \in O \subseteq B$.

**1.3 Definition** [2]:A subset $A$ of space $X$ is called r – compact if every r – open cover of $A$ has a finite sub cover. If $A=X$ then $X$ is called an r – compact space.

**1.4 Definition [2]:**

(i) A subset $A$ of space $X$ is called r- relative compact if $\overline{A}$ is r – compact.

(ii) A space $X$ is called r– locally r – compact if every point in $X$ has an r – relative compact r– neighborhood.

**1.5 Definition** [2]: Let $X$ and $Y$ be spaces and $f: X \rightarrow Y$ be a function. Then:

(i) $f$ is called regular continuous (r– continuous) function if $f^{-1}(A)$ is an r – open set in $X$ for every open set $A$ in $Y$.

(ii) $f$ is called regular irresolute (r – irresolute) function if $f^{-1}(A)$ is an r – open set in $X$ for every r- open set $A$ in $Y$.

**1.6 Definition** [2]: Let $(\chi_{d})_{d \in D}$ be a net in a space $X$ , $x \in X$. Then :

i) $(\chi_{d})_{d \in D}$ is called r – converges to $x$ (written $\chi_{d} \xrightarrow{r} x$) if $(\chi_{d})_{d \in D}$ is eventually in every r – neighborhood of $x$ . The point $x$ is called an r – limit point of $(\chi_{d})_{d \in D}$, and the notation "$\chi_{d} \xrightarrow{r} \infty$" is mean that $(\chi_{d})_{d \in D}$ has no r – convergent subnet.

ii) $(\chi_{d})_{d \in D}$ is said to have $x$ as an r – cluster point [written $\chi_{d} \overset{r}{\alpha} x$] if $(\chi_{d})_{d \in D}$ is frequently in every r - neighborhood of $x$ .

**1.7 Proposition** [2]: A space $(X, T)$ is an r – compact space if and only if every net in $X$ has r – cluster point in $X$.

**1.8 Proposition** [2]: Let $X$ be a space and $A \subseteq X$, $x \in X$. Then $x \in \overline{A}^{r}$ if and only if there exists a net $(\chi_{d})_{d \in D}$ in $A$ and $\chi_{d} \xrightarrow{r} x$ .

**1.9 Proposition** Let $X$ be a topological space .A point $x_o$ in $X$ is a cluster point of a net $(\chi_d)_{d\in D}$ if and only if there exists a subnet$(\chi_{dm})_{dm\in D}$ which converges to $x_o$ .

**1.10 Remark** [2]: For any space $X$ :

(i) If $(\chi_d)_{d\in D}$ is a net in $X$, $x\in X$ such that $\chi_d \longrightarrow x$ then $\chi_d \xrightarrow{r} x$ .

(ii) If $(\chi_d)_{d\in D}$ is a net in $X$, $x\in X$ such that $\chi_d \ \alpha\ x$ then $\chi_d \overset{r}{\alpha} x$.

(iii) If $(\chi_d)_{d\in D}$ is a net in $X$, $x\in X$. Then $\chi_d \xrightarrow{r} x$ in $(X, \mathrm{T})$ if and only if $\chi_d \to x$ in $(X, \mathrm{T}^r)$, and $\chi_d \overset{r}{\alpha} x$ in $(X, \mathrm{T})$ if and only if $\chi_d \ \alpha\, x$ in $(X, \mathrm{T}^r)$.

**1.11 Theorem**: Let $(\chi_d)_{d\in D}$ be a net in a space $(X, \mathrm{T})$ and $x_o$ in $X$. Then $\chi_d \overset{r}{\alpha} x_o$ if and only if there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{r} x_o$.

**Proof:** By Proposition (1.9) and Remark (1.10,.iii).

**1.12 Remark** :

(i)A function $f$:$(X, \mathrm{T})\to(Y,\tau)$ is r–continuous function if and only if $f$:$(X, \mathrm{T}^r)\to(Y,\tau^r)$ is continuous.

