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T-FILTER IN BCK-ALGEBRA
راشح T في جبر (BCK)

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Abstract

In this paper , we introduce a new types of filters they are called T-filter , complete T-filter (c-T-filter) of BCK-algebra , and also we stated and prove some theorems which determine the relationship between these notions ,furthermore we investigate many relations and theorems between T-filter , complete T-filter and each of BCK-filter and complete BCK- filter .

المستخلص

في هذه البحث، قدمنا أنواع جديدة من المرشحات الذي يطلق عليه راشح T (T-filter) ، مكمل راشح T (c-T-filter) في جبر (BCK) ، و أيضا ذكرنا و إثبتنا بعض النظريات التي تحدد العلاقة بين هذه المفاهيم ، ثم درسنا بعض العلاقات والمبرهنات بين المفاهيم راشح T (T-filter)،مكمل راشح T (c-T-filter) , وبين راشح BCK و مكمل راشح BCK

1. Introduction

In 1966, Y.Imai, and K.Iseki introduced the a new notation called a BCK-algebra [7] ,thereafter in 1980 , E.Y.Deeba[1] introduced the notation of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. In 1996 J.Meng [3] introduced the notion of BCK-filter in BCK-algebra , The paper is organized as follows, in section 1 we introduced some definitions and results on BCK-algebra which we use in this paper . In section 2 we introduced definitions of T-filter, completely T-filter, and relationship between them .

2. preliminaries

In this section, we give some basic concepts about BCK-algebra , BCK-filter and basic concepts that we need in our work .

Definition (2.1): [6]

A BCK-algebra is a set X with a binary operation " $*$ " and constant element "0" in X which satisfies the following axioms :

1-$((x*y)*(x*z))*(z*y)=0$,

2- $\big(x*(x*y)\big)*y=0$,

3- $x*x=0$,

4- $0*x=0$,

5- $x*y=0$ *and* $y*x=0$ *imply* $x=y$, for all $x,y,z\in X$.

Remark (2.2): [7]

A BCK-algebra can be (partially) ordered by $x\leq y$ if and only if $x*y=0$

Theorem (2.3): [7]

In any BCK-algebra , the following hold :

(1) $x * 0 = x,$

(2) $x * y \leq x$,

(3) $(x * y) * z = (x * z) * y,$

(4) $(x * z) * (y * z) \leq x * y$,

(5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

For all $x , y , z \in X$.

Definition (2.4): [5]

If there is a special element e of a BCK-algebra X satisfying $x \leq e$ for all $x \in X$ then e is called a unit of X . A BCK-algebra with unit is called bounded BCK-algebra . In a bounded BCK-algebra X , we denoted $e * x$ by x^* for every $x \in X$

Definition (2.5):-[5]

Let $(X,* ,0)$ be BCK-algebra and $\bar{X} = X \cup \{e\}$ such that $\leq e$. then $(\bar{X},* ,0)$ is bounded BCK-algebra is called the Iseki's extension of X .

Definition (2.6):-[3]

For any $x, y \in X$ denoted by $x \wedge y = y * (y * x)$.

Definition (2.7):-[3]

A BCK-algebra X is said to be **commutativ**e if satisfies $x * (x * y) = y * (y * x)$, i.e., $x \wedge y = y \wedge x$.

Definition (2.8):-[3]

For a bounded BCK-algebra X, we put $x \vee y = (x^* \wedge y^*)^*$ for all $x, y \in X$.

Definition (2.9):-[5]

For a bounded BCK-algebra , if an element x satisfies $(x^*)^* = x$, then x is called an involution . If every element of X is an involution, we call X is an involutory BCK-algebra .

Proposition (2.10):- [4]

A bounded commutative BCK-algebra is involutory,

Proposition (2.11):- [5]

In a bounded BCK-algebra , we have

$1 - e^* = 0$ *and* $0^* = e$,

$2 - y \leq x$ *implies* $x^* \leq y^*$,

$3 - x^* * y^* \leq y * x$,

$4 -$ $x^* * y^* = y * x$, when X is invoultory (commutative)

$5 - (x^*)^*)^* = x^*$,

Definition (2.12):-[4]

A nonempty subset $F$of a bounded BCK-algebra X is called BCK-filter if

1- $e \in F$

2- $(x^* * y^*)^* \in F$, $y \in F$ implies $x \in F$.

