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RESEARCH ARTICLE

Embedded Schemes of the Runge-Kutta Type for the Direct Solution of Fourth-Order Ordinary Differential Equations

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ABSTRACT

This paper introduces an innovative approach for solving fourth-order ordinary differential equations (ODEs) of the form $\mu^{(iv)} = f(x, \mu, \mu')$ and $\mu^{(iv)} = f(x, \mu, \mu', \mu'')$. We present the embedded Runge-Kutta (RK) Direct Explicit (ERKDGF) method, a family of embedded direct explicit RK type methods tailored specifically for this purpose. Through meticulous application of Taylor expansion, we have derived algebraic equations with order conditions up to the sixth order, ensuring the accuracy and reliability of our proposed integrator. We have developed two key variants within this method, namely RKDF5(4) and ERKDGF5(4), with orders five and four, respectively. Our approach is strategically designed, with the higher-order method ensuring exceptional accuracy, and the lower-order counterpart providing optimal error estimates. To facilitate practical implementation, we have devised a variable step-size code based on these methods, which was applied to solve a set of fourth-order problems. Our method's performance was rigorously assessed through numerical experiments, with comparisons to existing embedded RK pairs that necessitate problem reduction into a system of first-order ODEs. The results unequivocally demonstrate the efficiency of our ERKDGF method, both in terms of accuracy and the number of function evaluations required. This research marks a significant advancement in the field, offering a robust and efficient solution for directly solving fourth-order ordinary differential equations.

Keywords: Runge-Kutta type method, General fourth-order, Ordinary differential equations, Embedded method, Order conditions

1. Introduction

Differential equations (DEs) represent a fundamental tool in the field of mathematical modelling, with numerous scientific applications. Fourth-order DEs perform an important role in a variety of practical sciences such as mechanics [1], quantum chemistry [2], beam theory [3], and even the intricate workings of neural networks [4].The prevalence of these equations emphasises their importance for comprehending intricate physical and engineering systems.

The current methods for solving DEs are limited in their ability to directly solve all types. Therefore, we are researching the development of more straightforward numerical approaches for this proposal. An ODE of higher order must be converted into a system of first-order ODEs before using an indirect numerical method to solve it. Numerous researchers created the family of Runge-Kutta methods for solving first-, second-, third-, and fourth-order ODEs. Nevertheless, it would be more effective if the issue could be resolved by using numerical techniques (see [5, 6]).

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Recent research has made significant strides in in solving third-order differential equations (ODEs). Several scholars have focused on developing direct integration methods based on RK types, specifically tailored for specialised third-order ODEs. This pioneering work is exemplified by studies in [7–9].

Then, Mechee et al. [10, 11] introduced a direct RK method designed explicitly for solving specific third-order ODEs. Similarly, Senu et al. [12] delved into the realm of one-step embedded explicit methods, emphasising RK-type techniques for the direct solution of third-order ODEs. Kasim et al. [13] made significant contributions by constructing an improved RK method specifically tailored for solving specialised third-order ODEs directly.

Recent advancements in this area include the work of Fawzi et al. [14], who developed a one-step method of fourth-order precision for direct solutions of third-order ODEs. Additionally, [15] proposed a one-step technique of fifth-order precision, further simplifying the direct solution of third-order ODEs. [16] demonstrate how to solve exceptional 3^{rd} ODEs using a fitted exponential-diagonally implicit RK technique. Furthermore, researchers have explored the incorporating variable step-size methodologies within one-step direct integrators of RK type, as demonstrated in [17–19].

Nonetheless, there has also been significant advancements in the investigation of solving particular fourth-order ordinary differential equations (ODEs). Researchers have successfully developed one-step direct numerical integrators of the RK type with constant step size, designed specifically for solving unique fourth-order ODEs (see [20–25]). Another substantial advancement was accomplished by [26] who built two embedded pairs of RK-type procedures. These techniques represent a significant improvement in the discipline because they were created expressly for the direct solution of special fourth-order ordinary differential equations. This study emphasises the ongoing work to improve the effectiveness and precision of numerical approaches, ensuring a complete toolkit for solving various classes of fourth-order ODEs.

To solve fourth-order ODEs, we introduce embedded pairs of RK type methods in this paper's two-part methodology. The techniques are separated into two groups, with the first group addressing equations of the form $\mu^{(iv)} = f(x, \mu, \mu')$ and the second group addressing equations of the type $\mu^{(iv)} = f(x, \mu, \mu', \mu'')$. This division allows us to comprehensively address and provide direct solutions for a wider range of fourth-order ODEs. The main objective behind devising embedded pairs within explicit RK methods is to obtain a cost-effective estimation of local errors, specifically intended for integration into the variable step-size algorithm.

2. Preliminary

This paper establishes the framework for a general quasi-linear fourth-order ordinary differential equation (ODE) defined as:

$$\mu^{(iv)}(x) = f\Big(x, \mu(x), \mu'(x), \mu''(x), \mu'''(x)\Big), \quad x \geq x_0 \tag{1}$$

Eq. (1) encompasses various special forms, and this study focuses on deriving numerical methods of Runge-Kutta (RK) type specifically tailored for two of these special cases.

2.1. Category I: Quasi-linear fourth-order ordinary differential equations

In this part we focus on the numerical integration of fourth-order ordinary differential equations, specifically those in the form:

$$\mu^{(iv)}(x) = f\Big(x, \mu(x), \mu'(x)\Big), \quad x \geq x_0 \tag{2}$$

with initial conditions

$$\mu(x_0) = \mu_0, \quad \mu'(x_0) = \mu_0', \quad \mu''(x_0) = \mu_0'', \quad \mu'''(x_0) = \mu_0'''$$

where $\mu, \mu', \mu'' \in \Re^d$, $f : \Re \times \Re^d \to \Re^d$ represents a continuously valued vector function. The RKDF method, designed with a general form comprising s-stages for solving the initial value problems (IVPs) described in

Eq. (2), can be expressed as follows:

$$\mu_{n+1} = \mu_n + \hbar\,\mu'_n + \frac{1}{2}\hbar^2\,\mu''_n + \frac{1}{6}\hbar^3\,\mu'''_n + \hbar^4\sum_{i=1}^{s} b_i k_i,$$

$$\mu'_{n+1} = \mu'_n + \hbar\,\mu''_n + \frac{1}{2}\hbar^2\,\mu'''_n + \hbar^3\sum_{i=1}^{s} b'_i k_i,$$

$$\mu''_{n+1} = \mu''_n + \hbar\,\mu'''_n + \hbar^2\sum_{i=1}^{s} b''_i k_i,$$

$$\mu'''_{n+1} = \mu'''_n + \hbar\sum_{i=1}^{s} b'''_i k_i. \tag{3}$$

where

$$k_1 = f(x_n, \mu_n, \mu'_n),$$

$$k_i = f\Big(x_n + c_i\hbar,\ \mu_n + c_i\,\hbar\,\mu'_n + \frac{\hbar^2}{2}c_i^2\,\mu''_n + \frac{\hbar^3}{6}c_i^3\,\mu'''_n + \hbar^4\sum_{j=1}^{s} a_{ij}k_j,\ \mu'_n + c_i\,h\,\mu''_n + \frac{\hbar^2}{2}c_i^2\,\mu'''_n + \hbar^3\sum_{j=1}^{s}\bar{a}_{ij}k_j\Big), \tag{4}$$

for $i = 2, 3, \ldots, s$.The variables $b_i, b'_i, b''_i, b'''_i, a_{ij}, \bar{a}_{ij}$ and c_i in the new approach are considered to be real, where $i, j = 1, 2, \ldots, s$. The method is explicit if $a_{ij} = 0$ and $\bar{a}_{ij} = 0$ for $i \leq j$, and implicit otherwise. The coefficients for the Generalised RK Method (RKDF) can be represented using Butcher notation, as detailed in Table 1.