(ii)A function $f$: $(X, \mathrm{T}) \to(Y,\tau)$ is r – irresolute function if and only if $f$:$(X, \mathrm{T}^r)\to(Y, \tau^r)$ is continuous.

**1.13 Proposition**: Let $f$: $X\to Y$ be a function, $x\in X$. Then:

(i) $f$ is r – continuous at $x$ if and only if whenever a net $(\chi_d)_{d\in D}$ in $X$ and $\chi_d \xrightarrow{r} x$ then $f(\chi_d)\longrightarrow f(x)$.

(ii) $f$ is r – irresolute at $x$ if and only if whenever a net $(\chi_d)_{d\in D}$ in $X$ and $\chi_d \xrightarrow{r} x$ then $f(\chi_d) \xrightarrow{r} f(x)$.

**Proof:** (i) $\Rightarrow$ Let $x\in X$ and $(\chi_d)_{d\in D}$ be a net in $X$ such that $\chi_d \xrightarrow{r} x$ [To prove that $f(\chi_d)\longrightarrow f(x)$]. Let $V$ be an open neighborhood of $f(x)$. Since $f$ is r – continuous, then $f^{-1}(V)$ is r – neighborhood of $x$, but $\chi_d \xrightarrow{r} x$, then there is $\beta\in D$ such that $\chi_d \in f^{-1}(V)$, $\forall\ d \geq \beta$. Then $f(\chi_d) \in f(f^{-1}(V)) \subseteq V$. Thus $f(\chi_d)$ is eventually in every open neighborhood of $f(x)$, then $f(\chi_d)\longrightarrow f(x)$.

$\Leftarrow$ Suppose that $f$ is not r – continuous. Then there exists $x\in X$ such that $f$ is not r – continuous at $x$. Then there exists an open set $B$ in $Y$ such that $f(x) \in B$ and $f(A) \not\subset B$ for each $A$ is an r – open containing $x$ in $X$ . Thus there exists $\chi_A\in A$ and $f(\chi_A)\notin B$ for each $A$ is r – open in $X$. Then $\chi_A \xrightarrow{r} x$. But $f(\chi_A) \notin B$ for each $A \in N_r(x)$, then $f(\chi_A)$ is not convergent to $f(x)$ and this is a contradiction. Then $f$ is r – continuous.

(ii) $\Rightarrow$ Let $x \in X$ and $(\chi_d)_{d\in D}$ be a net in $X$ such that $\chi_d \xrightarrow{r} x$ .Then by Remark(1.10,iii) $\chi_d \longrightarrow x$ in $(X, T^r)$. Since $f$: $(X,T) \longrightarrow (Y,\tau)$ is r – irresolute then by Remark(1.12,ii) $f$: $(X, T^r) \longrightarrow (Y, \tau^r)$ is continuous. Thus $f(\chi_d) \longrightarrow f(x)$ in $(Y,\tau^r)$, so by Remark (1.10,iii) $f(\chi_d) \xrightarrow{r} f(x)$.

$\Leftarrow$By Remark (1.10,iii) and Remark (1.12,ii) we have $f$: $(X, \tau^r) \longrightarrow (Y, \tau^r)$ is continuous . Then $f$ is r – irresolute.

**<u>1.14 Definition</u>** <u>[3]:</u> A topological transformation group is a triple $(G,X,\varphi)$ where $G$ is a $T_2$–topological group, $X$ is a $T_2$ – topological space and $\varphi : G\times X \to X$ is a continuous function such that:

(i) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ for all $g_1, g_2 \in G$ , $x \in X$.

(ii) $\varphi(e, x) = x$ for all $x \in X$ , where $e$ is the identity element of $G$.

We shall often use the notation $g.x$ for $\varphi(g,x)$ $g.(h,x)= (gh).x$ for $\varphi(g, \varphi(h,x))= \varphi(gh,x)$. Similarly for $H \subseteq G$ and $A \subseteq X$ we put $HA = \{ga / g \in H, a \in A\}$ for $\varphi(H, A)$. A set $A$ is said to be invariant under $G$ if $GA = A$.