Lemma (2.13):-[6]

Suppose X is bounded BCK-algebra and F is BCK-filter, then $x \in F$ if and only if $(x^*)^* \in F$.

Definition (2.14):[2]

A subset F of a bounded BCK-algebra X is said to be complete BCK-filter (c-BCK-filter) , if,

1- $e \in F$

2- $(x^* * y^*)^* \in F$,$\forall y \in F$ implies $x \in F$.

Definition(2.15):[5]

Let f be a mapping from a BCK-algebra Y into a BCK-algebra . Then f is called

1- Homomorphism if $f(x * y) = f(x) * f(y)$ for all $x , y \in Y$.

2- Epimorphism if f is homomorphism and onto .

3- Monomorphism if f is homomorphism and one to one .

4- Isomorphism if f is epimorphism and monomorphism .

Lemma(2.16):[5]

If f is a homomorphism from BCK-algebra X into BCK-algebra Y, then f is isotone i.e, $x \leq y \Longrightarrow f(x) \leq f(y), for\ all\ x, y \in X$.

Lemma(2.17):[6]

Suppose f is an epimorphism from BCK-algebra X into BCK-algebra Y, then $f(e_x) = e_y$ where e_x, e_y are the units of X and Y respectively .

Lemma(2.18):

let f be an epimorphism from BCK-algebra X into BCK-algebra Y, then

1) $f(x^*) = (f(x))^*$, for all $x \in X$, and

2) $f^{-1}(y^*) = (f^{-1}(y))^*$, for all $y \in Y$. If is isomorphism

Proof:

1) $f(x^*) = f(e_x * x) = f(e_x) * f(x) = e_y * f(x) = (f(x))^*$ (by Lemma(2.17))

2) $f^{-1}(y^*) = f^{-1}(e_y * y) = f^{-1}(e_y) * f^{-1}(y) = e_x * f^{-1}(y) = (f^{-1}(y))^*$ (by Lemma (2.17))

3. The main results

In this section ,we provide a definitions of T-filter , complete T-filter. And, we study its relationship with BCK-filter in BCK-algebra .

Definition (3.1):

A subset F of a bounded BCK-algebra X is said to be T-filter , if

1- $e \in F$

2- $(x^* * y^*)^* \in F$, $y \in F$ implies $(x^*)^* \in F$.

Example (3.2):

Let $X = \{0,1,2,3,4\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	3	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 4 (see[5]) and $F_1 = \{4\}, F_2 = \{0,3,4\}, F_3 = \{0,1,3,4\}, F_4 = X$ are T-filter .

Example (3.3):

Let $X = \{0,1,2,3\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 3 (see[5]) and $F = \{2, 3\}$ is not T-filter since $(1^* * 2^*)^* = 2 \in F$ but $(1^*)^* = 1 \notin F$.

Proposition(3.4):

Let X be a bounded BCK-algebra and F be a subset of . If $e * x \in F, for\ all\ x \in X$ then F is T-filter .

Proof : clear

Corollary(3.5):

Let X be any BCK-algebra and $\bar{X}$ be the Iseki's extension of X. Then every subset contain 0 and e is T-filter.

Proposition(3.6):

In bounded BCK-algebra X every BCK-filter is a T-filter .

Proof :

Let F be a BCK-filter and $(x^* * y^*)^* \in F$, $y \in F$. Since F is BCK-filter, then $\in F$.

Thus $(x^*)^* \in F$ (by Lemma(2.10)) , then F is T-filter .

Remark (3.7):

The converse of proposition (3.6) needs not be true in general as in the example.

Example (3.8):

Let $X = \{0,1,2,3,4\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	2	1	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 4 , and $F = \{4, 2\}$ is T-filter but it's not BCK-filter , since $(3^* * 2^*)^* = 4 \in F$ but $3 \notin F$.

Corollary(3.9):
In general $\{e\}$, X are trivial T-filter .

Proposition(3.10):
Every T-filter in bounded involutory BCK-algebra X is BCK-filter.
Proof:
Let F be T-filter and $(x^* * y^*)^* \in F$, $y \in F$. Then $(x^*)^* \in F$, since X is involutory then $x \in F$ Thus F is BCK-filter .