Table 1. Butcher representation of embedded RKDF method.

$$\begin{array}{c|c|c} c & A & \bar{A} \\ \hline & b^T & \\ & b'^T & \\ & b''^T & \\ & b'''^T & \\ \hline & \hat{b}^T & \\ & \hat{b}'^T & \\ & \hat{b}''^T & \\ & \hat{b}'''^T & \end{array} \tag{5}$$

2.2. Category II: Quasi-linear fourth-order ordinary differential equations

In the subsequent section, our attention is directed towards the numerical integration of fourth-order ordinary differential equations, specifically those presented in the following form:

$$\mu_{n+1} = \mu_n + \hbar\,\mu'_n + \frac{1}{2}\hbar^2\,\mu''_n + \frac{1}{6}\hbar^3\,\mu'''_n + \hbar^4\sum_{i=1}^{s} b_i k_i,$$

$$\mu'_{n+1} = \mu'_n + \hbar\,\mu''_n + \frac{1}{2}\hbar^2\,\mu'''_n + \hbar^3\sum_{i=1}^{s} b'_i k_i,$$

$$\mu''_{n+1} = \mu''_n + \hbar\,\mu'''_n + \hbar^2 \sum_{i=1}^{s} b''_i k_i,$$

$$\mu'''_{n+1} = \mu'''_n + \hbar \sum_{i=1}^{s} b'''_i k_i. \tag{6}$$

where

$$k_1 = f\left(x_n, \mu_n, \mu'_n, \mu''_n\right),$$

$$k_i = f\Bigg(x_n + c_i\hbar,\ \mu_n + c_i\,\hbar\,\mu'_n + \frac{\hbar^2}{2}\,c_i^2\,\mu''_n + \frac{\hbar^3}{6}\,c_i^3\,\hbar'''_n + \hbar^4 \sum_{j=1}^{s} a_{ij}k_j,\ \mu'_n + c_i\,h\,\mu''_n + \frac{\hbar^2}{2}\,c_i^2\,\mu'''_n$$
$$+ \hbar^3 \sum_{j=1}^{s} \bar{a}_{ij}k_j,\ \mu''_n + c_i\,\hbar y'''_n + \hbar^2 \sum_{j=1}^{s} \bar{\bar{a}}_{ij}k_j\Bigg), \tag{7}$$

for $i = 2, 3, \ldots, s$. The variables $b_i, b'_i, b''_i, b'''_i, a_{ij}, \bar{a}_{ij}, \bar{\bar{a}}_{ij}$ and c_i in the new approach are presumed to be real and are utilised for $i, j = 1, 2, \ldots, s$. The method is explicit when $a_{ij} = 0,\ \bar{a}_{ij} = 0$ and $\bar{\bar{a}}_{ij} = 0$ for $i \le j$ and implicit otherwise. The coefficients for the Generalized Runge-Kutta Method (ERKDF) can be expressed through Butcher notation, outlined in Table 2.

Table 2. Butcher representation of embedded ERKDF method.

$$\begin{array}{c|c|c|c} c & A & \bar{A} & \bar{\bar{A}} \\ \hline & b^T & & \\ & b'^T & & \\ & b''^T & & \\ & b'''^T & & \\ \hline & \hat{b}^T & & \\ & \hat{b}'^T & & \\ & \hat{b}''^T & & \\ & \hat{b}'''^T & & \end{array} \tag{8}$$

This study introduces the RKDF 5(4) and RKT5(4) methods designed for solving $\mu^{(iv)} = f(x, \mu, \mu')$ and $\mu^{(iv)} = f(x, \mu, \mu', \mu'')$ respectively. These methods possess fifth and fourth orders of accuracy. The primary motivation behind developing this embedded pair of explicit RKDF and RKT methods is to generate a small local error estimation, crucial for the implementing variable step-size algorithms. The methods compute $\mu_{n+1}, \mu'_{n+1}, \mu''_{n+1}$, and μ'''_{n+1} to approximate $\mu(x_{n+1}), \mu'(x_{n+1}), \mu''(x_{n+1})$, and $\mu'''(x_{n+1})$, where μ_{n+1} represents the calculated solution, and $\mu(x_{n+1})$ represents the exact solution.

The following techniques are employed to enhance an optimised embedded RKDF and RKTF methods:

(a) The objective is to minimise the quantities $\| \tau_g^{(p+1)} \|_2$ and $\| \hat{\tau}_g^{(p+1)} \|_2$ for orders both higher and lower RKDF and RKTF formulas, ensuring their values are as small as possible,

where

$$\| \tau_g^{(p+1)} \|_2 = \left(\sum_{i=1}^{n_1} \left(\tau_i^{(p+1)}\right)^2 + \sum_{i=1}^{n_2} \left(\tau_i'^{(p+1)}\right)^2 + \sum_{i=1}^{n_3} \left(\tau_i''^{(p+1)}\right)^2 + \sum_{i=1}^{n_4} \left(\tau_i'''^{(p+1)}\right)^2 \right)^{\frac{1}{2}},$$

and

$$\| \hat{\tau}_g^{(q+1)} \|_2 = \left(\sum_{i=1}^{n_1} \left(\hat{\tau}_i^{(q+1)} \right)^2 + \sum_{i=1}^{n_2} \left(\hat{\tau}_i^{\prime(q+1)} \right)^2 + \sum_{i=1}^{n_3} \left(\hat{\tau}_i^{\prime\prime(q+1)} \right)^2 + \sum_{i=1}^{n_4} \left(\hat{\tau}_i^{\prime\prime\prime(q+1)} \right)^2 \right)^{\frac{1}{2}} \tag{9}$$

and $\tau^{(p+1)}$, $\tau'^{(p+1)}$, $\tau''^{(p+1)}$ and $\tau'''^{(p+1)}$ are the local truncation error norms for μ, μ', μ'' and μ''' respectively.

(b) The local error estimation at the point x_{n+1} is determined by the following formula:

$$LTE = max\{\| \xi_{n+1} \|_\infty, \| \xi'_{n+1} \|_\infty, \| \xi''_{n+1} \|_\infty, \| \xi'''_{n+1} \|_\infty\},$$

where

$$\xi_{n+1} = \hat{\mu}_{n+1} - \mu_{n+1}, \quad \xi'_{n+1} = \hat{\mu}'_{n+1} - \mu'_{n+1}, \quad \xi''_{n+1} = \hat{\mu}''_{n+1} - \mu''_{n+1},$$
$$\xi'''_{n+1} = \hat{\mu}'''_{n+1} - \mu'''_{n+1},$$

Here, μ_{n+1}, μ'_{n+1} μ''_{n+1} and μ'''_{n+1} represent solutions obtained using the higher-order formula, while $\hat{\mu}_{n+1}$, $\hat{\mu}'_{n+1}$ $\hat{\mu}''_{n+1}$ and $\hat{\mu}'''_{n+1}$ denote solutions obtained using the lower-order formula. These local error estimations, LTE, can be utilised to adjust the step size $\hbar$ based on the standard formula as outlined in [26].

$$\hbar_{n+1} = 0.9\, \hbar_n \left(\frac{Tol}{LTE} \right)^{\frac{1}{p+1}}, \tag{10}$$

Here, the value 0.9 acts as a safety factor and represents the local truncation error (LTE) at each step. Tol signifies the desired efficiency, indicating the maximum permissible local error. If $LTE \le Tol$, the step is considered acceptable, and the process proceeds with local extrapolation. This implies that a more efficient approximation is applied to enhance the integration, and h is updated using the Eq. (10). However, if $LTE > Tol$, the step is rejected. In this case, Eq. (10) provides a useful estimate for reducing the step size, making it smaller than h_n, for a repeated step.

3. Determination of order conditions and coefficients

To establish the order conditions and coefficients for the initial proposed numerical integrator outlined in Eqs. (4) and (5), we employed Taylor's series expansion technique on the RKDF formula. After performing necessary algebraic adjustments, this expanded form was set equal to the solution obtained through Taylor expansion. By directly expanding the error at each step, we determined the overall order conditions for the RKDF method, building upon the order conditions derived for the RK method as presented in [7]. Similarly, utilising Eqs. (6) and (7), we computed the specific order conditions (11)–(29) for the proposed direct integrators of RKTF type, which were implemented employing Maple Software to ensure wide accessibility and ease of replication.

The order terms for μ:

4^{th}-order

$$\sum b_i = \frac{1}{24}. \tag{11}$$

5^{th}-order

$$\sum b_i c_i = \frac{1}{120}. \tag{12}$$

6^{th}-order

$$\sum b_i c_i^2 = \frac{1}{360}, \quad \sum b_i \bar{\bar{a}}_{ij} = \frac{1}{720}. \tag{13}$$

The order terms for μ':

3^{rd}-order

$$\sum b'_i = \frac{1}{6}. \tag{14}$$

4^{th}-order

$$\sum b'_i c_i = \frac{1}{24}. \tag{15}$$

5^{th}-order

$$\sum b'_i c_i^2 = \frac{1}{60}, \quad \sum b'_i \bar{\bar{a}}_{ij} = \frac{1}{120}. \tag{16}$$

6^{th}-order

$$\sum b'_i c_i^3 = \frac{1}{120}, \quad \sum b'_i \bar{a}_{ij} = \frac{1}{720}, \quad \sum b'_i \bar{\bar{a}}_{ij} c_j = \frac{1}{720}, \quad \sum b'_i \bar{\bar{a}}_{ij} c_i = \frac{1}{240}. \tag{17}$$

The order terms for μ'':