**<u>1.15 Definition</u>** <u>[3]:</u> Let $X$ be a $G$ – space and $x \in X$. Then:

(i) The function $\varphi$ is called an action of $G$ on $X$ and the space $X$ together with $\varphi$ is called a $G$ – space ( or more precisely left $G$ – space ).

(ii) The subspace $\{g.x \,/\, g \in G\}$ is called the orbit (trajectory) of $x$ under $G$, which denoted by $Gx$ [or $\gamma(x)$], and for every $x \in X$ the stabilizer subgroup $G_x$ of $G$ at $x$ is the set $\{g \in G / \, gx = x\}$.

(iii) $Ag= r_g(A) =\{ag : a \in A\}$;$Ag$ is called the left translate of $A$ by $g$.

(ix) $gA= L_g(A) =\{ga: a \in A\}$; $gA$ is called the right translate of $A$ by $g$.

**<u>1.16 Proposition</u>**:Let $G$ be a topological group and $(g_d)_{d\in D}$ be a net in $G$. Then:

(i)If $g_d \xrightarrow{r} e$, where $e$ is identity element of $G$, then $gg_d \xrightarrow{r} g$ (or $g_d\, g \xrightarrow{r} g$ ) for each $g \in G$.

(ii) If $g_d \xrightarrow{r} \infty$, then $gg_d \xrightarrow{r} \infty$ (or $g_d\, g \xrightarrow{r} \infty$) for each $g \in G$.

(iii) If $g_d \xrightarrow{r} \infty$, then $g_d^{-1} \xrightarrow{r} \infty$.

**Proof:** i) Since $r_g:G \to G$ is continuous and open , where $r_g$ is right translation by $g$. then $r_g$ is r – irresolute. Thus by Proposition (1.12,ii) $g_d\, g \xrightarrow{r} g$ for each $g \in G$.

ii) Let $g_d \xrightarrow{r} \infty$ and $g \in G$. suppose that $g_d g \xrightarrow{r} g_1$, for some $g_1 \in G$. Since $r_g$ is r-irresolute, then by Proposition(1.12,ii) $r_g^{-1}(g_d g) \xrightarrow{r} r_g^{-1}(g_1)$ .Then $g_d \xrightarrow{r} g_1 g^{-1}$, a contradiction. Thus $g_d\, g \xrightarrow{r} \infty$.

iii) Let $g_d^{-1} \xrightarrow{r} g$ . Since the inversion map of a topological group G, $v$: $G \to G$ is r – irresolute, then $g_d \xrightarrow{r} g^{-1}$. Thus if $g_d \xrightarrow{r} \infty$, then $g_d^{-1} \xrightarrow{r} \infty$ .

**1.17 Proposition:** If $(G, X, \varphi)$ is a topological transformation group, then $\varphi$ is r – irresolute.

**Proof:** Let $A \times B$ be an open set in $G \times X$, then $\varphi\,(A \times B) = AB$. Since $AB = \{\, x \in X \,/\, x = ab,\, a \in A,\, b \in B\} = \bigcup_{a \in A} aB = \bigcup_{a \in A} \varphi(B)$ . Since $\varphi_a$: $X \to X$ is homeomorphism from $X$ on itself such that $a \in G$. Then $aB$ is an open set in $X$, so $\bigcup_{a \in A} aB = AB$ is open. Since $\varphi$ is continuous and open function, then its clear that the action $\varphi$ is an r – irresolute

## *2 – Regular limit sets of a point:*

From now on, in this section by $G$ – space is meant a topological $T_2$ – space $X$ on which an r – locally r – compact, non – compact, $T_2$ – topological group $G$ acts continuously on the left.

**2.1 Definition**: Let $X$ be a $G$ – space and $x \in X$. Then:

(i) $\Lambda^r(x) = \{y \in X$: there is a net $(g_d)_{d \in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ such that $g_d x \xrightarrow{r} y\}$ is called regular limit set of $x$.