Corollary(3.11):
Every T-filter in bounded commutative BCK-algebra X is BCK-filter.

Proposition(3.12):
If F is T-filter then for all $x \in F$ implies $(x^*)^* \in F$.
Proof:
Let F be T-filter and $x \in F$. Then$(x^* * x^*)^* = e \in F$,Thus $(x^*)^* \in F$.

Remark(3.13):
The converse of proposition (3.12) is not true in general as in the example (3.2), $(1^*)^* = 3 \in F_2$ but $1 \notin F_2$.

Proposition(3.14):
Let F be T-filter . Then , if $x \leq y$ and $x \in F$ $implies$ $(y^*)^* \in F$.
Proof:
Let F be T-filter and $\leq y$. Then $y^* \leq x^*$, so $(y^* * x^*)^* = e \in F$, thus $(y^*)^* \in F$.

Proposition(3.15):
Let X be bounded BCK-algebra and F be T-filter . Then
1) If $0 \in F$ $then(x^*)^* \in F$ for all $x \in X$.
2) If $x \in F$ and $x^* = e$ $then$ $0 \in F$.
Proof : 1) it is clear by $(x^* * 0^*)^* = e \in F$.
2) Since by $(0^* * x^*)^* = e \in F$.

Proposition (3.16):
The intersection of a family of T-filter is a T-filter .
Proof : clear

Remark (3.17):
Not that the union of two T-filter is not necessary to be T-filter as shown in the following example .
Example (3.18):
Let $X = \{0,1,2,3,4\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	4	3	0	0
4	4	4	3	2	0

It is clear that $(X ,* ,0)$ is a bounded BCK-algebra with unit 4 (see[5]) and . Now let $F_1 = \{2 , 4\}$ and $F_2 = \{3 , 4\}$, we can show easily that F_1 and F_2 are T-filters but $F_1 \cup F_2 = \{2 , 3 , 4\}$ is not T-filter , since $(1^* * 2^*)^* = 3 \in F_1 \cup F_2$, $2 \in F_1 \cup F_2$, but $(1^*)^* = 0 \notin F_1 \cup F_2$.

Theorem(3.19):

If f is an isomorphism from bounded BCK-algebra X into bounded BCK-algebra Y, then the image of a T-filter is a T-filter.

Proof:

Let f be an isomorphism from BCK-algebra X into BCK-algebra Y, and let A be a T-filter in X . since $e_x \in A$ then $f(e_x) = e_y \in f(A)$, (by Lemma(2.17))

Now let $(x^* * y^*)^* \in f(A)$, $y \in f(A)$, then $f^{-1}((x^* * y^*)^*) \in A, f^{-1}(y) \in A$, (since f is onto) , but $f^{-1}((x^* * y^*)^*) = (f^{-1}(x^* * y^*))^* = (f^{-1}(x^*) * f^{-1}(y^*))^* = ((f^{-1}(x))^* * (f^{-1}(y))^*)^*$ by (Lemma(2.18(2)))

Therefore , $((f^{-1}(x))^* * (f^{-1}(y))^*)^* \in A$, $f^{-1}(y) \in A$.

Since A is a T-filter in X , then $(f^{-1}(x)^*)^* = f^{-1}((x^*)^*) \in A$, thus $(x^*)^* \in f(A)$, that means $f(A)$ is a T-filter in Y .

Theorem(3.20):

If f is an epimorphism from bounded BCK-algebra X into bounded BCK-algebra Y, then the inverse image of a T-filter is a T-filter .

Proof:

Let f be an epimorphism from BCK-algebra X into BCK-algebra Y, and let B be a T-filter in Y. since $e_y \in B$ then $f(e_x) = e_y \in B$, (by Lemma(2.17)) , thus $e_x \in f^{-1}(B)$,

Now let $(x^* * y^*)^* \in f^{-1}(B)$, $y \in f^{-1}(B)$, so $f((x^* * y^*)^*) = ((f(x))^* * (f(y))^*)^* \in B$, $f(y) \in B$, (by Lemma (2.18(1))

Then $(f(x)^*)^* \in B$, since B is a T-filter , thus $(x^*)^* \in f^{-1}(B)$, that means $f^{-1}(B)$ is a T-filter in$X$.

