Solving Linear Systems of Partial Differential Equations of the first Order by using Al-Tememe Transformation

حل الأنظمة الخطية للمعادلات التفاضلية الجزئية من الرتبة الاولى بأستخدام التحويل التميمي

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Abstract:

Our aim in this paper is to find a solution of Linear Systems of Partial Differential Equations (LSPDE) with variable coefficients subjected to some initial conditions by using Al-Tememe transform ($\mathcal{T}$.T) through generalized the methods that found in [1].

المستخلص

هدفنا من هذا البحث هو ايجاد الحل للأنظمة الخطية للمعادلات التفاضلية الجزئية ذات معاملات متغيرة والمتضمنة بعض الشروط الابتدائية بواسطة استخدام التحويل التميمي من خلال الطرق المعممة التي توجد في [1] .

Introduction:

Integral transformations are an important role to solve the linear partial Differential equations (LPDE) with constant coefficients and variable coefficients. We will use Al-Tememe Transform ($\mathcal{T}$.T) to solve systems of linear partial differential equations of a first- order with variable coefficients. And the method summarized by taking ($\mathcal{T}$.T) to both sides of the equations then we take ($\mathcal{T}^{-1}.T$) to both sides of the equations and by using the given initial conditions we find the functions.

Definition 1: [2]

Let f is defined function at period (a, b) then the integral transformation for f whose it's symbol $F(s)$ is defined as:

$$F(s) = \int_{a}^{b} k(s,x)\, f(x)\, dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or ∓∞, such that the above integral is convergent.

Definition (2): [3]

The Al-Tememe transformation for the function $f(x)$; $x > 1$ is defined by the following integral:

$$\mathcal{T}\,[f(x)] = \int_{1}^{\infty} x^{-s} f(x)\, dx = F(s)$$

such that this integral is convergent, s is positive constant. From the above definition we can write:

$$T(u(x,t)) = \int_1^\infty t^{-s} u(x,t)dt$$
$$= v(x,s)$$

such that $u(x,t)$ is a function of x and t.

Property (1): [3]

This transformation is characterized by the linear property, that is

$\mathcal{T}\,[\mathrm{A}u_1(x,t) + \mathrm{B}u_2(x,t)] = A\mathcal{T}[u_1(x,t)] + B\mathcal{T}[u_2(x,t)]$ **,**

Where A and B, are constants while, the functions $u_1(x,t)$, $u_2(x,t)$ are defined when $(t > 1)$.

The Al-Tememe transform for some fundamental functions are given in table (1) **[3]:**

ID	$Function, f(x)$	$F(s) = \int_1^\infty x^{-s} f(x)\,dx = \mathcal{T}\,[f(x)]$	$Regional\ of\ convergence$
1	$k\,;\ k = constant$	$\frac{k}{(s-1)}$	$s > 1$
2	$x^n\,, n \in R$	$\frac{1}{(s-(n+1))}$	$s > n+1$
3	lnx	$\frac{1}{(s-1)^2}$	$s > 1$
4	$x^n lnx\,, n \in R$	$\frac{1}{[s-(n+1)]^2}$	$s > n+1$
5	$sina(lnx)$	$\frac{a}{(s-1)^2+a^2}$	$s > 1$
6	$cosa(lnx)$	$\frac{s-1}{(s-1)^2+a^2}$	$s > 1$
7	$sinha(lnx)$	$\frac{a}{(s-1)^2-a^2}$	$\lvert s-1 \rvert > a$
8	$cosha(lnx)$	$\frac{s-1}{(s-1)^2-a^2}$	$\lvert s-1 \rvert > a$

From the Al-Tememe definition and the above table, we get:

Theorem (1): [3]

If $\mathcal{T}\,(u(x,t)) = v(x,s)$ and a is constant, then $\mathcal{T}(u(x,t^{-a})) = v(x,s+a)$.

