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RESEARCH ARTICLE

New Three-Parameter Exponentiated Benini Distribution: Properties and Applications

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ABSTRACT

Modelling some rare environmental events and obtaining accurate predictions is quite important to determine their nature and to take action accordingly. In this article, a new three-parameter exponentiated Benini distribution based on the exponential family is proposed for this purpose. It is called the three-parameter exponentiated Benini distribution. As an additional motivation, although new distributions are derived by various methods in the literature, there is no study on variants of the Benini distribution. We first investigate its mathematical properties and then evaluate its performance in real applications by comparing it with other distributions. In particular, the corresponding probability density and hazard rate functions, quantile function, stochastic ordering, moment and incomplete moment, and ordered statistics, are presented in detail. The analysis of the shape of the distribution is also provided by calculating the skewness and kurtosis coefficients for different parameter values. In addition, several classical estimates for the parameters of the three-parameter exponential Benini distribution are computed, including maximum likelihood estimates, maximum product of distance estimates, Anderson-Darling estimates, right and left tail Anderson-Darling estimates, and Cramér-von Mises estimates. We examine the performance of these methods with simulation studies via different scenarios and sample sizes. Applying the proposed distribution to two real environmental data sets, we find that the three-parameter exponential Benini distribution gives the best results compared to the Benini, Ramous Louzada, inverse Pareto, Pareto, Pareto type I, Pareto type II, Beta Pareto and Weibull Pareto distributions. Overall, it is particularly adapted to the analysis of environmental data, mainly due to its long-tailed and flexible structure.

Keywords: Lifetime distribution, Exponentiated Benini distribution, Hazard rate function, Moments, Order statistics

1. Introduction

As a major reference, [1] proposed the exponentiated exponential distribution and introduced a novel approach to generating probability distributions based on the power transformation. The proposed exponentiated distribution consists of two parameters similar to the Weibull and gamma distributions. It is also shown in [2] that some of the properties of the distribution are also similar to that of the Weibull and gamma distributions. However, since the corresponding functions cannot be obtained in closed form when the shape parameter of the gamma distribution is not an integer, it loses its popularity. In addition, the exponential distribution and its mathematical properties are often used to analyze data sets in applied sciences. In the literature, based on the approach of [1], many exponentiated distributions have been proposed, such as the exponentiated beta

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Pareto distribution by [3], exponentiated Pareto distribution by [4], exponentiated log normal distribution by [5], exponentiated Fréchet distribution by [6], exponentiated Gumbel distribution by [7], exponentiated gamma distribution by [8], generalized exponentiated gamma distribution by [9], exponentiated Lomax Poisson distribution by [10], modified exponentiated Weibull distribution by [11], generalized exponentiated class of distributions by [12], exponentiated Weibull-Pareto distribution by [13], type I half-logistic odd Weibull distribution by [37], extended modified Weibull distribution by [38] and exponentiated Kumaraswamy distribution by [14].

On the other hand, in a series of famous articles, Pareto was the first scientist to attempt to discuss this problem quantitatively, and also to model experimentally the long tail of the cumulative income distribution of the richest part of a country (see [15]). The Pareto income distribution has been shown to be valid for different countries and epochs (see [16]). Despite the experimental success of this income distribution, the characterization of the low-income parts, which constitute the majority of a country, remains an unsolved problem. Various functions with many parameters have been proposed by economists to describe the low part or the whole part of the income distribution. Also [17] proposed the Benini distribution, which is a generalization of the Pareto distribution, to become a solution to this kind of problems in economy. Since the Benini distribution has a long tail, it allows the characterization of income inequality in income distributions.

In this study, we propose the exponentiated Benini (E-B) distribution, which is a new three-parameter extension of the Benini distribution. First, we examine the probability density, survival and hazard functions, order statistics, skewness and kurtosis. Accordingly, it is found that the corresponding probability density function (pdf) behaves differently depending on the parameter values, i.e., it has a flexible structure, and also a long tail structure depending on the parameter values. On the other hand, the model fit of the proposed distribution to a real data set is investigated, and it is found that the E-B distribution provides a better fit than the other distributions compared. Considering the flexible and long tail structure of the distribution and its applicability, it can be said that the proposed distribution can be used for modelling as an alternative to other distributions, especially in different data structures such as earthquakes, economic crises and waiting periods.

Section 2 gives a brief overview of the Benini distribution, while Section 3 presents the E-B distribution and its mathematical properties in detail. Section 4 mathematically examines the classical approaches traditionally used to obtain parameter estimates for the proposed distribution. Section 5 performs a simulation study to determine which approach performs better. In Section 6, an application is made with two different data sets to investigate the applicability of the corresponding model, and its performance is examined with other models known from the literature. The last section, Section 7, summarises the advantages and properties of the proposed distribution, the limitations of the study and future studies.

2. Benini distribution

The Benini distribution, developed in [17], is widely used for modelling income in probability, statistics, economics and actuarial science. The pdf and the cumulative distribution function (cdf) of the (generalized) Benini distribution are given by

$$f_B(x) = \frac{1}{x}\left[\alpha + 2\beta \log\left(\frac{x}{\sigma}\right)\right] \exp\left\{-\alpha \log\left(\frac{x}{\sigma}\right) - \beta\left[\log\left(\frac{x}{\sigma}\right)\right]^2\right\} \tag{1}$$

and

$$F_B(x) = 1 - \exp\left\{-\alpha \log\left(\frac{x}{\sigma}\right) - \beta\left[\log\left(\frac{x}{\sigma}\right)\right]^2\right\}. \tag{2}$$

Thus defined, the Benini distribution has three parameters, $\alpha, \beta, \sigma > 0$, and takes its values over the interval $[\sigma, \infty)$, i.e., the expressions of $f_B(x)$ and $F_B(x)$ in Eqs. (1) and (2), respectively, holds for $x \geq \sigma$, and they are equal to 0 elsewhere. Also, α and β are shape parameters and they determine the behavior of the pdf. Since σ determines the scale for the pdf of the Benini distribution, it is called as the “scale parameter”. The pdf of the Benini distribution can be monotonically decreasing or unimodal. The tail of the pdf can be fat, meaning an algebraic decrease instead of a multiple decrease for the large values of the pdf, or it can be thin.

The Benini distribution is related to other well-known distributions. For example, it is a natural generalization of the Pareto distribution; it becomes the Pareto distribution for $\beta = 0$. The Benini distribution is also a transformation of the Rayleigh distribution: if a random variable X has a Rayleigh distribution with parameter σ, then the random variable $Y = \exp(X)$ has the Benini distribution with parameters $\alpha = 0$, $\beta = 1/\sigma^2$ and $\sigma = 1$. Because the Benini distribution is sometimes similar to the pdf of the logarithm of a variable with a Weibull distribution, it is called the log-Weibull distribution. The Benini distribution is therefore closely related to the gamma, exponential, max-stable, min-stable, Gumbel, Fréchet and uniform distributions. We can refer to [20] for more details.

For the simple case $\alpha = 0$, the pdf and cdf of the two-parameter Benini distribution are, respectively, given by

$$f_B(x) = \frac{2\beta}{x} \exp\left\{-\beta \left[\log\left(\frac{x}{\sigma}\right)\right]^2\right\} \log\left(\frac{x}{\sigma}\right) \tag{3}$$

and

$$F_B(x) = 1 - \exp\left\{-\beta \left[\log\left(\frac{x}{\sigma}\right)\right]^2\right\}, \tag{4}$$

where $x \geq \sigma > 0$. As a secondary remark, these functions can also be expressed more concisely without the exponential function, as

$$f_B(x) = \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right), \quad F_B(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}. \tag{5}$$

3. E-B distribution

The basis of the proposed distribution is the exponential family of (continuous) distributions introduced by [21]. It consists of adding an extra parameter to a parent distribution. That is, by denoting $F(x)$ the cdf of the parent distribution, the cdf of the exponentiated family of distributions is defined by

$$G(x) = F(x)^{\gamma}, \tag{6}$$

where $\gamma > 0$ is an additional shape parameter. Because of this parameter, the proposed exponentiated distribution has a more flexible structure than the parent distribution. This article proposes the E-B distribution to achieve a more flexible structure than the former Benini distribution.

Hereafter, X denotes a random variable that follows the proposed E-B distribution with three parameters.

That is, based on Eqs. (3), (4) and (6), the pdf and cdf of X are obtained as

$$f_{E-B}(x) = \gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma - 1} \tag{7}$$

and

$$F_{E-B}(x) = \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma}, \tag{8}$$

respectively, where σ is the scale parameter of the distribution, while β and γ are the shape parameters, $\alpha, \beta, \sigma > 0$, and $x \geq \sigma$.

We now show the plots of the above pdf for different parameter values in Fig. 1.

It can be inferred from this figure that the E-B distribution has a flexible structure, as the pdf has different shapes, showing reversed J and lower truncated bell shape for the different parameter values.

In the following subsections of this section, various statistical properties such as survival and hazard rate functions, stochastic ordering, skewness and kurtosis, moments and incomplete moments, order statistics, which provide important information about the structure of the distribution, are explained in detail.