2^{nd}-order

$$\sum b''_i = \frac{1}{2}. \tag{18}$$

3^{rd}-order

$$\sum b''_i c_i = \frac{1}{6}. \tag{19}$$

4^{th}-order

$$\sum b''_i c_i^2 = \frac{1}{12}, \quad \sum b'_i \bar{\bar{a}}_{ij} = \frac{1}{24}. \tag{20}$$

5^{th}-order

$$\sum b''_i c_i^3 = \frac{1}{20}, \quad \sum b''_i \bar{a}_{ij} = \frac{1}{120}, \quad \sum b''_i \bar{\bar{a}}_{ij} c_j = \frac{1}{120}, \quad \sum b''_i \bar{\bar{a}}_{ij} c_i = \frac{1}{40}. \tag{21}$$

6^{th}-order

$$\sum b''_i c_i^4 = \frac{1}{30}, \quad \sum b''_i \bar{a}_{ij} c_j = \frac{1}{720}, \quad \sum b''_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} = \frac{1}{720}, \quad \sum b''_i \bar{\bar{a}}_{ij} c_i^2 = \frac{1}{60}, \tag{22}$$

$$\sum b''_i \bar{a}_{ij} c_i = \frac{1}{180}, \quad \sum b''_i a_{ij} = \frac{1}{720} \quad \sum b''_i \bar{\bar{a}}_{ij} c_j^2 = \frac{1}{360}, \tag{23}$$

$$\sum b''_i c_i \bar{\bar{a}}_{ij} c_j = \frac{1}{180}, \quad \frac{1}{2} \sum b''_i \bar{\bar{a}}_{ij}^2 + \sum b''_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{240}. \tag{24}$$

The order terms for μ''':

1^{st}-order

$$\sum b'''_i = 1. \tag{25}$$

2^{nd}-order

$$\sum b_i''' c_i = \frac{1}{2}. \tag{26}$$

3^{th}-order

$$\sum b_i''' c_i^2 = \frac{1}{3}, \quad \sum b_i''' \bar{a}_{ij} = \frac{1}{6}. \tag{27}$$

4^{th}-order

$$\sum b_i''' c_i^3 = \frac{1}{4}, \quad \sum b_i''' \bar{\bar{a}}_{ij} c_j = \frac{1}{24}, \quad \sum b_i''' \bar{a}_{ij} c_i = \frac{1}{8}, \quad \sum b_i''' \bar{a}_{ij} = \frac{1}{24}. \tag{28}$$

5^{th}-order

$$\begin{aligned}
&\sum b_i''' c_i^4 = \frac{1}{5}, \quad \sum b_i''' \bar{a}_{ij} c_j = \frac{1}{120}, \quad \sum b_i''' \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} = \frac{1}{120}, \quad \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} = \frac{1}{10},\\
&\sum b_i''' \bar{a}_{ij} c_i = \frac{1}{30}, \quad \sum b_i''' a_{ij} = \frac{1}{120}, \quad \sum b_i''' \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} + \frac{1}{2} \sum b_i''' \bar{\bar{a}}_{ij}^2 = \frac{1}{40},\\
&\sum b_i''' c_i \bar{\bar{a}}_{ij} c_j = \frac{1}{30}, \sum b_i''' \bar{\bar{a}}_{ij} c_j^2 = \frac{1}{60}, \quad \sum b_i''' \bar{\bar{a}}_{ij} c_j^2 + \sum b_i''' c_i \bar{\bar{a}}_{ij} c_j = \frac{1}{20},\\
&\frac{1}{2} \sum b_i''' \bar{\bar{a}}_{ij} c_j^2 + \sum b_i''' \bar{a}_{ij} c_j = \frac{1}{60}, \quad \frac{1}{2} \sum b_i''' c_i^2 \bar{a}_{ij} + \sum b_i''' c_i \bar{a}_{ij} = \frac{1}{12},\\
&\frac{1}{2} \sum b_i''' \bar{\bar{a}}_{ij} c_j^2 + \sum b_i''' c_i \bar{\bar{a}}_{ij} c_j = \frac{1}{24}.
\end{aligned} \tag{29}$$

6^{th}-order

$$\begin{aligned}
&\frac{1}{2} \sum b_i''' c_i \bar{\bar{a}}_{ij}^2 + \sum b_i''' c_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{48}, \quad \sum b_i''' c_i^5 = \frac{1}{6}, \quad \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 = \frac{1}{120},\\
&\sum b_i''' c_i \bar{\bar{a}}_{ij} c_j^2 = \frac{1}{72}, \quad \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j = \frac{1}{36}, \quad \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j + \frac{1}{2} \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 = \frac{23}{720},\\
&\frac{1}{2} \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j + \frac{1}{6} \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 = \frac{11}{720}, \quad \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 + \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j = \frac{13}{360},\\
&\sum b_i''' c_i \bar{a}_{ij} c_j = \frac{1}{144}, \quad \sum b_i''' c_i a_{ij} = \frac{1}{144}, \quad \sum b_i''' c_i \bar{a}_{ij} c_j + \sum b_i''' \bar{a}_{ij} c_j^2 = \frac{7}{720},\\
&\sum b_i''' \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} c_k = \frac{1}{720}, \quad \frac{1}{2} \sum b_i''' a_{ij} c_j^3 + \sum b_i''' c_i \bar{a}_{ij} c_j = \frac{1}{90}, \quad \sum b_i''' c_i^3 \bar{\bar{a}}_{ij} = \frac{1}{12},\\
&\sum b_i''' c_i^2 \bar{a}_{ij} = \frac{1}{36}, \quad \sum b_i''' \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} c_j + \sum b_i''' \bar{\bar{a}}_{ij}^2 c_j + \sum b_i''' \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} c_k = \frac{1}{72},\\
&\sum b_i''' c_i \bar{a}_{ij} c_j + \frac{1}{2} \sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j = \frac{1}{48}, \quad \sum b_i''' \bar{a}_{ij} c_j^2 + \frac{1}{2} \sum b_i''' c_i \bar{\bar{a}}_{ij} c_j^2 = \frac{7}{720},\\
&\sum b_i''' \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} c_i + \sum b_i''' \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} c_j = \frac{1}{90}, \quad \sum b_i''' \bar{\bar{a}}_{ij} \bar{a}_{ik} + \sum b_i''' \bar{\bar{a}}_{ij} \bar{a}_{jk} = \frac{1}{360},\\
&\sum b_i''' c_i^2 \bar{a}_{ij} + \frac{1}{2} \sum b_i''' c_i^3 \bar{\bar{a}}_{ij} = \frac{5}{72}, \quad \frac{1}{6} \sum b_i''' c_i^3 \bar{\bar{a}}_{ij} + \sum b_i''' c_i a_{ij} = \frac{1}{48},\\
&\sum b_i''' c_i a_{ij} = \frac{1}{144}, \quad \frac{1}{2} \sum b_i''' c_i^2 \bar{a}_{ij} + \sum b_i''' c_i a_{ij} = \frac{1}{48}, \quad \sum b_i''' c_i^2 \bar{a}_{ij} = \frac{1}{36},\\
&\frac{1}{6} \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 + \sum b_i''' a_{ij} c_j = \frac{1}{360}, \quad \frac{1}{2} \sum b_i''' \bar{a}_{ij} c_j^2 + \sum b_i''' a_{ij} c_j = \frac{1}{360},
\end{aligned}$$