(ii) $J^r(x) = \{y \in X$: there is a net $(g_d)_{d \in D}$ in $G$ and there is a net $(\chi_d)_{d \in D}$ in $X$ with $g_d \xrightarrow{r} \infty$ and $\chi_d \xrightarrow{r} x$ such that $g_d x \xrightarrow{r} y\}$ is called regular first prolongation limit set of $x$.

**2.2 Proposition:** Let $X$ be a $G$ – space and $x \in X$. Then:

(i) $\Lambda^r(x)$ and $J^r(x)$ are invariant sets under $G$.

(ii) The orbit $\gamma(x)$ is r – closed if and only if $\Lambda^r(x)$ is a subset of $\gamma(x)$.

(iii) If $x \notin \Lambda^r(x)$, then the stabilizer subgroup $G_x$ of $G$ is r – compact.

(iv) if $\Lambda^r(x) = \phi$, for each $x \in X$. Then the orbit $\gamma(x)$ is not r – compact.

(v) $\overline{\gamma(x)}^r = \gamma(x) \cup \Lambda^r(x)$

(vi) $y \in J^r(x)$ if and only if $x \in J^r(y)$.

(vii) If $X$ is discrete $G$ – space, then $\Lambda^r(x) = J^r(x)$ for each $x \in X$.

(viii) If $x \in J^r(x)$, then for each $y \in \gamma(x)$, $y \in J^r(y)$.

(ix) (viii) If $y \in J^r(x)$, then for each $z \in \gamma(x)$, $y \in J^r(z)$.

(x) $g\Lambda^r(x) = \Lambda^r(gx) = \Lambda^r(x)$ and $gJ^r(x) = J^r(gx) = J^r(x)$ for each $g \in G$.

**Proof:** i) Let $y \in \Lambda^r(x)$ and $g \in G$. Then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ and $g_d x \xrightarrow{r} y$. It is clear that $(gg_d)_{d\in D}$ is a net in $G$ with $gg_d \xrightarrow{r} \infty$.Since the action is r – irresolute, thus $(gg_d).x \xrightarrow{r} gy$ which implies that $gy \in \Lambda^r(x)$ and hence $\Lambda^r(x)$ is invariant. The proof of $J^r(x)$ is similar.

ii)$\Rightarrow$ Let $y \in \Lambda^r(x)$, then there is a net $(g_d)_{d\in D}$ in $G$ such that $g_d \xrightarrow{r} \infty$ and $g_d x \xrightarrow{r} y$. Since $g_d x \in \gamma(x)$ and $(g_d x)_{d\in D}$ is a net in $\gamma(x)$, then by Proposition (1.8) $y \in \overline{\gamma(x)}^r$. But $\gamma(x)$ is r – closed then $y \in \gamma(x)$, so $\Lambda^r(x) \subseteq \gamma(x)$.

$\Leftarrow$ Let $y \in \overline{\gamma(x)}^r$. Then there exists $(y_d)_{d\in D}$ is a net in $\gamma(x)$ such that $y_d \xrightarrow{r} y$, then $\forall\, d \in D$ there is $g_d \in G$ such that $y_d = g_d x$. Then $(g_d)_{d\in D}$ is a net in $G$ and $g_d x \xrightarrow{r} y$. Now either $g_d \xrightarrow{r} g$ or $g_d \xrightarrow{r} \infty$. If $g_d \xrightarrow{r} g$ then $g_d x \xrightarrow{r} gx = y$, which implies that $y \in \gamma(x)$. If $g_d \xrightarrow{r} \infty$, then $y \in \Lambda^r(x) \subseteq \gamma(x)$, then $\gamma(x)$ is r – closed.

(iii) Let $x \notin \Lambda^r(x)$ and suppose that $G_x$ is not r – compact . Then there is a net $(g_d)_{d\in D}$ in $G_x$ such that $g_d \xrightarrow{r} \infty$. Since $g_d x = x$, i.e. $g_d x \xrightarrow{r} x$ then $x \in \Lambda^r(x)$ which is a contradiction, thus $G_x$ is r – compact.