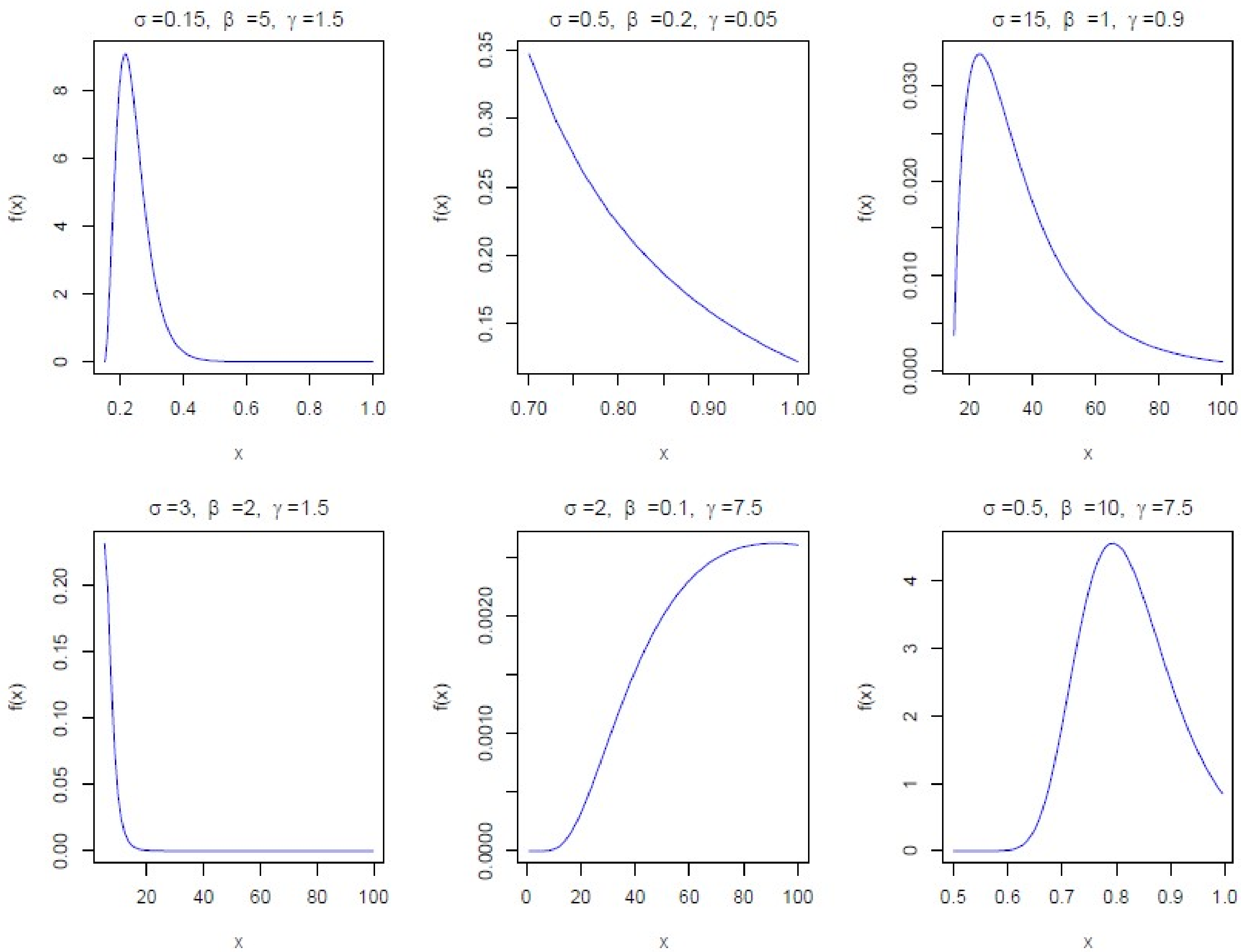

Fig. 1. The pdf of the E-B Distribution.

3.1. Survival and hazard rate functions of the E-B distribution

As a characteristic property, the survival function (sf) of the E-B distribution can be written by means of the following formula: $S_{E-B}(x) = 1 - G_{E-B}(x)$. It is given as

$$S_{E-B}(x) = 1 - \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma}. \tag{9}$$

Now, we present the plot of this sf in Fig. 2 for different parameter values.

As another characteristic property of the E-B distribution, the hrf is obtained as $h_{E-B}(x) = f_{E-B}(x)/S_{E-B}(x)$, that is

$$h_{E-B}(x) = \gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \frac{\left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma-1}}{1 - \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma}}. \tag{10}$$

In Fig. 3, we illustrate the hrf of the E-B distribution for different parameter values.

This figure shows the decreasing, increasing and revered bathtub shapes for the corresponding hrf, which are required properties for modelling purposes.

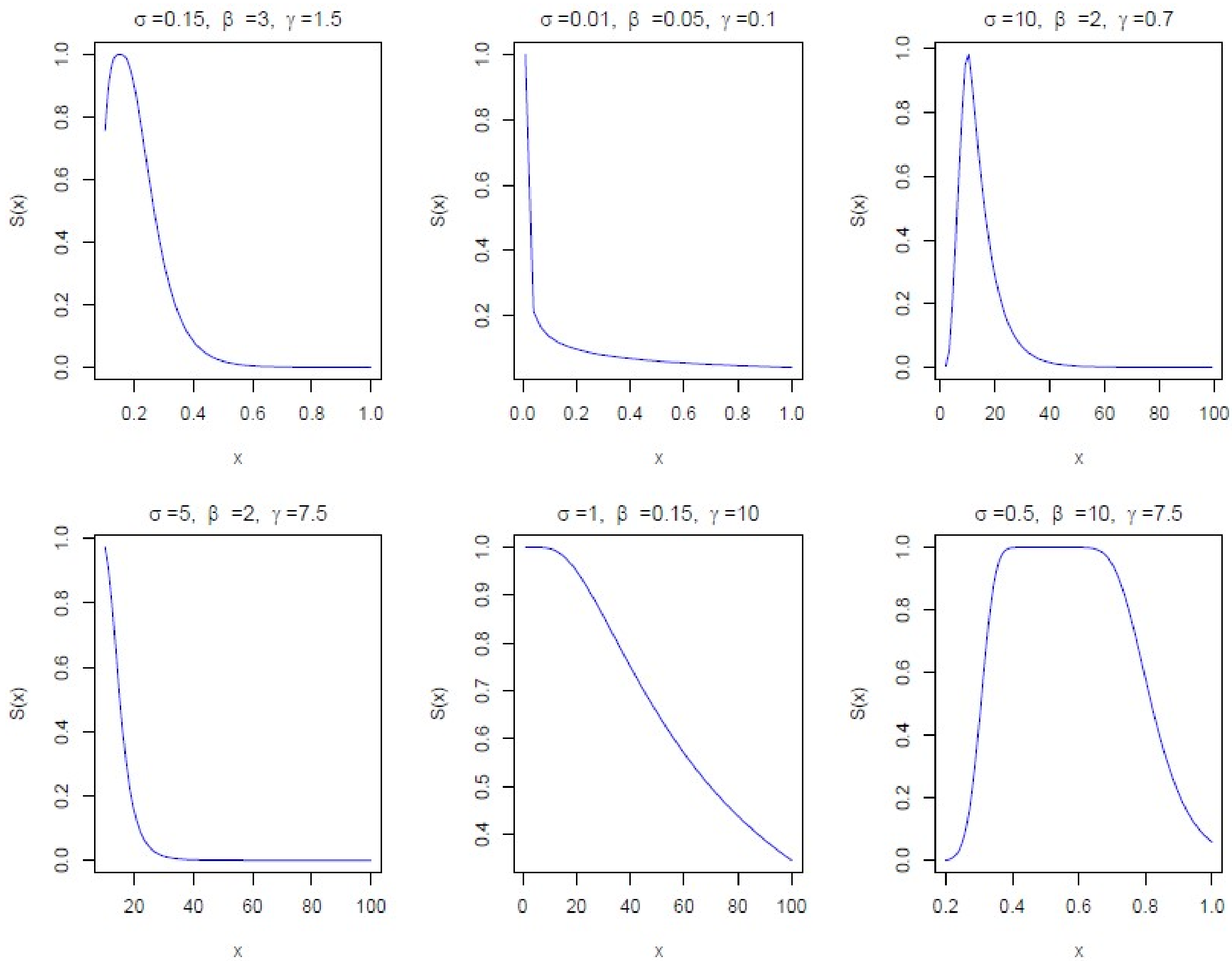

Fig. 2. The sf of the E-B distribution.

3.2. Stochastic ordering results

Here, we present some stochastic ordering results using the cdf of the E-B distribution. These results are useful to provide a hierarchical understanding of the related model (also called first-order stochastic dominance). Let $F(x; \sigma, \gamma, \beta)$ be the cdf of the E-B distribution as defined by Eq. (8), where only the names of the parameters are given. Then the following inequalities hold:

- For $\sigma_2 \geq \sigma_1$, we have $F(x; \sigma_2, \gamma, \beta) \leq F(x; \sigma_1, \gamma, \beta)$ (for $x \geq \sigma_2 \geq \sigma_1$).
- For $\beta_2 \geq \beta_1$, we have $F(x; \sigma, \gamma, \beta_1) \leq F(x; \sigma, \gamma, \beta_2)$.
- For $\gamma_2 \geq \gamma_1$, we have $F(x; \sigma, \gamma_2, \beta) \leq F(x; \sigma, \gamma_1, \beta)$.

Also, for a stochastic ordering result comparing the former Benini distribution and the E-B distribution, one has, if $\gamma \geq 1$,

$$F_{E-B}(x) = F(x; \sigma, \gamma, \beta) \leq F(x; \sigma, 1, \beta) = F_E(x).$$

The reversed inequality holds for $\gamma \in (0, 1)$.