$$\sum b_i''' c_i \bar{a}_{ij} c_j + \frac{1}{2}\sum b_i''' \bar{a}_{ij} c_j^2 = \frac{1}{120}, \quad \frac{1}{2}\sum b_i''' \bar{a}_{ij} c_j^2 + \frac{1}{2}\sum b_i''' c_i \bar{\bar{a}}_{ij} c_j^2 = \frac{1}{120},$$
$$\sum b_i''' \bar{a}_{ij} c_j^2 = \frac{1}{360}, \quad \sum b_i''' c_i \bar{a}_{ij} c_j + \frac{1}{2}\sum b_i''' c_i^2 \bar{a}_{ij} c_j + \frac{1}{2}\sum b_i''' \bar{a}_{ij} c_j^3 = \frac{1}{40},$$
$$\sum b_i''' c_i^2 \bar{a}_{ij} + \sum b_i''' c_i^3 \bar{a}_{ij} = \frac{1}{9}, \quad \sum b_i''' a_{ij} c_j = \frac{1}{720}, \quad \sum b_i''' \bar{a}_{ij} \bar{a}_{jk} c_k = \frac{1}{720},$$
$$\sum b_i''' \bar{a}_{ij} \bar{\bar{a}}_{jk} + \sum b_i''' \bar{\bar{a}}_{ik} \bar{a}_{ij} + \sum b_i''' \bar{\bar{a}}_{ij} \bar{a}_{ij} = \frac{1}{72}, \quad \sum b_i''' c_i \bar{a}_{ij} c_j = \frac{1}{144},$$
$$\sum b_i''' c_i^2 \bar{\bar{a}}_{ij} c_j + \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 = \frac{13}{360}, \quad \sum b_i''' \bar{a}_{ij} c_j^2 + \frac{1}{2}\sum b_i''' c_i \bar{\bar{a}}_{ij} c_j^2 = \frac{7}{720},$$
$$\sum b_i''' c_i^2 \bar{a}_{ij} + \frac{1}{2}\sum b_i''' c_i^3 \bar{a}_{ij} = \frac{5}{72}, \quad \sum b_i''' c_i^2 \bar{a}_{ij} c_j + \frac{1}{2}\sum b_i''' \bar{a}_{ij} c_j^3 = \frac{23}{720},$$
$$\frac{1}{2}\sum b_i''' c_i^2 \bar{a}_{ij} c_j + \frac{1}{2}\sum b_i''' c_i \bar{a}_{ij} c_j^2 + \sum b_i''' c_i \bar{a}_{ij} c_j + \sum b_i''' \bar{a}_{ij} c_j^2 = \frac{11}{360},$$
$$\sum b_i''' c_i \bar{a}_{ij} c_j + \sum b_i''' \bar{a}_{ij} c_j^2 = \frac{7}{720}, \quad \sum b_i''' \bar{\bar{a}}_{ij} c_j^3 = \frac{1}{120},$$
$$\sum b_i''' c_i \bar{a}_{ij} \bar{a}_{jk} + \sum b_i''' \bar{a}_{ik} \bar{a}_{ij} c_j + \sum b_i''' \bar{a}_{ik} \bar{a}_{ij} c_k + \sum b_i''' \bar{a}_{ij} \bar{a}_{jk} c_j + \sum b_i''' \bar{a}_{ij}^2 = \frac{1}{40}. \tag{30}$$

4. Derivation of the proposed methods

4.1. Derivation of embedded 5(4) RKDF method

In this subsection, a fifth-order three-stage RKDF method is derived. Algebraic conditions up to the fifth-order must be resolved, resulting in a system of 19 nonlinear equations. By solving this system simultaneously, a family of solutions in terms of a_{21}, a_{31} and b_2 is obtained as follows:

$$a_{21} = a_{21}, \ a_{31} = a_{31}, \ a_{32} = -\frac{131}{125}a_{2,1} - \frac{16\sqrt{6}}{125}a_{2,1} + \frac{12}{625} + \frac{3\sqrt{6}}{2500} - a_{3,1}, \ \bar{a}_{21} = \frac{27}{500} - \frac{19\sqrt{6}}{1000},$$
$$\bar{a}_{31} = \frac{33}{2500} + \frac{51\sqrt{6}}{5000}, \ \bar{a}_{32} = \frac{51}{1250} + \frac{11\sqrt{6}}{1250}, \ b_1' = \frac{1}{18}, b_2' = \frac{1}{18} + \frac{\sqrt{6}}{48}, \ b_3' = \frac{1}{18} - \frac{\sqrt{6}}{48},$$
$$c_2 = \frac{3}{5} - \frac{\sqrt{6}}{10}, \ c_3 = \frac{3}{5} + \frac{\sqrt{6}}{10}, b_1 = \frac{1}{40} + \frac{2}{5}b_2 + \frac{\sqrt{6}}{360} - \frac{2\sqrt{6}}{5}b_2, \ b_2 = b_2,$$
$$b_3 = \frac{1}{60} - \frac{\sqrt{6}}{360} - \frac{7}{5}b_2 + \frac{2\sqrt{6}}{5}b_2, b_1'' = \frac{1}{9}, \ b_2'' = \frac{7}{36} + \frac{\sqrt{6}}{18}, \ b_3'' = \frac{7}{36} - \frac{\sqrt{6}}{18},$$
$$b_1''' = \frac{1}{9}, \ b_2''' = \frac{4}{9} + \frac{\sqrt{6}}{36}, b_3''' = \frac{4}{9} - \frac{\sqrt{6}}{36}.$$

Using the minimise command in Maple, we obtained $a_{21} = 0.000201150115696978$, $a_{31} = 0.0114730199865407$, and $b_2 = 0.0245113423910039$, resulting in a minimum local truncation error of 0.00748935976364954044. To express these optimised values in fractional form, we choose $a_{21} = \frac{2}{10000}$, $a_{31} = \frac{1}{100}$, and $b_2 = \frac{2}{100}$. Consequently, the coefficients for the three-stage 5^{th}-order RKDF approach, represented by RKDF5, can be expressed in the following manner:

$$c_2 = \frac{3}{5} - \frac{\sqrt{6}}{10}, \ c_3 = \frac{3}{5} + \frac{\sqrt{6}}{10}, \ a_{21} = \frac{1}{5000}, \ a_{31}\frac{1}{100}, \ a_{32} = \frac{5619}{625000} + \frac{367\sqrt{6}}{312500}, \ \bar{a}_{21} = \frac{27}{500} - \frac{19\sqrt{6}}{1000},$$
$$\bar{a}_{31} = \frac{33}{2500} + \frac{51\sqrt{6}}{5000}, \ \bar{a}_{32} = \frac{51}{1250} + \frac{11\sqrt{6}}{1250}, \ b_1 = \frac{33}{1000} - \frac{47\sqrt{6}}{9000}, \ b_2 = \frac{1}{50}, \ b_3 = -\frac{17}{1500} + \frac{47\sqrt{6}}{9000},$$
$$b_1' = \frac{1}{18}, \ b_2' = \frac{1}{18} + \frac{\sqrt{6}}{48}, \ b_3' = \frac{1}{18} - \frac{\sqrt{6}}{48}, \ b_1'' = \frac{1}{9}, \ b_2'' = \frac{7}{36} + \frac{\sqrt{6}}{18}, \ b_3'' = \frac{7}{36} - \frac{\sqrt{6}}{18}, \ b_1''' = \frac{1}{9},$$

$$b_2''' = \frac{4}{9} + \frac{\sqrt{6}}{36},\ b_3''' = \frac{4}{9} - \frac{\sqrt{6}}{36}.$$

Utilising the aforementioned solution for the values of $A, \bar{A}$, and c, we derive and formulate a three-stage ($s = 3$) fourth-order ($q = 4$) embedded formula. Solving the equations Eq. (11), Eqs. (14) and (15), Eqs. (18) to (20) and Eqs. (25) to (28) simultaneously yields a solution for $b_1'', b_2'', b_3'', b_1''', b_2'''$, and b_3''' using the same as fifth-order formula. Expressing the solution in terms of three free parameters, b_2, b_3, and b_3' are obtained as follows:

$$b_1' = \frac{1}{12} + \frac{2}{5}b_3' + \frac{2\sqrt{6}}{5}b_3' - \frac{\sqrt{6}}{72},\ b_2' = \frac{1}{12} - \frac{7}{5}b_3' - \frac{2\sqrt{6}}{5}b_3' + \frac{\sqrt{6}}{72},\ b_3' = b_3',\ b_1 = \frac{1}{24} - b_2 - b_3,\ b_2 = b_2,\ b_3 = b_3,$$

$$b_1'' = \frac{1}{9},\ b_2'' = \frac{7}{36} + \frac{\sqrt{6}}{18},\ b_3'' = \frac{7}{36} - \frac{\sqrt{6}}{18},\ \ b_1''' = \frac{1}{9},\ b_2''' = \frac{4}{9} + \frac{\sqrt{6}}{36},\ b_3''' = \frac{4}{9} - \frac{\sqrt{6}}{36}.$$

Letting $b_3' = 0$, and employing the "minimize" command in Maple, we attain $b_2 = 0.496306580640579$, $b_3 = -0.198687532794505$ and the minimum error is 0.001872873928. For the optimised value in fractional, then we choose $b_2 = \frac{49}{100}$ and $b_3 = -\frac{19}{100}$.

Ultimately, the coefficients of the three-stage fifth-order RKDF method, denoted as RKDF5(4), can be expressed as follows in Table 3.

4.2. Derivation of embedded 5(4) ERKDGF method

In this subsection, a four-stage RKTF technique of fifth-order will be derived. The algebraic conditions up to fifth-order need to be solved. The resulting system consists of 35 nonlinear equations solving the system simultaneously and the family of solutions in terms of a_{21}, a_{32}, a_{42}, a_{43}, $\bar{a}_{21}$, $\bar{a}_{42}$, $\bar{a}_{43}$, b_4', c_2, b_3 and b_4 are given as follows:

$$a_{21} = a_{21},\ a_{31} = -\frac{1}{10\left(16\,c_2^2 - 8\,c_2 + 1\right)\left(5\,c_2 - 4\right)\left(4\,c_2 - 1\right)^2}\big(12500\,a_{42}c_2^5 + 12500\,a_{43}c_2^5$$
$$- 220\,c_2^5 + 12800\,a_{32}c_2^5 - 23040\,a_{32}c_2^4 - 21250\,a_{42}c_2^4 - 21250\,a_{43}c_2^4 + 366\,c_2^4 - 192\,c_2^3$$
$$+ 15040\,a_{32}c_2^3 + 11250\,a_{42}c_2^3 + 11250\,a_{43}c_2^3 - 1875\,a_{42}c_2^2 - 1875\,a_{43}c_2^2 + 32\,c_2^2$$

Table 3. The RKDF(5)4 method.