(iv) Suppose that $\gamma(x)$ is r – compact. Since $\Lambda^r(x) = \phi$, then there is net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$, $(g_d x)_{d\in D}$ is a net in $\gamma(x)$. Since $\gamma(x)$ is r – compact, then by

Proposition (1.7) $g_d x \xrightarrow{r} y$ for some $y \in X$. Hence $y \in \Lambda^r(x)$, which is a contradiction with $\Lambda^r(x) = \phi$ for each $x \in X$.

(v) The proof of (v) is obvious

(vi) let $y \in J^r(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and there is a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$. Put $y_d = g_d\chi_d \xrightarrow{r} y$. Then by Proposition (1.16,iii) $g_d^{-1} \xrightarrow{r} \infty$ and $g_d^{-1} y_d = g_d^{-1} g_d\chi_d = \chi_d \xrightarrow{r} x$ thus $x \in J^r(x)$. The converse is similar.

(vii) The proof of (vii) is obvious.

(viii) Let $x \in J^r(x)$ and $y \in \gamma(x)$. Since $J^r(x)$ is invariant, then for each $y \in \gamma(x)$, $y \in J^r(x)$, therefore $x \in J^r(y)$ (by vi) and since $J^r(y)$ is invariant, then $y \in J^r(y)$.

(ix) Let $y \in J^r(x)$, from (v), $x \in J^r(y)$. Since $J^r(y)$ is invariant, then for each $z \in \gamma(x)$, $z \in J^r(y)$ and by (v) $y \in J^r(z)$ for each $z \in \gamma(x)$.

(x) The proof of (x) is obvious.

**2.3 Proposition:** Let $X$ be an r – locally r – compact $G$ – space and $x \in X$, then $x \notin \Lambda^r(x)$ if there is an r – neighborhood $U$ of $x$ and an r – compact r – neighborhood $V$ of $e$, $e$ is the identity in $G$, such that $gx \notin U$ for each $g \notin V$.

**Proof**

Let the statement be true. We suppose that $x \in \Lambda^r(x)$ then there is a net $(g_d)_{d\in D}$ in $G$ such that $g_d \xrightarrow{r} \infty$ and $g_d x \xrightarrow{r} x$, by hypothesis there exists $U$ be an r – neighborhood of $x$ and an r-compact r-nbd $V$ of e such that $gx \notin U$ for each $g \notin V$. Since $g_d x \xrightarrow{r} x$ then there is $d_o \in D$ such that $g_d x \in U$, for each $d \geq d_o$ therefore that $g_d \in V$, which is an r – compact, thus the net $(g_d)_{d\in D}$ has an r – convergent sub net, say itself, i.e., there is a point $g \in G$ such that $g_d \xrightarrow{r} g$ which is contradiction, since $(g_d)_{d\in D}$ has no r – convergent sub net, thus $x \notin \Lambda^r(x)$.

**2.4 Notation:** Let $X$ be a $G$ – space and $A$ , $B$ be two subset of $X$ . We mean by $((A, B))$ the set $\{g \in G \,/\, gA \cap B \neq \phi\}$.

**2.5 corollary:** Let $X$ be an r – locally r – compact $G$ – space and $x, y \in X$, then $y \notin \Lambda^r(x)$ if there is an r – neighborhood $U$ of $y$ and an r – compact r – neighborhood $V$ of $e$, such that $gx \notin U$ for each $g \notin V$.

Proof:

Let the statement be true. Suppose that $y \in \Lambda^r(x)$, then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ such that $g_d x \xrightarrow{r} y$. Then by hypothesis there is an r– neighborhood $U$ of $y$ and r – compact r – neighborhood $V$ of $e$, such that $g_d x \notin U$ for each $g \notin V$. Since $g_d x \xrightarrow{r} y$, then there is $d_o \in D$ such that $g_d x \in U$ for each $d_o \geq d$, therefore $g_d \in V$, which is r – compact, then $(g_d)_{d\in D}$ has an r – convergent subnet, which contradictions that $g_d \xrightarrow{r} \infty$. Hence $y \notin \Lambda^r(x)$.