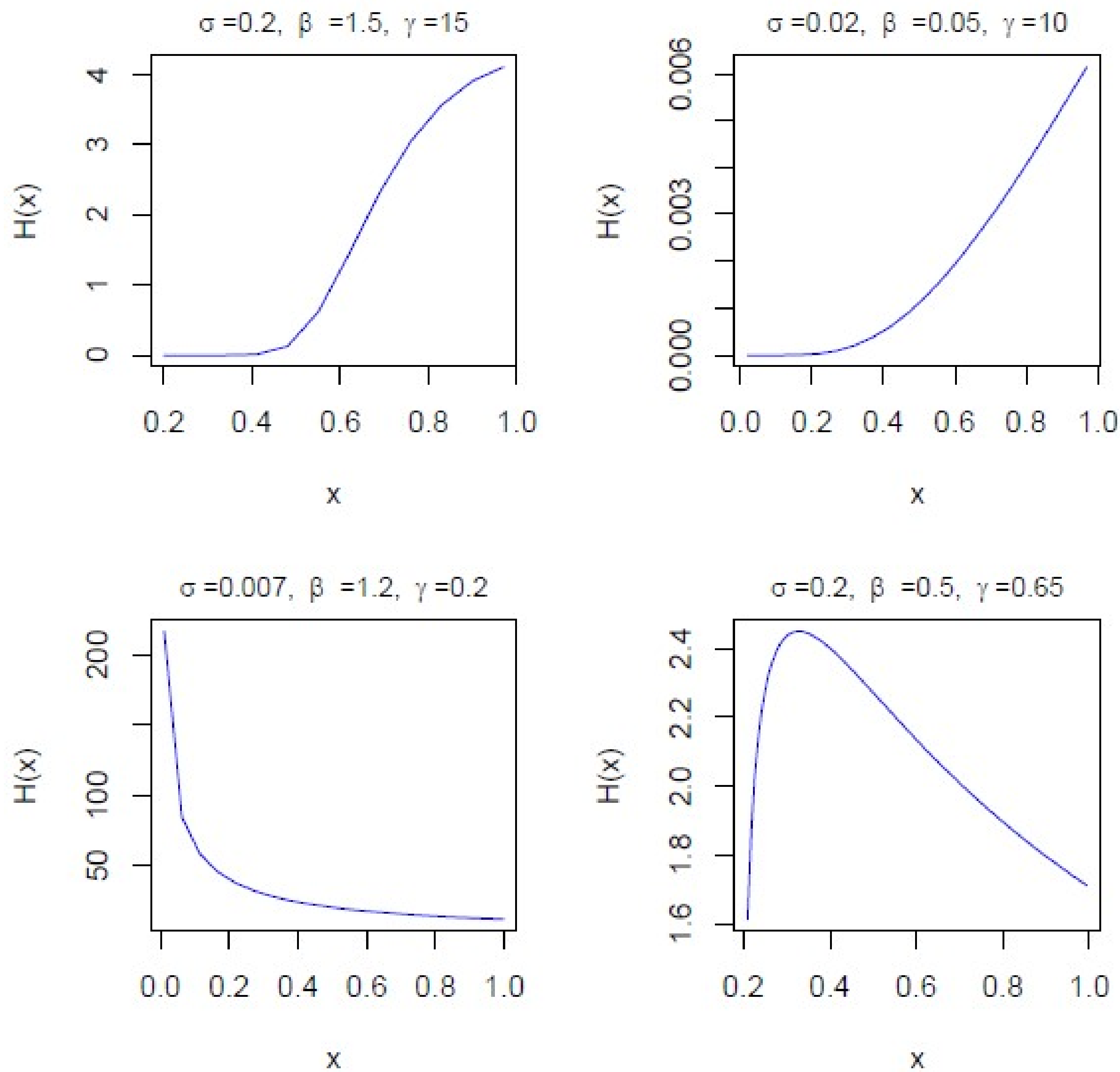

Fig. 3. The hrf of the E-B distribution.

3.3. Quantile function of the E-B distribution

The quantile function gives some of the important properties of a distribution. It is obtained by the inverse function of the corresponding cdf. Thus, thanks to Eq. (8) and after some algebra, we obtain

$$Q_{E-B}(u) = \sigma \exp\left[\sqrt{-\frac{1}{\beta}\log(1-u^{1/\gamma})}\right], \tag{11}$$

where $u \in (0,1)$. We can also calculate the quartiles Q_1, Q_2 and Q_3 of our distribution by using this quantile function. For the 25-th percent (quartile Q_1), we take $u = 0.25$, for the median (quartile Q_2), we take $u = 0.50$, and for the 75-th percent (quartile Q_3), we take $u = 0.75$ in Eq. (11).

From Eq. (11), skewness and kurtosis coefficients of the E-B distribution can be determined (the details are given in the next subsection), values of the E-B distribution can be generated via simulation techniques, and various quantile-derived functions characterizing the E-B distribution can be detailed. For example, the quantile density function can be derived by differentiating $Q_{E-B}(u)$ as a function of u. It is given as follows:

$$q_{E-B}(u) = \frac{\sigma}{2\beta\gamma} \frac{u^{1/\gamma-1} \exp\left[\sqrt{-\log(1-u^{1/\gamma})/\beta}\right]}{(1-u^{1/\gamma})\sqrt{-\log(1-u^{1/\gamma})/\beta}}, \tag{12}$$

where $u \in (0, 1)$. This function is of importance as it appears in many statistical objects (see [22]).

3.4. Skewness and kurtosis coefficients of the E-B distribution

The kurtosis coefficient is related to whether the distribution is heavy-tailed or light-tailed, while the skewness coefficient is related to the measure of symmetry. We can obtain the skewness coefficient of the E-B distribution using the Bowley measure (see [23]) as

$$S = \frac{Q_{E-B}(3/4) - 2Q_{E-B}(1/2) + Q_{E-B}(1/4)}{Q_{E-B}(3/4) - Q_{E-B}(1/4)} \tag{13}$$

and, using the Moor measure (see [24]), the kurtosis coefficient is also obtained as

$$K = \frac{Q_{E-B}(7/8) - Q_{E-B}(5/8) - Q_{E-B}(3/8) + Q_{E-B}(1/8)}{Q_{E-B}(6/8) - Q_{E-B}(2/8)}, \tag{14}$$

where $Q_{E-B}(u)$ is the quantile function given in Eq. (11). We calculate the skewness and kurtosis coefficients by using the values in Eqs. (13) and (14).

The shape of the distribution becomes symmetric when $S = 0$, and skewed to the right when $S > 0$, and skewed to the left when $S < 0$. As K increases, the tail of the distribution becomes heavier. As a benchmark, the kurtosis of the normal distribution is $K = 3$. Compared to the kurtosis of the normal distribution, $K > 3$ corresponds to a longer tail and $K < 3$ corresponds to a shorter tail for the E-B distribution.

Skewness, kurtosis, median, quartiles Q_1 and Q_3 values of the E-B distribution are given in Table 1 for different parameter values. Accordingly, it is observed that the all skewness and kurtosis values are positive for all parameter values, implying a skewed to right distribution with increasing skewness and lengthening tail by decreasing parameter values.

Table 1. Skewness, kurtosis, median and quartiles of the E-B distribution for different parameter values.

β	σ	γ	Q_1	Median	Q_3	Skewness	Kurtosis
0.05	1	0.1	1.0043	1.1500	2.9349	3.2319	5.3179
	1	0.05	1.0000	1.0043	1.2867	14.9851	7.9214
0.1	1	0.1	1.0030	1.1038	2.1411	4.7029	3.8329
	1	0.05	1.0000	1.0030	1.1951	21.436	6.6791
0.5	0.5	5	2.7264	6.9561	20.7003	2.0774	1.8669
	0.5	2	1.4617	2.6431	5.2681	3.1567	1.1527
	0.5	0.5	0.8549	1.1496	1.6229	6.2194	0.6466
	5	5	27.2637	69.5612	207.0034	2.0774	1.8669
	5	2	14.6166	26.4313	52.6811	3.1567	1.1527
	5	0.5	8.5488	11.4959	16.2298	6.2194	0.6465
5	10	5	17.0977	22.9919	32.4596	6.2194	0.6466
	10	2	14.0386	16.9309	21.0572	9.8249	0.4784
	10	0.5	11.8485	13.0119	14.5111	19.673	0.3395
	0.5	5	54.5275	139.1224	414.0068	2.0773	1.8669
	0.5	2	29.2332	52.8625	105.3623	3.1567	1.1527
	0.5	0.5	17.0977	22.9919	32.4596	6.2194	0.6466
10	10	5	14.6121	18.0165	18.0165	8.7875	0.5119
	10	2	12.7108	14.5111	16.9309	13.9012	0.3964
	10	0.5	11.2742	12.0462	13.0119	27.8413	0.2997
	0.5	5	0.7306	0.9008	1.1496	8.7879	0.5119
	0.5	2	0.6355	0.7255	0.8466	13.9012	0.3964
	0.5	0.5	0.5637	0.6023	0.6506	27.8414	0.2997

3.5. Moments of the E-B distribution

We now examine the crude moments of the E-B distribution, which are central to defining crucial probabilistic measures. For any integer r, the r-th crude moment of X exists (unlike the Pareto distribution) and is defined by

$$\mu_r' = E(X^r) = \int_\sigma^\infty x^r f_{E-B}(x)dx$$
$$= \int_\sigma^\infty x^r \left\{ \gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma-1} \right\} dx. \quad (15)$$

By applying the change of variable $y = \log(x/\sigma)$, we can simplify it as

$$\mu_r' = 2\sigma^r \gamma \beta \int_0^\infty y e^{yr} e^{-\beta y^2} \left[1 - e^{-\beta y^2}\right]^{\gamma-1} dy. \quad (16)$$

For fixed parameters, this integral can be determined using any mathematical software. Alternatively, one can propose a series expression of μ_r'. In fact, from the series expansion of the exponential distribution, the generalized binomial formula and the change of variable $z = \beta(\ell+1)y^2$ (for the last step), it follows that

$$\mu_r' = 2\sigma^r \gamma \beta \int_0^\infty y \left\{ \sum_{k=0}^{\infty} \frac{(yr)^k}{k!} \right\} e^{-\beta y^2} \left\{ \sum_{\ell=0}^{M(\gamma)} \binom{\gamma-1}{\ell} (-1)^\ell e^{-\beta \ell y^2} \right\} dy$$
$$= 2\sigma^r \gamma \beta \sum_{k=0}^{\infty} \sum_{\ell=0}^{M(\gamma)} \frac{r^k}{k!} \binom{\gamma-1}{\ell} (-1)^\ell \int_0^\infty y^{k+1} e^{-\beta(\ell+1)y^2} dy$$
$$= \sigma^r \gamma \sum_{k=0}^{\infty} \sum_{\ell=0}^{M(\gamma)} \frac{r^k}{k!} \binom{\gamma-1}{\ell} (-1)^\ell \beta^{-k/2} (\ell+1)^{-k/2-1} \Gamma\left(\frac{k}{2}+1\right), \quad (17)$$

where $\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt$ with $y > 0$ (the standard gamma function), and $M(\gamma) = (\gamma-1)!$ if γ is an integer greater to 1, and $M(\gamma) = \infty$ otherwise, and $\binom{\gamma-1}{\ell} = \prod_{i=1}^{\ell}(\gamma - i)$.