$\frac{3}{5} - \frac{\sqrt{6}}{10}$	$\frac{1}{5000}$			$\frac{27}{500} - \frac{19\sqrt{6}}{1000}$	
$\frac{3}{5} + \frac{\sqrt{6}}{10}$	$\frac{1}{100}$	$\frac{5619}{625000} + \frac{367\sqrt{6}}{312500}$		$\frac{33}{2500} + \frac{51\sqrt{6}}{5000}$	$\frac{51}{1250} + \frac{11\sqrt{6}}{1250}$
	$\frac{33}{1000} - \frac{47\sqrt{6}}{9000}$	$\frac{1}{50}$	$-\frac{17}{1500} + \frac{47\sqrt{6}}{9000}$		
	$\frac{1}{18}$	$\frac{1}{18} + \frac{\sqrt{6}}{48}$	$\frac{1}{18} - \frac{\sqrt{6}}{48}$		
	$\frac{1}{9}$	$\frac{7}{36} + \frac{\sqrt{6}}{18}$	$\frac{7}{36} - \frac{\sqrt{6}}{18}$		
	$\frac{1}{9}$	$\frac{4}{9} + \frac{\sqrt{6}}{36}$	$\frac{4}{9} - \frac{\sqrt{6}}{36}$		
	$-\frac{31}{120}$	$\frac{49}{100}$	$-\frac{19}{100}$		
	$\frac{1}{12} - \frac{\sqrt{6}}{72}$	$\frac{1}{12} + \frac{\sqrt{6}}{72}$	0		
	$\frac{1}{9}$	$\frac{7}{36} + \frac{\sqrt{6}}{18}$	$\frac{7}{36} - \frac{\sqrt{6}}{18}$		
	$\frac{1}{9}$	$\frac{4}{9} + \frac{\sqrt{6}}{36}$	$\frac{4}{9} - \frac{\sqrt{6}}{36}$		

$$-4640\,a_{32}c_2^2+690\,a_{32}c_2+110\,a_{21}c_2-40\,a_{32}-40\,a_{21}\big),\ a_{32}=a_{32},\ a_{41}=0,$$

$$a_{42}=a_{42},\ a_{43}=a_{43},\ \bar{a}_{31}=\frac{1}{10\left(16\,c_2^2-8\,c_2+1\right)\left(5\,c_2-4\right)\left(4\,c_2-1\right)^3}\big(50000\,c_2^6\bar{a}_{42}$$
$$-880\,c_2^5-1040\,c_2^2\bar{a}_{21}-18750\,c_2^3\bar{a}_{42}+11250\,c_2^3\bar{a}_{43}+800\,c_2^3\bar{a}_{21}+66250\,c_2^4\bar{a}_{42}$$
$$-21250\,c_2^4\bar{a}_{43}-97500\,c_2^5\bar{a}_{42}+12500\,c_2^5\bar{a}_{43}-40\,\bar{a}_{2,1}+320\,c_2^2-32\,c_2-1134\,c_2^3$$
$$+1684\,c_2^4+1875\,c_2^2\bar{a}_{42}-1875\,c_2^2\bar{a}_{43}+370\,c_2\bar{a}_{21}\big),$$

$$\bar{a}_{32}=\frac{1}{10\left(16\,c_2^2-8\,c_2+1\right)\left(5\,c_2-4\right)\left(4\,c_2-1\right)^3}\big((2\,c_2-1)c_2\big(-440\,c_2^3+622\,c_2^2-256\,c_2$$
$$+32+25000\,c_2^4\bar{a}_{42}-36250\,c_2^3\bar{a}_{42}+15000\,c_2^2\bar{a}_{42}-1875\,\bar{a}_{42}c_2+6250\,c_2^3\bar{a}_{43}$$
$$-7500\,{c_2}^2\bar{a}_{43}+1875\,c_2\bar{a}_{43}\big)\big),\ \bar{a}_{41}=-\frac{1}{125\,c_2\left(10\,c_2^2-12\,c_2+3\right)}\big(1250\,c_2^3\bar{a}_{43}-110\,c_2^3$$
$$+1250\,c_2^3\bar{a}_{42}-1500\,c_2^2\bar{a}_{43}+128\,c_2^2-1500\,c_2^2\bar{a}_{42}-32\,c_2+375\,c_2\bar{a}_{43}+375\,\bar{a}_{42}c_2$$
$$+20\,\bar{a}_{21}\big),\ \bar{a}_{42}=\bar{a}_{42},\ \bar{a}_{43}=\bar{a}_{43},\ \bar{\bar{a}}_{21}=\frac{1}{2}\,c_2^2,\ \bar{\bar{a}}_{31}=\frac{1}{2}\,\frac{c_2^2}{(4\,c_2-1)^2},\ \bar{\bar{a}}_{32}=0,$$

$$\bar{\bar{a}}_{41}=-\frac{4\left(50\,c_2^4-260\,{c_2}^3+321\,c_2^2-128\,c_2+16\right)}{625\,c_2^2\left(10\,c_2^2-12\,c_2+3\right)},\ \bar{\bar{a}}_{42}=\frac{\left(5\,c_2-4\right)\left(33\,c_2^2-34\,c_2+8\right)}{625\ \left(2\,c_2-1\right)c_2^2\left(10\,c_2^2-12\,c_2+3\right)},$$

$$\bar{\bar{a}}_{43}=\frac{\left(4\,c_2-1\right)^2\left(275\,c_2^3-430\,c_2^2+208\,c_2-32\right)}{625\ \left(2\,c_2-1\right)c_2^2\left(10\,{c_2}^2-12\,c_2+3\right)},\ b_1'=\frac{660\,c_2^2b_4+20\,c_2-768\,c_2b_4-5+192\,b_4}{300\,c_2^2},$$

$$b_2'=-\frac{1}{1200}\,\frac{-15\,c_2+1056\,c_2b_4-384\,b_4+10}{\left(2\,c_2-1\right)c_2^2},\ b_3'=-\frac{\left(-25\,c_2+480\,c_2b_4-384\,b_4+10\right)\left(4\,c_2-1\right)^2}{1200\ \left(2\,c_2-1\right)c_2^2},$$

$$b_4'=b_4',\ b_1=-\frac{-20\,c_2^2+9\,c_2-240\,b_3c_2-504\,b_4c_2+96\,b_4-1+480\,b_3c_2^2+480\,b_4c_2^2}{120\,c_2\left(4\,c_2-1\right)},$$

$$b_2=-\frac{1}{120}\,\frac{120\,b_3c_2+384\,b_4c_2-96\,b_4-4\,c_2+1}{c_2\left(4\,c_2-1\right)},\ b_3=b_3,\ b_4=b_4,\ b_1''=\frac{2\,c_2^2+4\,c_2-1}{48\,c_2^2},$$

$$b_2''=-\frac{c_2-1}{24\,c_2^2\left(5\,c_2-4\right)\left(2\,c_2-1\right)},\ ,\ b_3''=\frac{192\,c_2^4-208\,c_2^3+84\,c_2^2-15\,c_2+1}{24\ \left(2\,c_2-1\right)c_2^2\left(11\,c_2-4\right)},$$

$$b_4''=\frac{25}{48}\,\frac{10\,c_2^2-12\,c_2+3}{\left(11\,c_2-4\right)\left(5\,c_2-4\right)},\ b_1'''=\frac{2\,c_2^2+4\,c_2-1}{48\,c_2^2},\ b_2'''=\frac{1}{24\,c_2^2\left(5\,c_2-4\right)\left(2\,c_2-1\right)},$$

$$b_3'''=\frac{\left(4\,c_2-1\right)^2\left(16\,c_2^2-8\,c_2+1\right)}{24\ \left(2\,c_2-1\right)c_2^2\left(11\,c_2-4\right)},\ b_4'''=\frac{125}{48}\,\frac{10\,c_2^2-12\,c_2+3}{\left(11\,c_2-4\right)\left(5\,c_2-4\right)},\ c_4=\frac{4}{5},\ c_2=c_2,$$

$$c_3=\frac{c_2}{4\,c_2-1}.$$

By utilising the minimise command in Maple, we achieve $a_{21}=0.518475525248816$, $a_{32}=-0.412217645489556$, $a_{41}=0.000204911181723119$, $a_{42}=-.239252831207220$, $a_{43}=0.527446393740536$, $\bar{a}_{21}=0.00487550792349588$, $\bar{a}_{42}=0.000550566902448232$, $\bar{a}_{43}=0.0298499797533611$, $b_4'=0.0322254023634992$, $c_2=0.748893051614183$, $b_3=0.438487023975308$ and $b_4=1.17947883541692$ and the minimum local truncation error is 0.009351286738. For the optimised value in fractional form then we choose $a_{21}=\frac{5}{10}$, $a_{32}=-\frac{4}{100}$, $a_{41}=\frac{2}{10000}$,$a_{42}=-\frac{24}{100}$, $a_{43}=\frac{52}{100}$, $\bar{a}_{21}=\frac{5}{1000}$, $\bar{a}_{42}=\frac{1}{1000}$, $\bar{a}_{43}=\frac{3}{100}$, $b_4'=\frac{3}{100}$, $c_2=\frac{74}{100}$, $b_3=\frac{4}{10}$ and $b_4=\frac{11}{10}$. Ultimately we can express the coefficients of the fifth-order, four-stage RKTF method, which we will refer to as RKTF5.