**2.6 Theorem:** Let $X$ be r – locally r – compact $G$ – space and $x \in X$. Then $x \notin J^r(x)$ if and only if there is an r – neighborhood $U$ of $x$ and there is an r –compact r-neighborhood $V$ of $e$, where e is the identity element of $G$, such that $gU \cap U = \phi$ for each $g \notin V$.

**Proof:** $\Rightarrow$We suppose that the above statement is not true, i.e., for each r – neighborhood $U$ of $x$ and for each r – compact r – neighborhood $V$ of $e$ there is $g \notin V$ such that $gU \cap U \neq \phi$. We can choose $\{U_n\}_{n\in Z^+}$ to be sequence of an r – open neighborhood of $x$ such that $U_{n+1} \subset U_n \ldots$ and $\bigcap_{n\in Z^+} U_n = \{x\}$. Since $G$ is r – locally r – compact , then there is an r – compact r – neighborhood $V$ of $e$ , such that $G_x \subset V$. Thus for each $n$ there is $g_n \notin V$ such that $g_n U_n \cap U_n \neq \phi$ i.e., there is $\chi_n \in U_n$ and $g_n \chi_n \in U_n$. Since $\bigcap_{n\in Z^+} U_n = \{x\}$ , then we have $\chi_n \xrightarrow{r} x$ and $g_n \chi_n \xrightarrow{r} x$ and by hypothesis the sequence $(g_n)_{n\in N}$ has an r – convergent sub sequence , say itself, thus there is a point $g \in G$ such that $g_n \xrightarrow{r} g$ , and by Proposition (1.17) the action is an r – irresolute . Then by Proposition (1.13,ii) $g_n \chi_n \xrightarrow{r} gx = x$ and hence $g \in G_x \subset V$ . Therefore $g_n \in V$ for $n \geq n_o$ , which is a contradiction. Thus the statement is true.

Conversely: $\Leftarrow$ Let the statement be true, we suppose that $x \in J^r(x)$. Then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ and there is a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} x$. Then by hypothesis, there exists $U$ be an r – neighborhood of $x$ and $V$ be an r – compact r – neighborhood such that $gU \cap U = \phi$ for each $g \notin V$. Since$(\chi_d)_{d\in D}$ and $(g_d\chi_d)_{d\in D}$ are r – convergent to $x$ , thus there is $d_o \in D$ such that $\chi_d \in U$ and $g_d\chi_d \in U$ for each $d \geq d_o$ and hence $g_d \in ((U,U))$, therefore $g_d \in V$, which is r – compact , this it must have an r – convergent sub net which is a contradiction $x \notin J^r(x)$.

**2.7 corollary:** Let $X$ be an r – locally r – compact $G$ – space and $x, y$ be two points of $X$. Then $y \notin J^r(x)$ if there is an r – neighborhood $U$ of $x$, an r – neighborhood $W$ of $y$ and an r – compact r – neighborhood $V$ of $e$, where $e$ is the identity element of $G$ , such that $gU \cap W = \phi$ for each $g \notin V$.

Proof

Let the statement be true, suppose that $y \in J^r(x)$ . Then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ and a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$ . By hypothesis, there exist $U$ be r – neighborhood of $x$ , $W$ be r – neighborhood of $y$ and $V$ be an r – compact r – neighborhood of $e$ such that $gU \cap W = \phi$ for each $g \notin V$. Thus for $d_o \in D$ we have $\chi_d \in U$ and $g_d\chi_d \in W$ for each $d \geq d_o$ , then $g_d \in V$, which is r – compact . Therefore the net $(g_d)_{d\in D}$ has r – convergent subnet , which is contradiction. Thus

## *3 –Regular Cartan G - space*

**3.1 Definition:** Let $X$ be a $G$ – space .A subset $A$ of $X$ is said to be regular thin (r – thin) relative to a subset $B$ of $X$ if the set $((A, B)) = \{g \in G \mid gA \cap B \neq \phi\}$ has an r – neighborhood whose closure is r – compact in $G$. If $A$ is r – thin relative to itself, then it is called r – thin.