In this series expression, an efficient approximation of μ_r' is given by replacing the bound ∞ by any large integer. Such an approximation is sometimes less error-prone than computing the integral directly.

From the moments, we can derive several important quantities that provide valuable information about the characteristics of the E-B distribution, such as the mean of X given by μ_1', its variance given by $\sigma^2 = \mu_2' - (\mu_1')^2$, measures of skewness and kurtosis complementing those defined by S and K, and the general coefficient. In this respect, we can refer to the book by [25].

3.6. Incomplete moments of the E-B distribution

The incomplete moments of a distribution allow to define useful functions in various applied fields. Here, we study them in the context of the E-B distribution. For any integer r and $t \geq \sigma$, there exists the r-th incomplete moment of X defined at t, which is given by

$$\mu_r'(t) = E(X^r 1_{\{X \leq t\}}) = \int_\sigma^t x^r f_{E-B}(x)dx$$
$$= \int_\sigma^t x^r \left\{ \gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \left[1 - \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma-1} \right\} dx. \quad (18)$$

By proceeding as for the crude moments, we arrive at the following concise integral:

$$\mu_r'(t) = 2\sigma^r \gamma \beta \int_0^{\log(t/\sigma)} y e^{yr} e^{-\beta y^2} \left[1 - e^{-\beta y^2}\right]^{\gamma-1} dy. \quad (19)$$

If t and the parameters are fixed, we can at least evaluate this integral numerically. Alternatively, using the same technical steps as for the crude moments, a series expression of $\mu_r'(t)$ is given below

$$\mu_r'(t) = \sigma^r \gamma \sum_{k=0}^{\infty} \sum_{\ell=0}^{M(\gamma)} \frac{r^k}{k!} \binom{\gamma-1}{\ell} (-1)^{\ell} \beta^{-k/2} (\ell+1)^{-k/2-1} \times$$
$$\gamma\left(\frac{k}{2}+1, \beta(\ell+1)\left[\log\left(\frac{t}{\sigma}\right)\right]^2\right), \tag{20}$$

where $\gamma(y,z) = \int_0^z t^{y-1} e^{-t} dt$ with $y > 0$ (which is the lower incomplete gamma function).

Thanks to the incomplete moments, we can define, among others, the mean deviations, important income functions and indices (such as the Bonferroni curve, Bonferroni index, and Lorenz curve). All of them are described in full generality in [25].

3.7. Order statistics of the E-B distribution

Order statistics are often used in the fields of reliability and survival testing. On the other hand, it has a crucial importance in statistical estimation (see [19]). If a random variable X has pdf and cdf given as $f(x)$ and $F(x)$, respectively, then the pdf of the r-th order statistic of X, say $X_{(r)}$, is defined as

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}. \tag{21}$$

Then, based on Eqs. (7), (8) and (21), the r-th order statistic of the E-B distribution has the following pdf:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \times$$
$$\left[1-\left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma r-1} \left\{1-\left[1-\left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma}\right\}^{n-r}. \tag{22}$$

In particular, the maximum order statistics $X_{(n)}$ has the pdf given as

$$f_{X_{(n)}}(x) = n\gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \left[1-\left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma n-1} \tag{23}$$

and the minimum order statistics $X_{(1)}$ has the pdf given as

$$f_{X_{(1)}}(x) = n\gamma \frac{2\beta}{x} \left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)} \log\left(\frac{x}{\sigma}\right) \times$$
$$\left[1-\left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma-1} \left\{1-\left[1-\left(\frac{x}{\sigma}\right)^{-\beta \log\left(\frac{x}{\sigma}\right)}\right]^{\gamma}\right\}^{n-1}. \tag{24}$$

These expressions remain tractable. They can be used for further purposes.

4. Estimation methods

This section discusses various estimation methods for the E-B distribution. Traditional estimation techniques using order statistics are used to estimate the parameters of the distribution. The corresponding formulas can be derived by minimizing objective functions for the ordinary least-squares estimates (OLSEs), weighted least-squares estimates (WLSEs), Anderson-Darling estimates (ADEs), right-tail Anderson Darling estimates (RTADEs), left-tail Anderson Darling estimates (LTADEs), and Cramér-von Mises estimates (CVMEs). For the maximum likelihood estimates (MLEs) and maximum product of interval estimates (MPSEs) methods, the respective objective functions must be maximized.

We consider n observations of X, say $x_1, x_2, \ldots, x_n$, which are supposed to be independent in context, as well as their ordered values in increasing order, denoted by $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$.

4.1. OLSEs and WLSEs

The OLSE and WLSE methods were first used to estimate the parameters of the beta distribution (see [36]). To obtain the corresponding parameter estimates for the E-B distribution, we consider the following objective functions:

$$OLSEs = \sum_{i=1}^{n} [\ F_{E-B}\,(x_{i:n}) - \frac{i}{n+1}]^2 = \sum_{i=1}^{n} \left[\left(1 - \left(\frac{x_{i:n}}{\sigma}\right)^{-\beta \log\left(\frac{x_{i:n}}{\sigma}\right)} \right)^{\gamma} - \frac{i}{n+1} \right]^2 \tag{25}$$

and

$$\begin{aligned} WLSEs &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{E-B}\,(x_{i:n}) - \frac{i}{n+1} \right]^2 \\ &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\left(1 - \left(\frac{x_{i:n}}{\sigma}\right)^{-\beta \log\left(\frac{x_{i:n}}{\sigma}\right)} \right)^{\gamma} - \frac{i}{n+1} \right]^2. \end{aligned} \tag{26}$$

Minimizing these function give the OLSEs and WLSEs of the parameters, respectively.

4.2. MLEs

Based on Eq. (7), the likelihood function of the E-B distribution is given by

$$L(\beta, \gamma, \sigma) = \prod_{i=1}^{n} \left\{ \gamma \frac{2\beta}{x_i} \left(\frac{x_i}{\sigma}\right)^{-\beta \log\left(\frac{x_i}{\sigma}\right)} \log\left(\frac{x_i}{\sigma}\right) \left[1 - \left(\frac{x_i}{\sigma}\right)^{-\beta \log\left(\frac{x_i}{\sigma}\right)} \right]^{\gamma-1} \right\} \tag{27}$$

and the log-likelihood function is defined as follows:

$$\begin{aligned} \log L(\beta, \gamma, \sigma) = {} & n \log\gamma + n\log 2 + n\log\beta - \sum_{i=1}^{n} \log x_i - \beta \sum_{i=1}^{n} \left[\log\left(\frac{x_i}{\sigma}\right) \right]^2 \\ & + \sum_{i=1}^{n} \log\left[\log\left(\frac{x_i}{\sigma}\right) \right] + (\gamma - 1) \sum_{i=1}^{n} \log\left[1 - \left(\frac{x_i}{\sigma}\right)^{-\beta \log\left(\frac{x_i}{\sigma}\right)} \right]. \end{aligned} \tag{28}$$

The MLEs are obtained by maximizing this function. Then the known theory can be applied to this method. In this respect, the book by [26] can be useful.

4.3. MPSEs

The MPSE method is used as an alternative to the MLE approach. The corresponding objective function is given as

$$\begin{aligned} MPSEs &= \frac{1}{n+1} \sum_{i=1}^{n} \log\left[F_{E-B}\,(x_{i:n}) - F_{E-B}(x_{i:n-1}) \right] \\ &= \frac{1}{n+1} \sum_{i=1}^{n} \log\left\{ \left[1 - \left(\frac{x_{i:n}}{\sigma}\right)^{-\beta \log\left(\frac{x_{i:n}}{\sigma}\right)} \right]^{\gamma} - \left[1 - \left(\frac{x_{i:n-1}}{\sigma}\right)^{-\beta \log\left(\frac{x_{i:n-1}}{\sigma}\right)} \right]^{\gamma} \right\}. \end{aligned} \tag{29}$$

The MPSEs are obtained by maximizing this function.