$$c_2=\frac{37}{50},\ c_3=\frac{37}{98},\ c_4=\frac{4}{5},\ a_{21}=\frac{1}{2},\ a_{31}=\frac{29560597}{288240050},\ a_{32}=-\frac{1}{25},$$

$$a_{41}=\frac{1}{5000},\ a_{42}=-\frac{6}{25},\ a_{43}=\frac{13}{25},\ \bar{a}_{21}=\frac{1}{200},\ \bar{a}_{31}=\frac{23408341}{4519603984},\ \bar{a}_{32}=-\frac{20407091}{4519603984},$$

$$\bar{a}_{41}=\frac{77969}{3737000},\ \bar{a}_{42}=\frac{1}{1000},\ \bar{a}_{43}=\frac{3}{100},\ \bar{\bar{a}}_{21}=\frac{1369}{5000},\ \bar{\bar{a}}_{31}=\frac{1369}{19208},\ \bar{\bar{a}}_{32}=0,$$

$$\bar{\bar{a}}_{41}=\frac{3347324}{17283625},\ \bar{\bar{a}}_{42}=\frac{2277}{553076},\ \bar{\bar{a}}_{43}=\frac{8449119}{69134500},\ b_1=-\frac{1107}{14504},$$

$$b_2=-\frac{30067}{21756},\ b_3=\frac{2}{5},\ b_4=\frac{11}{10},\ b'_1=\frac{116911}{2053500},\ b'_2=-\frac{13529}{394272},\ b'_3=\frac{5620741}{49284000},$$

$$b'_4=\frac{3}{100},\ b''_1=\frac{1273}{10952},\ b''_2=-\frac{40625}{295704},\ b''_3=\frac{7176589}{20403576},\ b''_4=\frac{2525}{14904},\ b'''_1=\frac{1273}{10952},$$

$$b'''_2=-\frac{78125}{147852},\ b'''_3=\frac{5764801}{10201788}\ b'''_4=\frac{12625}{14904}.$$

Building upon the previously obtained values for A, $\bar{A}$, $\bar{\bar{A}}$ and c, we derive and formulate a fourth-order ($q = 4$) embedded formula with four stages ($s = 4$). Solving the equations Eq. (11), Eqs. (14) and (15), Eqs. (18) to (20) and Eqs. (25) to (28) simultaneously yields a solution for $\hat{b}'''_1, \hat{b}'''_2, \hat{b}'''_3$ and $\hat{b}'''_4$ using the same as fifth-order formula. The solution in six free parameters in terms of $\hat{b}_2, \hat{b}'_2, \hat{b}'_4, \hat{b}_3, \hat{b}''_4$, and $\hat{b}_4$ is obtained as follows:

$$\hat{b}'_1=\frac{25}{444}+\frac{24}{25}\hat{b}'_2+\frac{207}{185}\hat{b}'_4,\ \hat{b}'_2=\hat{b}'_2,\ \hat{b}'_3=-\frac{49}{25}\hat{b}'_2+\frac{49}{444}-\frac{392}{185}\hat{b}'_4,\ \hat{b}'_4=\hat{b}'_4,$$

$$\hat{b}_1=\frac{1}{24}-\hat{b}_2-\hat{b}_3-\hat{b}_4,\ \hat{b}_2=\hat{b}_2,\ \hat{b}_3=\hat{b}_3,\ \hat{b}_4=\hat{b}_4,\ \hat{b}''_1=\frac{1081}{8214}-\frac{621}{6845}\hat{b}''_4,$$

$$\hat{b}''_2=\frac{625}{8214}-\frac{1725}{1369}\hat{b}''_4,\ \hat{b}''_3=\frac{2401}{8214}+\frac{2401}{6845}\hat{b}''_4,\ \hat{b}''_4=\hat{b}''_4,\ \hat{b}'''_1=\frac{1273}{10952},$$

$$\hat{b}'''_2=-\frac{78125}{147852},\ \hat{b}'''_3=\frac{5764801}{10201788},\ \hat{b}'''_4=\frac{12625}{14904}.$$

By the minimise command in Maple we obtain $\hat{b}_2 = -0.667785556227001$, $\hat{b}_3 = 0.490362563807150$, $\hat{b}_4 = -.159234278111095$, $\hat{b}'_2 = 0.137906168938096$, $\hat{b}'_4 = -.106677825541037$ and the minimum error is 0.005850684564. For the optimised value in fractional form we choose $b_2 = -\frac{33}{500}$, $b_3 = \frac{49}{100}$, $\hat{b}_4 = -\frac{4}{25}$, $\hat{b}'_2 = \frac{7}{50}$ and $\hat{b}'_4 = -\frac{1}{10}$.

Finally, all the coefficients of four-stage fifth-order RKTF method denoted by RKTF5(4) can be written as follows in Table 4.

5. Implementations

In this section, we evaluate two categories of problems. We compare the numerical findings with current approaches by converting the identical problem set into a system of 1^{st}-order equations and solving them with established RK methods of the same order.

5.1. First Proposed Method

This section tests some problems involving $\mu^{(iv)} = f(x, \mu, \mu')$. The results obtained numerically are compared with those obtained when the same problem set is converted into a system of first-order equations and solved with a well-known RK method of equal order.

Table 4. The RKTF(5)4 method.

$\frac{37}{50}$	$\frac{1}{2}$				$\frac{1}{200}$			$\frac{1369}{5000}$		
$\frac{37}{98}$	$\frac{29560597}{288240050}$	$-\frac{1}{25}$			$\frac{23408341}{4519603984}$	$-\frac{20407091}{4519603984}$		$\frac{1369}{19208}$	0	
$\frac{4}{5}$	0	$-\frac{6}{25}$	$\frac{13}{25}$		$\frac{77969}{3737000}$	$\frac{1}{1000}$	$\frac{3}{100}$	$\frac{3347324}{17283625}$	$\frac{2277}{553076}$	$\frac{8449119}{69134500}$
	$-\frac{1107}{14504}$	$-\frac{30067}{21756}$	$\frac{2}{5}$	$\frac{11}{10}$						
	$\frac{116911}{2053500}$	$-\frac{13529}{394272}$	$\frac{5620741}{49284000}$	$\frac{3}{100}$						
	$\frac{1273}{10952}$	$-\frac{40625}{295704}$	$\frac{7176589}{20403576}$	$\frac{2525}{14904}$						
	$\frac{1273}{10952}$	$-\frac{78125}{147852}$	$\frac{5764801}{10201788}$	$\frac{12625}{14904}$						
	$-\frac{667}{3000}$	$-\frac{33}{5000}$	$\frac{49}{100}$	$-\frac{4}{25}$						
	$\frac{21871}{277500}$	$\frac{7}{50}$	$\frac{13279}{277500}$	$-\frac{1}{10}$						
	$\frac{12581}{102675}$	$-\frac{205}{4107}$	$\frac{33614}{102675}$	$\frac{1}{10}$						
	$\frac{1273}{10952}$	$-\frac{78125}{147852}$	$\frac{5764801}{10201788}$	$\frac{12625}{14904}$						

- RKDF5(4): This work derives a pair of RK type 5(4).
- RKF5(4): RK 5(4) pair as given in Fehlberg [27].
- DOPRI(5)4: RK 5(4) pair introduced in Butcher [5].