**3.2 Remark:** The r – thin sets have the following properties:

(i) Since $(gA \cap B) = g(A \cap g^{-1}B)$ it follows that if $A$ is r – thin relative to $B$, then $B$ is r – thin relative to $A$.

(ii) Since $(gg_1A \cap g_2B) = g_2(g_2^{-1} g\, g_1A \cap B)$ it follows that if $A$ is r – thin relative to $B$, then so are any translates $gA$ and $gB$.

(iii) If $A$ and $B$ are r – relative thin and $K_1 \subseteq A$ and $K_2 \subseteq B$ , then $K_1$ and $K_2$ are r – relatively thin.

(iv) Let $X$ be a $G$ – space and $K_1$, $K_2$ be r – compact subset of $X$, then $((K_1, K_2))$ is r – closed in $G$.

(v) If $K_1$ and $K_2$ are r – compact subset of $G$ – space $X$ such that $K_1$ and $K_2$ are r – relatively thin, then $((K_1, K_2))$ is an r – compact subset of $G$.

**Proof:** The prove of (i), (ii), (iii) and (v) are obvious.

(iv) Let $g \in \overline{((K_1, K_2))}^r$ . Then there is a net $(g_d)_{d\in D}$ in $((K_1, K_2))$ such that $g_d \xrightarrow{r} g$. Then we have net $(k_d^1)_{d\in D}$ in $K_1$, such that $g_d k_d^1 \in K_2$ , since $K_2$ is r – compact, then by Theorem (1.11) there exists a subnet $(g_{d_m} k_{d_m}^1)$ of $(g_d k_d^1)$ such that $g_{d_m} k_{d_m}^1 \xrightarrow{r} k_o^2$, where $k_o^2 \in K_2$. But $(k_{d_m}^1)$ in $K_1$ and $K_1$ is r – compact , thus there is a point $k_o^1 \in K_1$ and a subnet of $k_{d_m}^1$ say itself such that $k_{d_m}^1 \xrightarrow{r} k_o^1$. Then $g_{d_m} k_{d_m}^1 \xrightarrow{r} gk_o^1 = k_o^2$, which mean that $g \in ((K_1, K_2))$, therefore $((K_1, K_2))$ is r – closed in $G$.

**3.3 Theorem:** Let $X$ be r – locally r – compact $G$ – space and $x \in X$. Then $x \in J^r(x)$ if $x$ has no r – thin r – neighborhood.

**Proof:** $\Rightarrow$ Let $x \in J^r(x)$ and suppose that $x$ has r – thin r – neighborhood , there is an r – neighborhood $U$ of $x$ such that the set $((U,U))$ has r – compact closure .By hypothesis $x \in J^r(x)$ , then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d \xrightarrow{r} \infty$ and a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d \xrightarrow{r} x$ such that $g_d \chi_d \xrightarrow{r} x$, since $U$ is a r – neighborhood of $x$ , thus there is $d_o \in D$ such that $\chi_d \in U$ and $g_d \chi_d \in U$ for each $d \geq d_o$ . Thus $g_d \in \overline{((U,U))}$ ,$\forall\ d \geq d_o$ , which is r – compact , and hence the net $(g_d)_{d\in D}$ must have an r – convergent subset , which is a contradiction . Therefore $x$ has no r – thin r – neighborhood.

**3.4 Proposition:** Let $X$ be an r – locally r – compact $G$ – space . Then $J^r(x) = \phi$ for each $x \in X$ if every pair of point of $X$ has r – relatively thin r – neighborhood.