4.4. ADEs, RTADEs and LTADEs

The Anderson-Darling method is a form of minimum distance estimation achieved by minimizing an Anderson-Darling statistic, as referenced in [18]. For the E-B distribution, we can write the objective functions related to the ADEs, RTADEs and LTADEs as follows:

$$ADEs = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log[F_{E-B}(x_{i:n})] + \log[S_{E-B}(x_{i:n})]\right]$$

$$= n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma} + \log\left(1-\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma}\right)\right),$$

$$RTADEs = \frac{n}{2} - 2\sum_{i=1}^{n} F_{E-B}(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log[S_{E-B}(x_{i:n})]$$

$$= \frac{n}{2} - 2\sum_{i=1}^{n}\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1-\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma}\right)$$

and

$$LTADEs = -\frac{3}{2}n + 2\sum_{i=1}^{n} F_{E-B}(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log[F_{E-B}(x_{i:n})]$$

$$= -\frac{3}{2}n + 2\sum_{i=1}^{n}\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right]^{\gamma}.$$

The ADEs, RTADEs and LTADEs are obtained by minimizing these functions, respectively.

4.5. CVMEs

The CVME method was first introduced by [35]. To obtain the parameter estimates for the E-B distribution, we consider the following objective function:

$$CVMEs = CV = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{E-B}(x_{i:n}) - \frac{2i-1}{2n}\right]^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left(1-\left(\frac{x_{i:n}}{\sigma}\right)^{-\beta\log\left(\frac{x_{i:n}}{\sigma}\right)}\right)^{\gamma} - \frac{2i-1}{2n}\right]^2.$$

The CVMEs are obtained by minimizing this function.

5. Numerical simulation

This section is devoted to the simulation work, which is considered a crucial aspect of our research. Our aim is to evaluate the effectiveness of different estimation methods proposed to estimate the parameters of the E-B model by analyzing detailed simulation results. To achieve this goal, we have conducted a series of experiments using different sample sizes ($n = 10, 30, 50, 100, 250$) and different parameter values, including $\sigma = \{3,\ 4,\ 0.75\}$, $\beta = \{0.5,\ 0.75,\ 0.8\}$ and $\gamma = \{2,\ 0.5,\ 4.5\}$. In addition, we generated $N = 5000$ random samples from the E-B model using Eq. (2) to obtain a substantial data set in the R programming language. Finally, we computed the average absolute biases (|BIAS|), root mean square errors (RMSEs), and mean relative

Table 2. The results of the simulation for $\sigma = 3$, $\beta = 0.5$ and $\gamma = 2$.

		|Bias|			RMSE			MRSE		
Method	n	β	σ	γ	β	σ	γ	β	σ	γ
MLEs	10	2.1090	1.6009	0.6535	4.8746	2.6113	2.0677	4.2287	0.7030	1.1252
	30	1.9800	1.5921	0.5600	4.7236	2.5564	2.0515	4.2072	0.6600	1.0766
	50	1.8810	1.5785	0.5547	4.5882	2.5161	1.9977	4.0884	0.6270	1.0571
	100	1.6095	1.5529	0.5441	4.3789	2.4662	1.8495	3.7649	0.6182	0.9476
	250	1.5100	1.5213	0.5191	4.1311	2.4274	1.7649	3.2191	0.5033	0.9404
OLSEs	10	0.0499	0.3874	4.2576	1.5067	2.6112	7.4713	0.0998	0.0166	0.0250
	30	0.0386	0.1746	2.3141	0.1589	2.0236	6.4468	0.0772	0.0129	0.0393
	50	0.0261	0.1471	1.5764	0.1370	1.8226	4.0175	0.0633	0.0105	0.0158
	100	0.0202	0.0926	1.1976	0.1108	1.5935	3.0564	0.0403	0.0067	0.0101
	250	0.0094	0.0576	0.7466	0.0906	1.3550	2.1848	0.0187	0.0031	0.0047
MPSEs	10	0.1152	0.6932	9.3154	0.2563	2.9172	8.4914	0.2303	0.0384	0.0576
	30	0.1086	0.6596	9.1443	0.1900	2.1193	6.9750	0.2172	0.0362	0.0543
	50	0.1032	0.4871	8.7425	0.1772	1.8584	5.9819	0.1980	0.0330	0.0495
	100	0.0948	0.3585	8.9106	0.1623	1.6776	5.7305	0.1897	0.0310	0.0474
	250	0.0811	0.2481	8.2351	0.1370	1.3505	3.0395	0.1621	0.0270	0.0405
CVMEs	10	0.1530	0.8546	4.9505	0.5683	2.4585	8.3091	0.3060	0.0510	0.0765
	30	0.0213	0.4443	1.9305	0.1746	1.9124	5.5598	0.0427	0.0071	0.0107
	50	0.0088	0.3077	1.3632	0.1395	1.7498	3.7399	0.0176	0.0048	0.0044
	100	0.0064	0.1951	0.9593	0.1084	1.5165	2.7577	0.0127	0.0021	0.0016
	250	0.0050	0.0525	0.7814	0.0855	1.2967	2.1779	0.0100	0.0006	0.0009
WLEs	10	0.0465	0.2221	5.0411	0.3346	2.5669	9.4191	0.0979	0.0163	0.0245
	30	0.0419	0.1347	3.5029	0.1576	1.9382	7.8403	0.0837	0.0140	0.0209
	50	0.0378	0.1242	2.3282	0.1334	1.7122	5.1689	0.0756	0.0126	0.0189
	100	0.0270	0.1121	1.8584	0.1142	1.5457	4.2540	0.0541	0.0105	0.0176
	250	0.0249	0.0889	1.4021	0.0903	1.2332	3.0722	0.0498	0.0083	0.0163
ADEs	10	0.0343	0.8097	4.7153	0.2763	2.6831	9.1744	0.0685	0.0107	0.0161
	30	0.0311	0.3878	4.5766	0.1567	1.9920	6.5365	0.0641	0.0099	0.0153
	50	0.0296	0.2868	2.6218	0.1349	1.6843	5.7393	0.0594	0.0090	0.0140
	100	0.0250	0.2416	2.4409	0.1132	1.4145	5.4940	0.0454	0.0088	0.0135
	250	0.0227	0.1218	1.4422	0.0869	1.1250	3.1976	0.0421	0.0076	0.0113
RTADEs	10	0.1140	0.9225	7.5830	0.5514	2.6994	9.2141	0.2280	0.0380	0.0570
	30	0.0262	0.2589	6.2315	0.1940	2.1281	8.3577	0.0524	0.0092	0.0126
	50	0.0229	0.1711	5.5292	0.1451	1.9182	7.1611	0.0466	0.0084	0.0107
	100	0.0198	0.0640	3.8545	0.1243	1.7453	5.3224	0.0382	0.0078	0.0099
	250	0.0160	0.0250	1.9740	0.1023	1.5186	4.9145	0.0320	0.0062	0.0080
LTADEs	10	2.1426	1.7725	9.2849	3.5955	2.2448	7.4134	3.3992	0.5665	0.8498
	30	1.8565	1.4144	8.8881	3.2023	2.1683	7.2533	2.8025	0.4667	0.7004
	50	1.7443	1.3349	7.5953	3.0648	2.0649	6.9407	2.3798	0.1746	0.4471
	100	1.6996	1.3220	6.9194	2.5485	1.6988	6.8602	1.7886	0.0442	0.3698
	250	1.4012	1.1290	6.5286	2.3000	1.3138	5.9945	1.0478	0.0130	0.2269

errors (MREs) for all parameters using the following formulas:

$$|BIAS| = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta_i} - \theta\right),$$

$$RMSEs = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta_i} - \theta\right)^2}$$

and

$$MRSEs = \frac{1}{N}\sum_{i=1}^{N}\frac{\widehat{\theta_i} - \theta}{\theta},$$

Table 3. The results of the simulation for $\sigma = 4$, $\beta = 0.75$ and $\gamma = 0.5$.