Definition (3): [3]

Let $u(x,t)$ be a function where $(t > 1)$ and $\mathcal{T}(u(x,t)) = v(x,s)$, $u(x,t)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}\big(v(x,s)\big) = u(x,t)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function. For example

$$\mathcal{T}^{-1}\left[\frac{2cosx}{(s-2)^3}\right] = cosxt(lnt)^2$$

Property (2): [3]

If $\mathcal{T}^{-1}\big(v_1(x,s)\big) = u_1(x,t)$, $\mathcal{T}^{-1}(v_2(x,s)) = u_2(x,t)$, … , $\mathcal{T}^{-1}\big(v_n(x,s)\big) = u_n(x,t)$ and $a_1, a_2, \dots a_n$ are constants then,

$$\mathcal{T}^{-1}[a_1v_1(x,s) + a_2v_2(x,s) + \cdots + a_nv_n(x,s)] = a_1u_1(x,t) + a_2u_2(x,t) + \cdots + a_n\, u_n(x,t)$$

Theorem (2): [3]

If the function $u(x,t)$ is defined for $t > 1$ and its derivatives $u_t(x,t), u_{tt}(x,t), \dots, u_t^{(n)}(x,t)$ are exist then:

$$\mathcal{T}\left[t^n u_t^{(n)}(x,t)\right] = -u_t^{(n-1)}(x,1) - (s-n)u_t^{(n-2)}(x,1) - \cdots$$
$$-(s-n)\,(s-(n-1))\,\dots(s-2)\,u(x,1) + (s-n)!\,v(x,s).$$

Solving Linear Systems of Partial Differential Equations of the first order by using Al - Tememe transformation

Let us consider we have a linear system of partial differential equation of first order with variable coefficients which we can write it by:

$$tu_{1t}(x,t) = a_{11}u_1(x,t) + a_{12}u_2(x,t) + g_1(x,t)$$
$$tu_{2t}(x,t) = a_{21}u_1(x,t) + a_{22}u_2(x,t) + g_2(x,t) \quad \dots(1)$$

Where a_{11}, a_{12}, a_{21} and a_{22} are constants , $u_{1t}(x,t)$ the first derivative of function $u_1(x,t)$ and , $u_{2t}(x,t)$ the first derivative of function $u_2(x,t)$, such that $u_1(x,t)$ and $u_2(x,t)$ are continuous functions and the $(\mathcal{T}.T)$ of $g_1(x,t)$ and $g_2(x,t)$ are known.

To solve the system (1) we take $(\mathcal{T}.T)$ to both sides of it, and after simplification we put $v_1(x,s) = \mathcal{T}\big(u_1(x,t)\big), v_2(x,s) = \mathcal{T}\big(u_2(x,t)\big)$, $G_1(x,s) = \mathcal{T}\big(g_1(x,t)\big)$, $G_2(x,s) = \mathcal{T}\big(g_2(x,t)\big)$ so by using Theorem 2, we get:

$$(s-1)v_1(x,s) - u_1(x,1) = a_{11}v_1(x,s) + a_{12}v_2(x,s) + G_1(x,s) \quad \dots(2)$$
$$(s-1)v_2(x,s) - u_2(x,1) = a_{21}v_1(x,s) + a_{22}v_2(x,s) + G_2(x,s) \quad \dots(3)$$

So

$$(s-1-a_{11})v_1(x,s) - a_{12}v_2(x,s) = u_1(x,1) + G_1(x,s) \quad \dots(4)$$
$$(s-1-a_{22})v_2(x,s) - a_{21}v_1(x,s) = u_2(x,1) + G_2(x,s) \quad \dots(5)$$