Proof

Let $x, y \in X,$ then by hypothesis, there are r – relative thin r – neighborhood $U$ of $x$ and $W$ of $y$. Thus $((U, W))$ has r – compact closure. If $V_1 = \overline{((U,W))}^r$ and $V_2$ be an r – compact r – neighborhood of $G_x$, then $V = V_1 \cup V_2$ is an r – compact r – neighborhood of $e$ and each $g\in V$, then $gU\cap W\neq\phi$ this means that $y\notin J^r(x)$. But $x$ and $y$ are arbitrary, thus we have $J^r(x)=\phi$ for each $x\in X$

**3.5 Definition:** A $G$ – space $X$ is said to be an r – Cartan $G$ – space if every point in $X$ has an r – thin r – neighborhood.

**3.6 Proposition:** If $X$ is r – Cartan $G$ – space, then each orbit of $x$ is r – closed in $X$ and stabilizer group of $G$ is r – compact.

Proof: Let $y\in\overline{\gamma(x)}^r$. Then there is a net $(y_d)_{d\in D}$ in $\gamma(x)$such that $y_d\xrightarrow{r}y$. Since $X$ is an r – Cartan $G$ – space, then $y$ has r – thin r – neighborhood $U$. Since $y_d\in\gamma(x)$, then there exists a net $(g_d)_{d\in D}$ in $G$ such that $y_d=g_dx$ for each $d\in D$. Fixed $d_o$ and $(g_d g_{d_o}^{-1})(g_{d_o}x)=g_dx$ so $g_d g_{d_o}^{-1}\in((U,U))$ , such that $g_d g_{d_o}^{-1}\xrightarrow{r}g$, then $g_dx\xrightarrow{r}g g_{d_o}x$ and $y=g g_{d_o}x$, so $y\in Gx$. Thus $\gamma(x)$ is r – closed in $X$. Now, let $x\in X$, then there exists an r – thin r – neighborhood $V$ of $x$. Clearly $G_x$ is r – closed in $G$ and since $G_x\subseteq((V,V))$. Hence $G_x$ is r – compact.

**3.7 Theorem:** Let $X$ be a $G$ – space. If $X$ is r – Cartan G – space then $x\notin J^r(x)$ for each $x\in X$.

Proof: $\Rightarrow$If $X$ is an r – Cartan $G$ – space. Let $x\in J^r(x)$, then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d\xrightarrow{r}\infty$ and there is a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d\xrightarrow{r}x$ such that $g_d\chi_d\xrightarrow{r}x$. Since $x\in X$ and $X$ is an r – Cartan $G$ – space, then $x$ has an r – open neighborhood $U$ such that $((U, U))$ is r – relative thin. Then $((U, U))$ is r - relative compact. Thus there is $d\in D$, $\chi_d$ and $g_d\chi_d$ are in $U$. So that $g_d$ is in $((U, U))$.Then $(g_d)_{d\in D}$ contains a convergent subnet, this is contradiction.

. **3.8 Proposition:** Let $X$ be an r – Cartan$G$ –space, then $\Lambda^r(x)=\phi$ for each $x\in X$.

Proof: Suppose that there is a point $y\in X$ such that $y\in\Lambda^r(x)$. Then there is a net $(g_d)_{d\in D}$ in $G$ with $g_d\xrightarrow{r}\infty$ such that $g_dx\xrightarrow{r}y$. Let $U_y$ be an r – thin r – neighborhood of $y$. Then there is $d_o\in D$ such that $g_dx\in U_y$ for each $d\geq d_o$, we get $g_d g_{d_1}^{-1}g_{d_1}x=g_dx\in U_y$, thus $g_d g_{d_1}^{-1}\in((U_y, U_y))$, which has r – compact closure. Hence the net $g_d g_{d_1}^{-1}$ has r – convergent subnet, say itself, i.e, there is $g_o\in G$ such that $g_d g_{d_1}^{-1}\xrightarrow{r}g_o$, then $g_d\xrightarrow{r}g_o g_{d_1}$, which is a contradiction, therefore $y\notin\Lambda^r(x)$, since $y$ is arbitrary, thus $\Lambda^r(x)=\phi$.

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