Problem 1: (Inhomogeneous Linear Problem)

$$\mu^{(iv)}(x) = \mu'(x) - \cos(x),$$

$$\mu(0) = -\frac{1}{2}, \quad \mu'(0) = \frac{1}{2}, \quad \mu''(0) = \frac{1}{2}, \quad \mu'''(0) = -\frac{1}{2},$$

The precise solution is expressed as $\mu(x) = \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)$.

Problem 2: (linear System Inhomogeneous)

$$\mu_1^{(iv)}(x) = \mu_1'(x) + 1, \qquad \mu_1(0) = 1, \quad \mu_1'(0) = 0, \quad \mu_1''(0) = -1, \quad \mu_1'''(0) = 0,$$

$$\mu_2^{(iv)}(x) = -y_2'(x) - \cos(x), \qquad \mu_2(0) = -\frac{1}{2}, \quad \mu_2'(0) = -\frac{1}{2}, \quad \mu_2''(0) = \frac{1}{2}, \quad \mu_2'''(0) = \frac{1}{2},$$

$$\mu_3^{(iv)}(x) = -y_3'(x) - \sin(x), \qquad \mu_3(0) = -\frac{1}{2}, \quad \mu_3'(0) = \frac{1}{2}, \quad \mu_3''(0) = \frac{1}{2}, \quad \mu_3'''(0) = -\frac{1}{2},$$

$$\mu_4^{(iv)}(x) = -y_4'(x)\sin(x), \qquad \mu_4(0) = 1, \quad \mu_4'(0) = -1, \quad \mu_4''(0) = -1, \quad \mu_4'''(0) = 1,$$

The precise solution is expressed as

$$\mu_1(x) = 1 - \frac{2\sqrt{3}}{3} e^{\frac{1}{2}x} \sin(\frac{1}{2}\sqrt{3}x) + x,$$

$$\mu_2(x) = -\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x),$$

$$\mu_3(x) = -\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x),$$

$$\mu_4(x) = \cos(x) - \sin(x),$$

Problem 3: (linear System Homogeneous)

$$\mu_1^{(iv)}(x) = -\mu_1'(x), \qquad \mu_1(0)=1, \quad \mu_1'(0)=2, \quad \mu_1''(0)=3, \quad \mu_1'''(0)=0,$$
$$\mu_2^{(iv)}(x) = -\mu_2'(x), \qquad \mu_2(0)=0, \quad \mu_2'(0)=-1, \quad \mu_2''(0)=1, \quad \mu_2'''(0)=2,$$
$$\mu^{(iv)}(x) = -\mu_3'(x), \qquad \mu_3(0)=2, \quad \mu_3'(0)=4, \quad \mu_3''(0)=4, \quad \mu_3'''(0)=5,$$
$$\mu_4^{(iv)}(x) = -\mu_4'(x), \qquad \mu_4(0)=1, \quad \mu_4'(0)=-1, \quad \mu_4''(0)=-1, \quad \mu_4'''(0)=1,$$

The exact solution is given by

$$\mu_1(x) = 1 + \frac{1}{3}e^{-x} + \frac{5\sqrt{3}}{3}e^{\frac{1}{2}x}\sin(\frac{\sqrt{3}}{3}x) - \frac{1}{3}e^{\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x),$$

$$\mu_2(x) = 2 - 2e^{\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x),$$

$$\mu_3(x) = 7 - \frac{4}{3}e^{-x} + \frac{7\sqrt{3}}{3}e^{\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x) - \frac{11}{3}e^{\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x),$$

$$\mu_4(x) = 2 - \frac{1}{3}e^{-x} - \frac{2\sqrt{3}}{2}e^{\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x) - \frac{2}{3}e^{\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x).$$

Table 5. Comparison of RKDF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 1 with $X_{end} = 2$.

h	Methods	F.N	MAXE
0.025	RKDF5(4)	243	9.816314(−13)
	RKF5(4)	1944	6.970813(−12)
	DOPRI(5)4	2268	2.853828(−12)
0.05	RKDF5(4)	120	3.160371(−11)
	RKF5(4)	960	2.266052(−10)
	DOPRI(5)4	1120	9.375184(−11)
0.075	RKDF5(4)	81	2.413710(−10)
	RKF5(4)	648	1.748493(−9)
	DOPRI(5)4	756	7.305133(−10)
0.1	RKDF5(4)	60	1.023045(−9)
	RKF5(4)	480	7.480006(−9)
	DOPRI(5)4	560	3.158440(−9)

Table 6. Comparison of RKDF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 2 with $X_{end} = 2$.

h	Methods	F.N	MAXE
0.025	RKDF5(4)	120	2.495226(−12)
	RKF5(4)	1944	2.384974(−2)
	DOPRI(5)4	2268	3.203662(−2)
0.05	RKDF5(4)	60	8.005052(−11)
	RKF5(4)	960	4.371991(−2)
	DOPRI(5)4	1120	5.877136(−2)
0.075	RKDF5(4)	42	6.231417(−10)
	RKF5(4)	648	6.377007(−2)
	DOPRI(5)4	756	8.580196(−2)
0.1	RKDF5(4)	33	2.684844(−9)
	RKF5(4)	480	7.759483(−2)
	DOPRI(5)4	560	1.045625(−1)

Table 7. Comparison of RKDF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 4 with $X_{end} = 5$.

h	Methods	F.N	MAXE
0.025	RKDF5(4)	600	2.368878(−10)
	RKF5(4)	4800	2.907342(−9)
	DOPRI(5)4	5600	8.770940(−10)
0.05	RKDF5(4)	303	7.683447(−9)
	RKF5(4)	2424	9.559413(−8)
	DOPRI(5)4	2828	2.848230(−8)
0.075	RKDF5(4)	201	5.754256(−8)
	RKF5(4)	1608	7.041820(−7)
	DOPRI(5)4	1876	2.064936(−7)
0.1	RKDF5(4)	153	2.480497(−7)
	RKF5(4)	1224	3.107898(−6)
	DOPRI(5)4	1428	9.018576(−7)

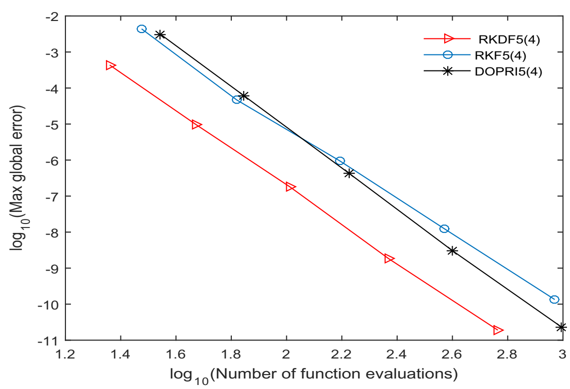

Fig. 1. Comparison for RKDF5(4), RKF5(4) and DOPRI(5)4 Problem 1 with $X_{end} = 5$.

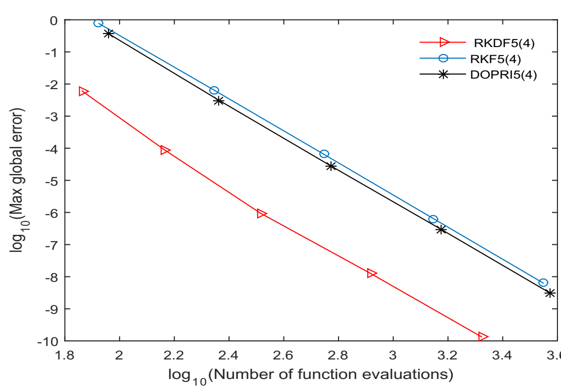

Fig. 2. Comparison for RKDF5(4), RKF5(4) and DOPRI(5)4 Problem 2 with $X_{end} = 10$.

5.2. Second proposed method

This subsection examines a few of the problems with $\mu^{(iv)} = f(x, \mu, \mu', \mu'')$. The numerical outcomes are compared to those obtained by reducing the same set of problems to a system of 1^{st}-order equations and using the current RK of the same order to solve it.

- ERKDGF5(4): RK type 5(4) pair developed in this manuscript.
- RKF5(4): RK 5(4) pair as given in Fehlberg [27].
- DOPRI(5)4: RK 5(4) pair presented in Butcher's work [5].

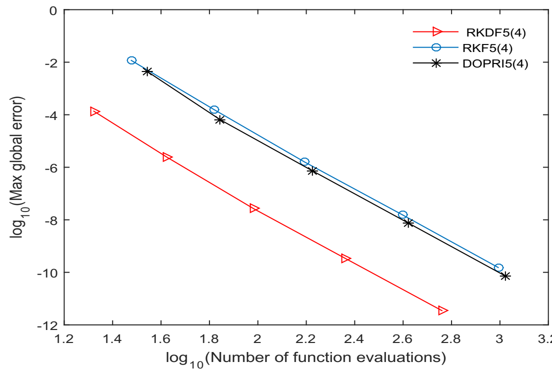

Fig. 3. Comparison for RKDF5(4), RKF5(4) and DOPRI(5)4 Problem 4 with $X_{end} = 3$.