		|BIAS|			*RMSEs*			*MRSEs*		
Method	n	β	σ	γ	β	σ	γ	β	σ	γ
MLEs	10	0.1984	1.7584	0.6094	1.6262	3.2186	3.3067	0.2646	0.0496	0.3969
	30	0.1359	1.5602	0.5276	0.9779	2.8970	2.9131	0.2646	0.0340	0.2719
	50	0.1093	1.4805	0.2271	0.7810	1.7440	1.9655	0.1915	0.0287	0.2186
	100	0.0768	1.4378	0.1390	0.6401	1.5108	1.1808	0.1457	0.0128	0.1536
	250	0.0069	1.3631	0.0513	0.5243	1.3634	0.9317	0.1024	0.0033	0.1025
OLSEs	10	0.0407	0.4579	0.7704	0.4245	1.0357	2.4111	0.0543	0.0102	0.0814
	30	0.0337	0.4050	0.6851	0.3029	0.9458	2.2944	0.0449	0.0084	0.0674
	50	0.0233	0.2585	0.2904	0.2136	0.7005	1.1423	0.0311	0.0058	0.0466
	100	0.0107	0.1013	0.0787	0.1633	0.3661	0.4432	0.0143	0.0027	0.0214
	250	0.0063	0.0239	0.0137	0.1012	0.1420	0.0845	0.0056	0.0006	0.0083
MPSEs	10	0.1590	0.3695	4.2546	0.4090	1.2781	2.8590	0.2120	0.0397	0.3179
	30	0.0780	0.1945	1.4240	0.2380	0.4934	2.4903	0.1040	0.0195	0.1560
	50	0.0470	0.0121	0.1257	0.1770	0.1483	0.1386	0.0627	0.0118	0.0941
	100	0.0326	0.0045	0.0146	0.1240	0.0724	0.0816	0.0478	0.0082	0.0653
	250	0.0181	0.0019	0.0077	0.0809	0.0278	0.0447	0.0200	0.0038	0.0300
CVMEs	10	0.3543	0.4304	1.8940	1.9305	1.3079	8.3434	0.4724	0.0886	0.7086
	30	0.0789	0.3056	0.4593	0.3073	0.8270	1.3829	0.0630	0.0118	0.0946
	50	0.0314	0.2349	0.3055	0.2309	0.6668	1.2229	0.0419	0.0092	0.0629
	100	0.0189	0.0782	0.0717	0.1597	0.3394	0.3815	0.0252	0.0058	0.0378
	250	0.0061	0.0272	0.0191	0.0989	0.1360	0.0837	0.0084	0.0016	0.0126
WLEs	10	0.0623	0.6207	1.6290	0.6769	1.3685	5.1974	0.0830	0.0156	0.1245
	30	0.0154	0.2644	0.4388	0.2809	0.7071	1.9754	0.0223	0.0042	0.0334
	50	0.0069	0.1091	0.2154	0.1983	0.5139	1.0711	0.0172	0.0039	0.0258
	100	0.0037	0.0245	0.0258	0.1629	0.1678	0.1465	0.0096	0.0018	0.0143
	250	0.0029	0.0059	0.0029	0.0855	0.0471	0.0488	0.0039	0.0007	0.0058
ADEs	10	0.0422	0.1882	1.1640	0.5572	1.1518	5.2838	0.0562	0.0105	0.0844
	30	0.0261	0.1293	0.2785	0.2511	0.5901	1.7524	0.0279	0.0046	0.0419
	50	0.0142	0.0768	0.1623	0.1983	0.4176	1.5191	0.0160	0.0035	0.0284
	100	0.0103	0.0215	0.0243	0.1278	0.1669	0.1401	0.0144	0.0027	0.0205
	250	0.0056	0.0060	0.0051	0.0853	0.0581	0.0517	0.0075	0.0012	0.0096
RTADEs	10	0.1509	0.3676	1.3323	0.8840	1.2295	6.3512	0.2012	0.0377	0.3018
	30	0.0125	0.2698	0.4954	0.3407	0.9411	4.6861	0.0103	0.0019	0.0250
	50	0.0077	0.2006	0.3235	0.2080	0.7864	2.4145	0.0098	0.0012	0.0146
	100	0.0061	0.1441	0.1639	0.1324	0.5129	0.9414	0.0049	0.0009	0.0122
	250	0.0021	0.0349	0.0206	0.0879	0.2100	0.1115	0.0029	0.0005	0.0043
LTADEs	10	0.4017	3.1181	3.4659	0.8592	1.6805	3.8806	0.5356	0.1004	0.6088
	30	0.3044	2.9873	2.6019	0.7902	1.5942	2.5672	0.3764	0.0776	0.5646
	50	0.2227	2.6780	2.3385	0.7125	1.5188	2.3874	0.2019	0.0627	0.4548
	100	0.1070	2.4620	1.5369	0.6716	1.4872	1.9876	0.1356	0.0557	0.3663
	250	0.0564	2.0628	1.1697	0.6197	1.3566	1.0151	0.0437	0.0398	0.3188

where the index i refers to the i-th sample, θ is the unknown parameter considered and $\widehat{\theta_i}$ is the indicated estimate of θ based on the i-th sample.

5.1. Simulation algorithm

1- Provide the initial parameter values.
2- Create a random sample of size 'n' using the inverse cdf described in Eq. (8).
3- Assess the estimates obtained from different estimation methods.
4- Calculate BIAS, MSEs and MREs for all parameters obtained using various estimation methods.

5.2. Simulation result

From Tables 2 to 4 and Fig. 4, we see that for different parameter values of the E-B distribution, all parameter estimation methods give better results (smaller side and RMSE values) in large samples compared to small samples.

Table 4. The results of the simulation for $\sigma = 3$, $\beta = 0.5$ and $\gamma = 2$.

		|BIAS|			*RMSEs*			*MRSEs*		
Method	n	β	σ	γ	β	σ	γ	β	σ	γ
MLEs	10	0.2199	0.2034	0.6699	0.8936	0.7413	1.6851	0.2749	0.2932	0.0489
	30	0.2029	0.1894	0.6577	0.8001	0.7260	1.6712	0.2537	0.2706	0.0451
	50	0.1515	0.1820	0.6428	0.7467	0.7064	1.6633	0.2110	0.2020	0.0337
	100	0.1179	0.1769	0.6359	0.7177	0.6910	1.6226	0.1917	0.1754	0.0330
	250	0.0957	0.1583	0.5949	0.6869	0.6669	1.5455	0.1229	0.1311	0.0218
OLSEs	10	0.7995	0.2858	0.2057	0.7996	0.8640	0.8371	0.9995	1.0661	0.1777
	30	0.7990	0.2353	0.1686	0.7991	0.8589	0.8297	0.9991	1.0657	0.1772
	50	0.7985	0.2002	0.1241	0.7966	0.8481	0.8175	0.9942	1.0630	0.1767
	100	0.7870	0.1303	0.0647	0.7903	0.8310	0.7986	0.9837	1.0493	0.1749
	250	0.7752	0.1115	0.0336	0.7859	0.8257	0.7802	0.9690	1.0336	0.1723
MPSEs	10	0.1952	0.2143	9.5216	0.4999	0.7468	8.7927	0.2440	0.2603	0.0434
	30	0.1862	0.1964	8.4273	0.3880	0.5966	8.1044	0.2328	0.2483	0.0410
	50	0.1736	0.1864	7.2801	0.3549	0.5510	7.3788	0.2170	0.2315	0.0386
	100	0.1586	0.1706	6.7554	0.3195	0.4858	6.4774	0.1983	0.2115	0.0353
	250	0.1422	0.1693	6.4797	0.2586	0.4109	6.0326	0.1778	0.1896	0.0316
CVMEs	10	0.1436	0.9547	5.1020	0.4810	2.5069	9.4984	0.2871	0.0479	0.0718
	30	0.0285	0.5723	1.7581	0.1762	1.9739	5.2974	0.0570	0.0095	0.0142
	50	0.0141	0.4130	1.4403	0.1595	1.7578	4.6527	0.0282	0.0047	0.0070
	100	0.0069	0.1776	1.0478	0.1062	1.5180	2.7972	0.0138	0.0023	0.0034
	250	0.0028	0.1557	0.6664	0.0911	1.3242	2.1964	0.0056	0.0009	0.0014
WLEs	10	0.1021	0.1670	24.7303	0.6944	0.5374	32.4495	0.1276	0.1361	0.0227
	30	0.0985	0.1617	20.0856	0.3262	0.4357	30.7602	0.1236	0.1314	0.0219
	50	0.0747	0.1050	15.0517	0.2839	0.4322	26.3758	0.0934	0.0996	0.0166
	100	0.0691	0.0975	11.8587	0.2470	0.4120	20.1116	0.0864	0.0921	0.0154
	250	0.0471	0.0555	6.8245	0.2005	0.3507	12.6073	0.0708	0.0755	0.0126
ADEs	10	0.0939	0.1353	17.0768	0.5869	0.6653	24.3289	0.1174	0.1252	0.0209
	30	0.0731	0.0969	14.1305	0.3200	0.4823	23.0001	0.0988	0.0913	0.0176
	50	0.0669	0.0894	11.7996	0.2824	0.4520	19.6600	0.0836	0.0892	0.0153
	100	0.0648	0.0877	9.3125	0.2407	0.4006	16.5649	0.0811	0.0863	0.0144
	250	0.0569	0.0817	7.6071	0.2009	0.3444	14.5147	0.0712	0.0759	0.0127
RTADEs	10	0.2622	0.0687	15.7062	1.1964	0.7727	24.0205	0.3277	0.3495	0.0583
	30	0.0516	0.0613	14.0218	0.4664	0.5933	23.8581	0.0645	0.0707	0.0115
	50	0.0472	0.0413	11.3686	0.3507	0.5393	21.3873	0.0590	0.0629	0.0086
	100	0.0430	0.0357	10.7343	0.2950	0.4890	18.4095	0.0413	0.0518	0.0073
	250	0.0311	0.0329	9.5612	0.2513	0.4460	17.4455	0.0389	0.0415	0.0069
LTADEs	10	0.7965	0.9050	0.0616	0.7964	0.8430	0.7965	0.9956	1.0620	0.1770
	30	0.7960	0.8453	0.0610	0.7962	0.7976	0.7959	0.9948	1.0617	0.1769
	50	0.7959	0.7898	0.0600	0.7955	0.7966	0.7951	0.9944	1.0607	0.1767
	100	0.7955	0.6906	0.0597	0.7952	0.7965	0.7949	0.9936	1.0598	0.1767
	250	0.7845	0.5910	0.0584	0.7945	0.7955	0.7945	0.9932	1.0594	0.1766

Table 5. Descriptive statistics of the two data sets.

Data	Min	Q_1	Median	Mean	Q_3	Max.
I	17.88	47.20	67.80	73.85	101.88	173.40
II	4.10	8.45	10.60	13.49	16.85	39.20

We also found that the CVME, WLE and ADE methods for different values of the parameter β, the WLE, ADE, CVME and RTADE methods for all different values of the parameter σ, and the MLE, OLSE and CVME methods for all different values of the parameter γ give less biased estimates than the other methods. As a result, we can say that the CVME approach gives less biased estimates for all parameters compared to other parameter estimation methods.

Table 6. Numerical values for analyzing the first data set.