By multiplying equation (4) by a_{21} and equation (5) by $(s-1-a_{11})$ and collecting the result terms we have

$$v_2(x,s) = \frac{h_2(x,s)}{N_2(s)};\quad N_2(s) \neq 0 \quad \dots(6)$$

By the similar method we find

$$v_1(x,s) = \frac{h_1(x,s)}{N_1(s)}; \quad N_1(s) \neq 0 \qquad \dots (7)$$

Where $h_1(x,s)$ and $h_2(x,s)$ are polynomials of x and s, $\mathrm{N}_1(\mathrm{s})$ and $\mathrm{N}_2(\mathrm{s})$ are Polynomials of s, such that the degree of $h_1(x,s)$ is less than the degree of $\mathrm{N}_1(\mathrm{s})$ and the degree of $h_2(x,s)$ is less than the degree of $\mathrm{N}_2(\mathrm{s})$. By taking the inverse of Al-Tememe transformation ($\mathcal{T}^{-1}$. T) to both sides of equations (6) and (7) we get:

$$u_1(x,t) = \mathcal{T}^{-1}\left[\frac{h_1(x,s)}{N_1(s)}\right]$$

$$\dots (8)$$

$$u_2(x,t) = \mathcal{T}^{-1}\left[\frac{h_2(x,s)}{N_2(s)}\right]$$

Equations (8) represent the general solution of system (1) which we can be written it as follows

$$u_1(x,t) = \sum_{i=1}^{m} A_i(x)B_i(t)$$

$$\dots (9)$$

$$u_2(x,t) = \sum_{i=1}^{m} C_i(x)D_i(t)$$

Where $A_i(x)$ and $C_i(x)$ are functions of x and $B_i(t)$ and $D_i(t)$ are functions of t where the number of *i* depend on the degree of $N(s)$. To find the forms of the functions $A_i(x)$ and $C_i(x)$ we use the initial conditions $u_1(x,1)$ and $u_2(x,1)$ in system but the conditions $u_1(x,1)$ *and* $u_2(x,1)$ are not enough to find out the above functions thus we find $u_{1t}(x,1), \dots, u_{1t}^{(m-1)}(x,1)$ and $u_{2t}(x,1), \dots, u_{2t}^{(m-1)}(x,1)$ by using system (1), so we get number of equations equal to 2m with initial conditions also we use the equation (9) for finding derivatives and by using the initial conditions we get 2m equations which is formed linear system, this linear system can be solved to obtain the forms of $A_1(x), A_2(x), \dots, A_m(x)$ and $C_1(x), C_2(x), \dots, C_m(x)$.

Example (1): To solve the linear system of partial differential equation

$$tu_{1t}(x,t) = u_1(x,t) + u_2(x,t) + x \quad ; \quad u_1(x,1) = 1$$

$$tu_{2t}(x,t) = 5\,u_1(x,t) - 3u_2(x,t) + xt^{-1} \quad ; \quad u_2(x,1) = -1$$

We take Al-Tememe transformation to the above system, so we get

$$\mathcal{T}\big(tu_{1t}(x,t)\big) = \mathcal{T}\big(u_1(x,t)\big) + \mathcal{T}\big(u_2(x,t)\big) + \mathcal{T}(x)$$

$$-u_1(x,1) + (s-1)v_1(x,s) = v_1(x,s) + v_2(x,s) + \frac{x}{s-1}$$

$$(s-2)v_1(x,s) - v_2(x,s) = \frac{x}{(s-1)} + 1 \qquad \dots (10)$$

$\mathcal{T}(tu_{2t}(x,t)) = 5\mathcal{T}\big(u_1(x,t)\big) - 3\mathcal{T}\big(u_2(x,t)\big) + x\mathcal{T}(t^{-1})$

$-u_2(x,1) + (s-1)v_2(x,s) = 5v_1(x,s) - 3v_2(x,s) + \frac{x}{s}$

$$(s+2)v_2(x,s) - 5v_1(x,s) = \frac{x}{s} - 1 \quad \dots (11)$$

by multiplying equation (10) by $(s+2)$ and equation (11) by 1

$$(s+2)(s-2)v_1(x,s) - (s+2)v_2(x,s) = \frac{x(s+2)}{(s-1)} + (s+2) \quad \dots (12)$$

$$(s+2)v_2(x,s) - 5v_1(x,s) = \frac{x}{s} - 1 \quad \dots (13)$$

from equation (12) and (13) we get

$$v_1(x,s) = \frac{h_1(x,s)}{s(s-1)(s-3)(s+3)} \quad , h_1(x,s) = xs(s+2) + (s+2)(s-1)s + x(s-1) - s(s-1)$$