Problem 1: (linear System Inhomogeneous)

$$\mu_1^{(iv)}(x) = -\mu_2''(x), \qquad \mu_1(0) = 1, \quad \mu_1'(0) = 1, \quad \mu_1''(0) = 1, \quad \mu_1'''(0) = 1,$$

$$\mu_2^{(iv)}(x) = -\mu_1''(x), \qquad \mu_2(0) = -1, \quad \mu_2'(0) = -1, \quad \mu_2''(0) = -1, \quad \mu_2'''(0) = -1,$$

$$\mu_3^{(iv)}(x) = -\mu_3''(x) - \mu_3(x) - \cos(x), \qquad \mu_3(0) = -1, \quad \mu_3'(0) = 0, \quad \mu_3''(0) = 1, \quad \mu_3'''(0) = 0,$$

$$\mu_4^{(iv)}(x) = -\mu_4''(x) - 2\cos(x), \qquad \mu_4(0) = -2, \quad \mu_4'(0) = 0, \quad \mu_4''(0) = 2, \quad \mu_4'''(0) = 0,$$

The exact solution is given by

$$\mu_1(x) = e^x$$

$$\mu_2(x) = -e^x$$

$$\mu_3(x) = -\cos(x),$$

$$\mu_4(x) = -2\cos(x),$$

Problem 2: (Inhomogeneous Linear Problem)

$$\mu^{(iv)}(x) = -\mu''(x),$$

$$\mu(0) = 0, \quad \mu'(0) = 1, \quad \mu''(0) = 2, \quad \mu'''(0) = 3,$$

The exact solution is given by $\mu(x) = 2 + 4x - 3\sin(x) - 2\cos(x)$.

Problem 3: (Homogeneous linear Problem)

$$\mu^{(iv)}(x) = -\mu''(x) - 2\cos(x),$$

$$\mu(0) = -1, \quad \mu'(0) = 0, \quad \mu''(0) = 1, \quad \mu''(0) = 0,$$

The exact solution is given by $\mu(x) = -2 + \cos(x) + \sin(x)x$.

6. Discussion and conclusions

This paper presents the introduction of RKDF5(4) and ERKDGF5(4) methods, which are embedded Runge-Kutta schemes specifically developed for directly solving fourth-order ordinary differential equations (ODEs) with variable step sizes. These methods were compared with single-step Runge-Kutta procedures, as other direct methods employ multi-step approaches. We have created variable step-size codes using RKDF5(4) and ERKDGF5(4) methods to solve fourth-order ordinary differential equations (ODEs) in the forms $\mu^{(iv)} = f(x, \mu, \mu')$ and $\mu^{(iv)} = f(x, \mu, \mu', \mu'')$. In order to achieve equitable comparisons, approaches of equivalent magnitude were examined in terms of numerical outcomes. The results of the comparison, displayed in

Table 8. Comparison of ERKDGF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 1 with $X_{end} = 10$.

h	Methods	F.N	MAXE
0.00625	ERKDGF5(4)	6404	6.002665(−9)
	RKF5(4)	38424	4.878530(−9)
	DOPRI5(4)	44828	7.250492(−9)
0.0125	ERKDGF5(4)	3204	1.786611(−8)
	RKF5(4)	19224	6.053233(−8)
	DOPRI5(4)	22428	1.931767(−8)
0.025	RKDF5(4)	1600	4.254471(−7)
	RKF5(4)	9600	1.917218(−6)
	DOPRI5(4)	11200	5.706643(−7)
0.1	ERKDGF5(4)	404	5.966861(−4)
	RKF5(4)	2424	2.064967(−3)
	DOPRI5(4)	2828	5.732670(−4)

Table 9. Comparison of ERKDGF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 2 with $X_{end} = 15$.

h	Methods	F.N	MAXE
0.00625	ERKDGF5(4)	9604	4.156675(−12)
	RKF5(4)	57624	1.926903(−12)
	DOPRI5(4)	67228	1.822542(−12)
0.0125	ERKDGF5(4)	4804	1.675460(−11)
	RKF5(4)	28824	1.501910(−11)
	DOPRI5(4)	33628	4.469314(−12)
0.025	RKDF5(4)	2400	5.439915(−10)
	RKF5(4)	14400	4.749152(−10)
	DOPRI5(4)	16800	1.448619(−10)
0.1	ERKDGF5(4)	604	4.687449(−7)
	RKF5(4)	3624	4.920205(−7)
	DOPRI5(4)	4228	1.489875(−7)

Table 10. Comparison of ERKDGF5(4), RKF5(4) and DOPRI(5)4 methods when solving Problem 3 with $X_{end} = 45$.

h	Methods	F.N	MAXE
0.00625	ERKDGF5(4)	28800	1.901710(−10)
	RKF5(4)	172800	1.709926(−10)
	DOPRI5(4)	201600	1.705764(−10)
0.0125	ERKDGF5(4)	14400	2.187335(−10)
	RKF5(4)	86400	2.998135(−10)
	DOPRI5(4)	100800	1.004494(−10)
0.025	RKDF5(4)	7204	6.196252(−9)
	RKF5(4)	43224	8.779388(−9)
	DOPRI5(4)	50428	2.690385(−9)
0.1	ERKDGF5(4)	1800	4.576414(−6)
	RKF5(4)	10800	9.009718(−6)
	DOPRI5(4)	12600	2.762210(−6)

Figs. 1–4 and Tables 5–10, clearly show that our new methods are superior to previous ones in terms of both maximum global error and the number of function evaluations. This holds true across different tolerance levels and evaluation situations. Figs. 1–4 and Tables 5–10 demonstrate that RKDF5(4) and ERKDGF5(4) have a reduced computational cost as they necessitate fewer function evaluations each step. The reduced computational burden results from the approaches' capacity to reach numerical convergence with fewer steps, enabling

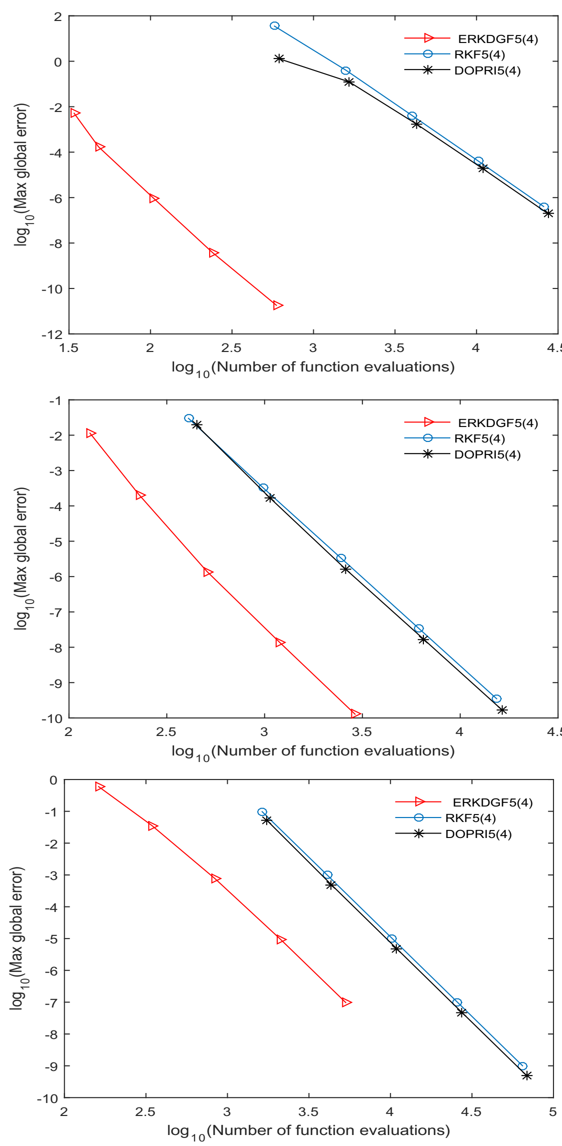

Fig. 4. Comparison for ERKDGF5(4), RKF5(4) and DOPRI(5)4 Problem 1,2 and 3 with $X_{end} = 5$, $X_{end} = 10$, and $X_{end} = 15$ respectively.

them to approximate exact solutions within set tolerances. In addition, due to the requirement of transforming fourth-order ODEs into a first-order system of ODEs, resulting in a fourfold increase in dimensionality, our new approaches are notably more efficient in solving fourth-order ODEs compared to the strategies currently described in the literature. Future work could explore the application of these methods to partial differential equations.

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Conflict of interest

The authors affirm that there is no conflict of interest associated with this paper's publication.

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