Model	A_1	A_2	A_3	A_4	D_1	D_2	D_3	$D_3(p)$	Est. parameters (SEs)
EB	236.102	237.365	239.508	236.959	0.36333	0.0492655	0.101799	0.970988	$\hat{\gamma} = 4.70893$ (7.62629) $\hat{\beta} = 0.431846$ (0.225768) $\hat{\sigma} = 7.21743$ (9.40747)
B	238.972	239.572	241.243	239.543	1.51977	0.290379	0.239414	0.143144	$\hat{\beta} = 0.383349$ (0.127748) $\hat{\sigma} = 13.8546$ (3.0486)
RL	245.888	246.079	247.024	246.174	2.72797	0.512763	0.299685	0.0321232	$\hat{\lambda} = 72.8116$ (15.4004)
IP	248.624	249.224	250.895	249.195	2.58989	0.483118	0.309539	0.0243718	$\hat{\alpha} = 53452.5$ (1.15541×10^6) $\hat{\theta} = 0.00104305$ (0.0225464)
P	251.376	251.567	252.512	251.662	3.66263	0.726424	0.35033	0.00706576	$\hat{\alpha} = 0.777412$ (0.162102)
PTI	305.472	305.663	306.608	305.758	8.08562	1.73726	0.510306	0.0000125494	$\hat{\alpha} = 0.239841$ (0.0500103)
PTII	247.892	248.492	250.163	248.463	2.72738	0.51258	0.299636	0.0321665	$\hat{\alpha} = 1.29098 \times 10^7$ (749.197) $\hat{\theta} = 9.53351 \times 10^8$ (10914×10^{-10})
BP	253.107	254.371	256.514	253.964	2.62449	0.4515	0.286575	0.0457477	$\hat{\gamma} = 1.53572$ (0.412996) $\hat{\beta} = 282.791$ (9333.69) $\hat{\sigma} = 0.00421776$ (0.139078)
WP	245.027	245.627	247.298	245.599	1.51882	0.206096	0.205353	0.286608	$\hat{\alpha} = 0.726655$ (0.0846417) $\hat{\theta} = 1.8237$ (0.334853)

Table 7. Numerical values for analyzing the second data set.

Model	A_1	A_2	A_3	A_4	D_1	D_2	D_3	$D_3(p)$	Est. parameters (SEs)
EB	382.33	382.766	388.562	384.763	0.273084	0.0443203	0.0622399	0.976268	$\hat{\gamma} = 1.76914$ (1.42196) $\hat{\beta} = 0.724401$ (0.147663) $\hat{\sigma} = 3.20966$ (1.36783)
B	409.107	409.321	413.262	410.729	13.1385	2.51183	0.322841	< 0.000001	$\hat{\beta} = 0.784942$ (0.234091) $\hat{\sigma} = 3.02028$ (0.462226)
RL	426.707	426.777	428.785	427.518	6.96233	1.32559	0.304805	0.00003467	$\hat{\lambda} = 12.2561$ (1.76747)
IP	426.646	426.86	430.801	428.268	7.51117	1.42369	0.292074	0.0000849	$\hat{\alpha} = 127587$ (1.03051×10^6) $\hat{\theta} = 0.0000811755$ (0.000655651)
P	416.146	416.216	418.223	416.957	6.72268	1.31179	0.273	0.00030315	$\hat{\alpha} = 0.95128$ (0.123846)
PTI	516.459	516.529	518.537	517.27	17.2069	3.63814	0.467417	< 0.000001	$\hat{\alpha} = 0.406545$ (0.0529276)
PTII	429.014	429.228	433.169	430.636	6.92151	1.3119	0.303445	0.0000383	$\hat{\alpha} = 1.56913 \times 10^7$ (515.7) $\hat{\theta} = 2.11653 \times 10^8$ (0.0000000245)
BP	396.888	397.324	403.12	399.321	1.09477	0.152467	0.123111	0.332872	$\hat{\gamma} = 2.58016$ (0.447702) $\hat{\beta} = 56.1482$ (264.018) $\hat{\sigma} = 0.0431205$ (0.200007)
WP	386.116	386.33	390.271	387.738	0.455329	0.0512657	0.0779402	0.866039	$\hat{\alpha} = 0.854343$ (0.0581366) $\hat{\theta} = 1.99664$ (0.20761)

6. Application

In this section, we demonstrate the application of the E-B distribution on two different real data sets to highlight its modelling behavior. We aim to show its consistency and reliability in dealing with different data sets. The first data set represents the number of millions of revolutions before failure for each of the 23 ball bearings in the life tests and was introduced by citeLawless. The second data set represents the monthly actual tax revenue in Egypt between January 2006 and November 2010 and introduced it citeNassar. The descriptive statistics of these data sets are given in Table 5.

The results of our research show the remarkable performance of the proposed model compared to other models that have been evaluated for their suitability in fitting the aforementioned data sets. The comparative models studied are those realted to the Benini (B) distribution by [17], the Ramous Louzada (RL) distribution by [30], the inverse Pareto (IP) distribution by [31], the Pareto (P) distribution by [32], the Pareto type I (PTI)

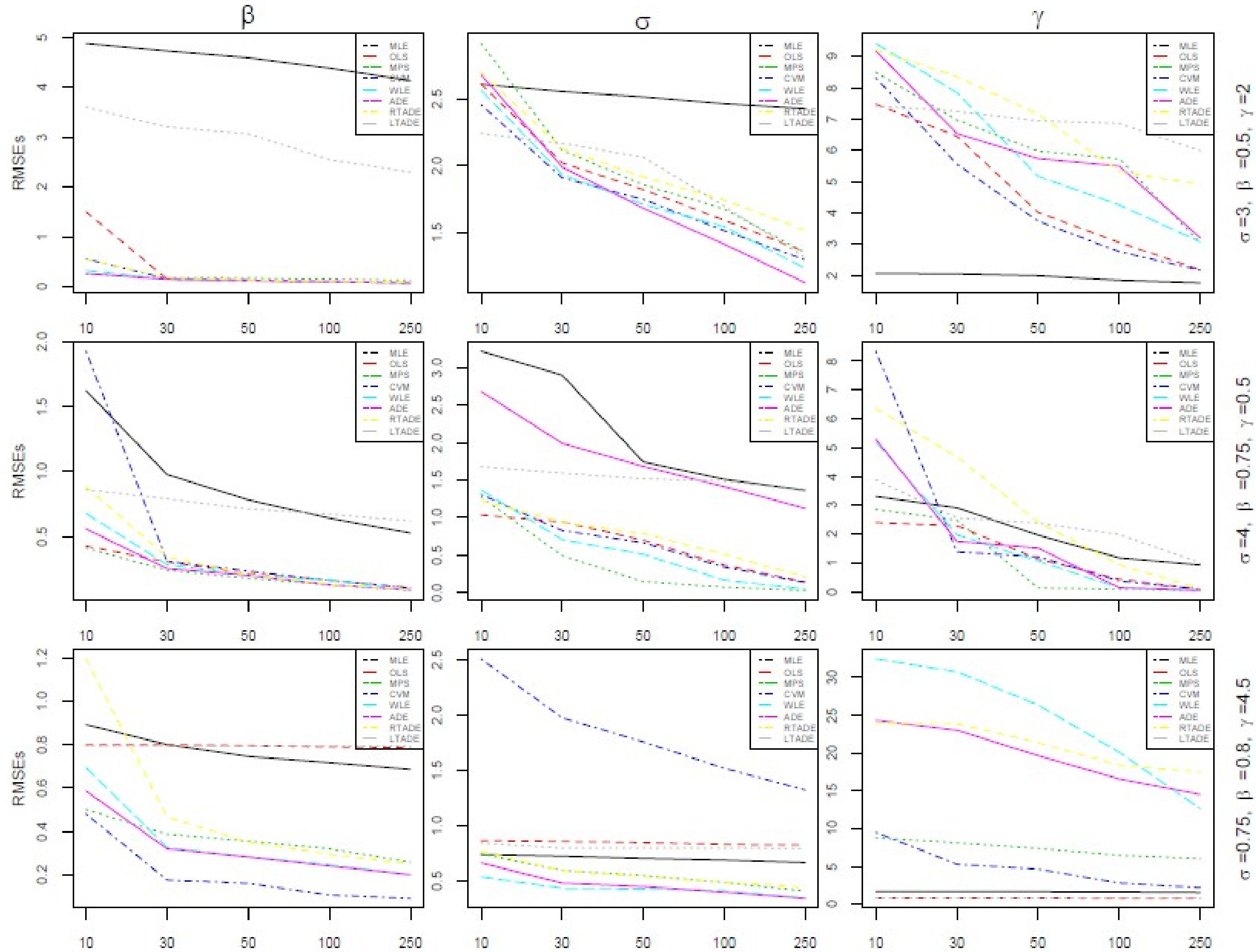

Fig. 4. RMSE values of the parameter estimation methods for different parameter values.

distribution by [32], the Pareto type II (PTII) distribution by [32], the beta Pareto (BP) distribution by [33], and the Weibull Pareto (WP) distribution by [34].

We focus on the MLEs for the parameter estimates, standard errors (SEs), and eight analytical information criteria and goodness-of-fit measures. Specifically, we investigate the effectiveness of four prominent analytical information criteria: Akaike information criterion (AIC) denoted as (A_1), corrected AIC (A_2), Bayesian information criterion (BIC - A_3), and Hannan-Quinn information criterion (HQIC or A_4). In addition, to ensure the robustness of the model assessments, we integrate several goodness-of-fit measures, including the Anderson-Darling measure (D_1), Cramér-von Mises measure (D_2), and Kolmogorov-Smirnov measure (D_3), along with the associated p-value ($D_3(p)$). By thoroughly evaluating these metrics, we aim to identify the most appropriate statistical models that effectively capture the underlying data patterns while accounting for model complexity.