And

$$v_2(x,s) = \frac{h_2(x,s)}{s(s-1)(s-3)(s+3)} \quad , h_2(x,s) = 5xs + x(s-1)(s-2) - s(s-7)(s-1)$$

Therefore after using $\mathcal{T}^{-1}$.T we get

$u_1(x,t) = A_1(x)t^{-1} + B_1(x) + C_1(x)t^2 + D_1(x)t^{-4}$,

and

$$u_2(x,t) = A_2(x)t^{-1} + B_2(x) + C_2(x)t^2 + D_2(x)t^{-4}$$

the conditions $u_1(x,1)$ and $u_2(x,1)$ are not enough to get the above functions from two equations so we will substitute the two conditions and their derivatives in the differential equations (linear system) to get :

$u_{1t}(x,1) = x$, $u_{1tt}\,(x,1) = 8 + x$, $u_{1ttt}\,(x,1) = -40 - x$

$u_{2t}(x,1) = 8 + x$, $u_{2tt}(x,1) = -32$, $u_{2ttt}\,(x,1) = 200 + 7x$

By substituting these initial conditions in the derivatives of the general solution of $u_1(x,t)$, $u_2\,(x,t)$ we get:

$A_1(x) + B_1(x) + C_1(x) + D_1(x) = 1$

$-A_1(x) + 2C_1(x) - 4D_1(x) = x$

$2A_1(x) + 2C_1(x) + 20D_1(x) = 8 + x$

$-6A_1(x) - 120D_1(x) = -40 - x$

By solving these equations we get:

$A_1(x) = -{}^{x}/_{9}\, , B_1(x) = -{}^{3x}/_{8}\, , C_1(x) = \frac{17x}{36} + \frac{2}{3}, D_1(x) = \frac{1}{3} + \frac{x}{72}$

$\Rightarrow u_1(x,t) = \frac{-x}{9}t^{-1} - \frac{3x}{8} + \left(\frac{2}{3} + \frac{17x}{36}\right)t^2 + \left(\frac{1}{3} + \frac{x}{72}\right)t^{-4}$

By the same method we find :

$$A_2(x) = 2x\,/\,9 \;, B_2(x) = -5x\,/\,8 \;, C_2(x) = \frac{2}{3} + \frac{17x}{36} \;, D_2(x) = \frac{-5}{3} - \frac{5x}{72}$$

$$\Rightarrow \quad u_2(x,t) = \frac{2x}{9}t^{-1} - \frac{5x}{8} + \left(\frac{2}{3} + \frac{17x}{36}\right)t^2 + \left(\frac{-5}{3} - \frac{5x}{72}\right)t^{-4}$$

Example (2): To solve the linear system of partial differential equation

$tu_{1t}(x,t) = 3u_1(x,t) - 2u_2(x,t) + x^2 lnt \quad ; \quad u_1(x,1) = 0$

$tu_{2t}(x,t) = 2u_1(x,t) - u_2(x,t) + x^2 t^4 \quad ; \quad u_2(x,1) = 0$

We take Al-Tememe transformation to both sides of the linear system

$\mathcal{T}(tu_{1t}(x,t)) = 3\mathcal{T}(u_1(x,t)) - 2\mathcal{T}(u_2(x,t)) + x^2\mathcal{T}(lnt)$

By using Theorem 2 we will have

$$-u_1(x,1) + (s-1)v_1(x,s) = 3v_1(x,s) - 2v_2(x,s) + \frac{x^2}{(s-1)^2}$$

$$(s-4)v_1(x,s) + 2v_2(x,s) = \frac{x^2}{(s-1)^2} \quad \dots (14)$$

$$\mathcal{T}\big(tu_{2t}(x,t)\big) = 2\mathcal{T}\big(u_1(x,t)\big) - \mathcal{T}\big(u_2(x,t)\big) + x^2\mathcal{T}(t^4)$$