Definition (3.21):

A subset F of a bounded BCK-algebra X is said to be complete T-filter (c-T-filter) if,

1- $e \in F$

2- $(x^* * y^*)^* \in F$, $\forall\, y \in F$ such that $y \neq e$ implies $(x^*)^* \in F$.

Example (3.22):

Let $X = \{0,1,2,3\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	3	3	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 4 , $F_1 = \{0, 3, 4\}$ is c-T-filter and $F_2 = \{1, 2, 4\}$ is not c-T-filter since $(3^* * 1^*)^* = 4 \in F_2$, $(3^* * 2^*)^* = 4 \in F_2$ but$(3^*)^* = 3 \notin F_2$.

Proposition(3.23):

In bounded BCK-algebra X every T-filter is a c-T-filter .

Proof :

Let F be T-filter and $(x^* * y^*)^* \in F$, $\forall\, y \in F$, such that $y \neq e$ .Since F is T-filter , then $(x^*)^* \in F$ .Thus F is c-T-filter .

Remark (3.24):

The converse of proposition (3.23) needs not be true in general as in the example (3.22) $F = \{0, 3, 4\}$ is c-T-filter but it's not T-filter , since $(1^* * 0^*)^* = 4 \in F$ but $(1^*)^* \notin F$.

Corollary(3.25):

In general $\{e\}$, X are trivial c-T-filter .

Remark (3.26):

In bounded BCK-algebra X in general not every c-BCK-filter is a c-T-filter as in the following example

Example (3.27):

Let $X = \{0,1,2,3\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 3 (see[5]) and

$F = \{ 1 , 3 \}$ is c-BCK-filter but not c-T-filter since$(2^* * 1^*)^* = 3 \in F$ but $(2^*)^* = 2 \notin F$.

Remark (3.28):

In bounded BCK-algebra X in general not every c-T-filter is a c-BCK-filter as in the example (3.2) , $F = \{ 0 , 1 , 3 , 4 \}$ is c-T-filter but not c-BCK-filter since$(2^* * 0^*)^* = 4 \in F$, $(2^* * 1^*)^* = 3 \in F$, $(2^* * 3^*)^* = 1 \in F$ and $(2^* * 4^*)^* = 0 \in F$ but $2 \notin F$.

Proposition(3.29):

Every c-T-filter in bounded involutory BCK-algebra X is c-BCK-filter.

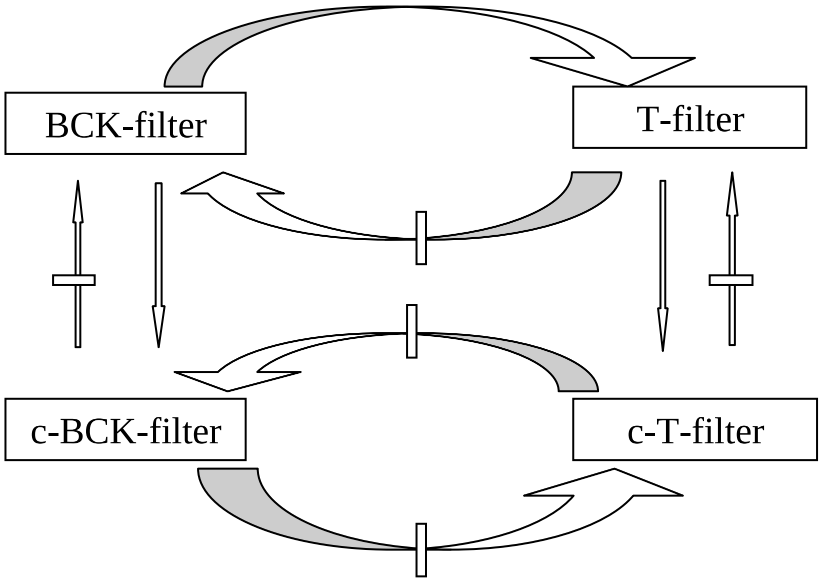
Proof:

Let F be c-T-filter and $(x^* * y^*)^* \in F$, $\forall\, y \in F$. Implies $(x^*)^* \in F$, since X is involutory then $x \in F$,Thus F is c-BCK-filter .

Corollary(2.30):

Every c-T-filter in bounded commutative BCK-algebra X is c-BCK-filter.

The following diagram shows the relation among BCK-filter , c-BCK-filter , T-filter and c-T-filter **.**



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