$$-u_2(x,1) + (s-1)v_2(x,s) = 2v_1(x,s) - v_2(x,s) + \frac{x^2}{(s-5)}$$

$$sv_2(x,s) - 2v_1(x,s) = \frac{x^2}{(s-5)} \quad \dots (15)$$

By multiplying equation (14) by s and equation (15) by 2 we get

$$s(s-4)v_1(x,s) + 2s\, v_2(x,s) = \frac{x^2 s}{(s-1)^2} \quad \dots (16)$$

$$2sv_2(x,s) - 4v_1(x,s) = \frac{2x^2}{s-5} \quad \dots (17)$$

From (16) and (17) we get

$$v_1(x,s) = \frac{h_1(x,s)}{(s-1)^2(s-2)^2(s-5)} \quad ; \quad h_1(x,s) = x^2 s(s-5) - 2x^2(s-1)^2$$

And

$$v_2(x,s) = \frac{h_2(x,s)}{(s-1)^2(s-2)^2(s-5)} \quad ; h_2(x,s) = 2x^2(s-5) + x^2(s-4)(s-1)^2$$

Therefore after using $\mathcal{T}^{-1}$. T we get

$$u_1(x,t) = A_1(x) + B_1(x)lnt + C_1(x)t + D_1(x)tlnt + E_1(x)t^4$$

And also, $u_2(x,t) = A_2(x) + B_2(x)lnt + C_2(x)t + D_2(x)tlnt + E_2(x)t^4$

The conditions $u_1(x,1)$ and $u_2(x,1)$ are not enough to get the above functions from the above two equations so we will substitute the two conditions and their derivatives in the differential equations (linear system) to get :-

$u_{1t}(x,1) = 0$, $u_{1tt}(x,1) = -x^2$, $u_{1ttt}(x,1) = -6x^2$, $u_{1t}^{(4)}(x,1) = -6x^2$

$u_{2t}(x,1) = x^2$, $u_{2tt}(x,1) = 2x^2$, $u_{2ttt}(x,1) = 4x^2$, $u_{2t}^{(4)}(x,1) = -4x^2$

By substituting these initial conditions in the derivatives of the general solution of $u_1(x,t)$, $u_2(x,t)$ we get:

$A_1(x) + C_1(x) + E_1(x) = 0$

$B_1(x) + C_1(x) + D_1(x) + 4E_1(x) = 0$

$2B_1(x) - D_1(x) + 24E_1(x) = -6x^2$

$-B_1(x) + D_1(x) + 12E_1(x) = -x^2$

$-6B_1(x) + 2D_1(x) + 24E_1(x) = -6x^2$

By solving these equations we get:

$A_1(x) = 3x^2$, $B_1(x) = x^2$, $C_1(x) = -25x^2/9$, $D_1(x) = {8x^2}/{3}$, $E_1(x) = {-2x^2}/{9}$

$\Longrightarrow\ u_1(x,t) = 3x^2 + x^2 lnt - \frac{25x^2}{9}t + \frac{8x^2}{3}tlnt - \frac{2x^2}{9}t^4$

By the same method we find:

$A_2(x) = 4x^2$, $B_2(x) = 2x^2$, $C_2(x) = -37x^2\ /\ 9$, $D_2(x) = 8x^2\ /\ 3$, $E_2(x) = x^2/9$

$\Longrightarrow\ u_2(x,t) = 4x^2 + 2x^2 lnt - \frac{37x^2}{9}t + \frac{8x^2}{3}tlnt + \frac{x^2}{9}t^4.$

REFERENCES

[1]. Herbert Kreyszig, Edward J. Norminton, “Advanced Engineering Mathematics” Ohio State University Columbus, Ohio, Tenth edition, 2011.

[2]. Gabriel Nagy, “Ordinary Differential Equations” Mathematics Department, Michigan State University, East Lansing, MI, 48824.October 14, 2014.

[3]. Mohammed, A.H., Athera Nema Kathem, “Solving Euler’s Equation by Using New Transformation”, Karbala university magazine for completely sciences, volume (6), number (4), (2008).

[4]. William F. Trench, “Elementary Differential Equations” Trinity University, 2013.