Tables 6 and 7 present the parameter estimates, standard errors (SEs) and the eight performance measures. Based on the observed results, it can be inferred that our proposed model exhibits superior fitting performance compared to the equivalent models considered for these real data sets. These results indicate the effectiveness of our proposed model in capturing the underlying data patterns and emphasize its potential as a more suitable choice for accurately representing the complexities inherent in the respective data sets.

Fig. 5 displays the estimated pdfs of the considered models for the first data set. Similarly, Fig. 6 presents the same for the second data set. In Figs. 7 and 8, the probability-probability (P-P) plots illustrate the comparison of the proposed model with all other models for the first and second data sets, respectively. Figs. 9 and 10 show

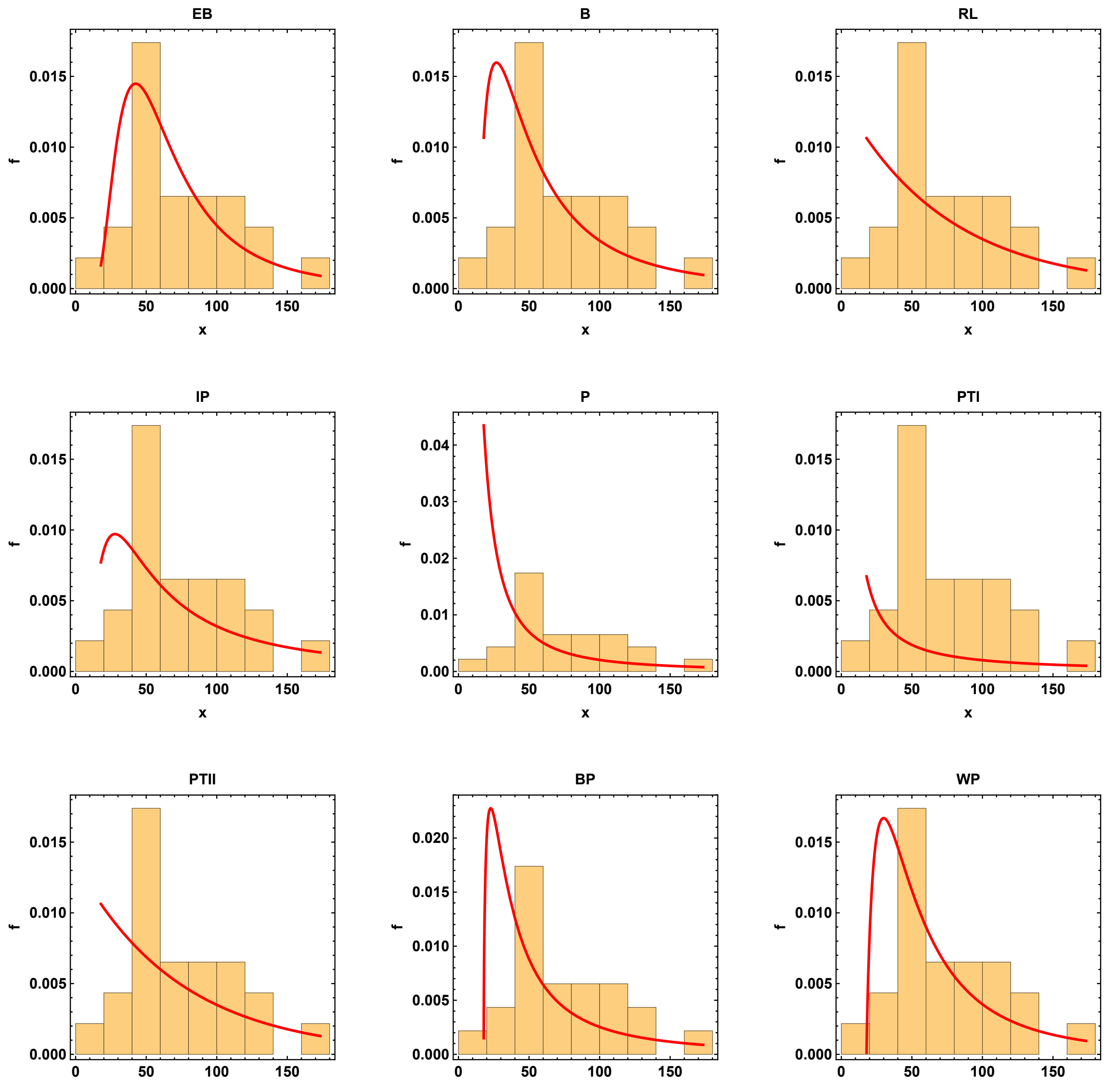

Fig. 5. Histogram of the first data set with the fitted pdfs of all compared models.

the empirical cdf plots for all models on the first and second data sets, respectively. Finally, Figs. 11 and 12 show the Kaplan-Meier survival curves and estimated sfs for both data sets, respectively.

7. Conclusion

Unlike most of the distribution articles in the literature, this article introduces a new distribution that will provide a significant advance in the field of statistics by deriving the three-parameter exponential Benini distribution, which has never been studied before. Thanks to its various flexible properties, the proposed distribution offers an alternative way to analyze real data sets in various fields, such as engineering, economics and natural sciences. Various classical estimate methods have been used to estimate unknown parameter values by introducing mathematical functions and properties of the distribution.

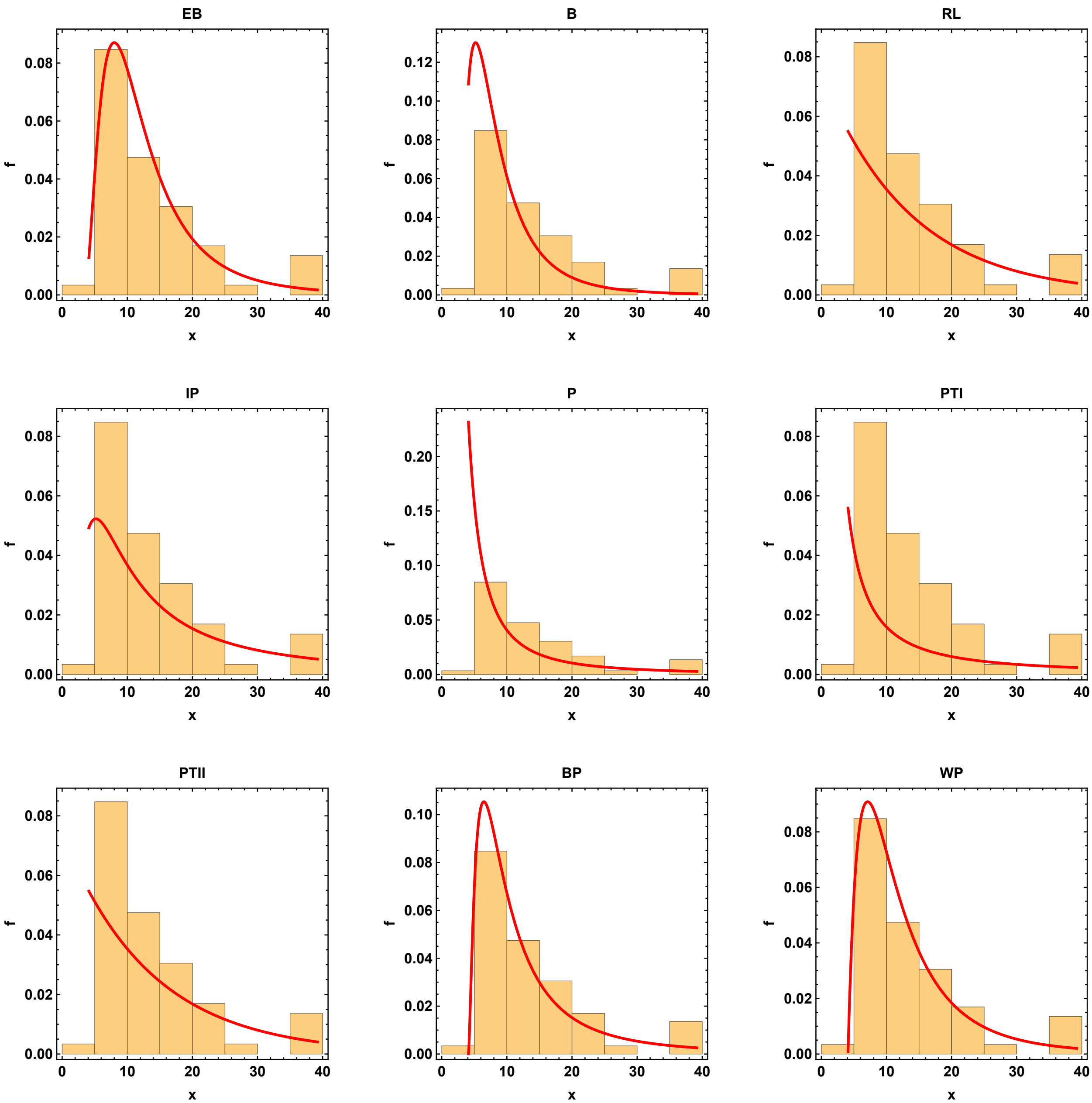

Fig. 6. Histogram of the second data set with the fitted pdfs of all compared models.

In particular, we examined the goodness of fit measures and prediction accuracy of the corresponding model on real data sets and found that it has superior performance compared to existing models. It can be said that the proposed distribution can be an important tool in analyzing different data structures encountered in real life, thanks to its ability to accurately model real data. Furthermore, considering the suitability of this new distribution for long-tailed data, it can be said that it can lead to significant advances in various areas of statistical modelling.

The mathematical model and prediction methods developed for the proposed distribution can be improved in several ways. The flexible structure of the distribution allows modelling in different data structures such as earthquakes, economic crises and waiting times. In addition, the unknown parameters can be estimated using different approaches, such as Bayesian statistics, and compared with classical approaches.

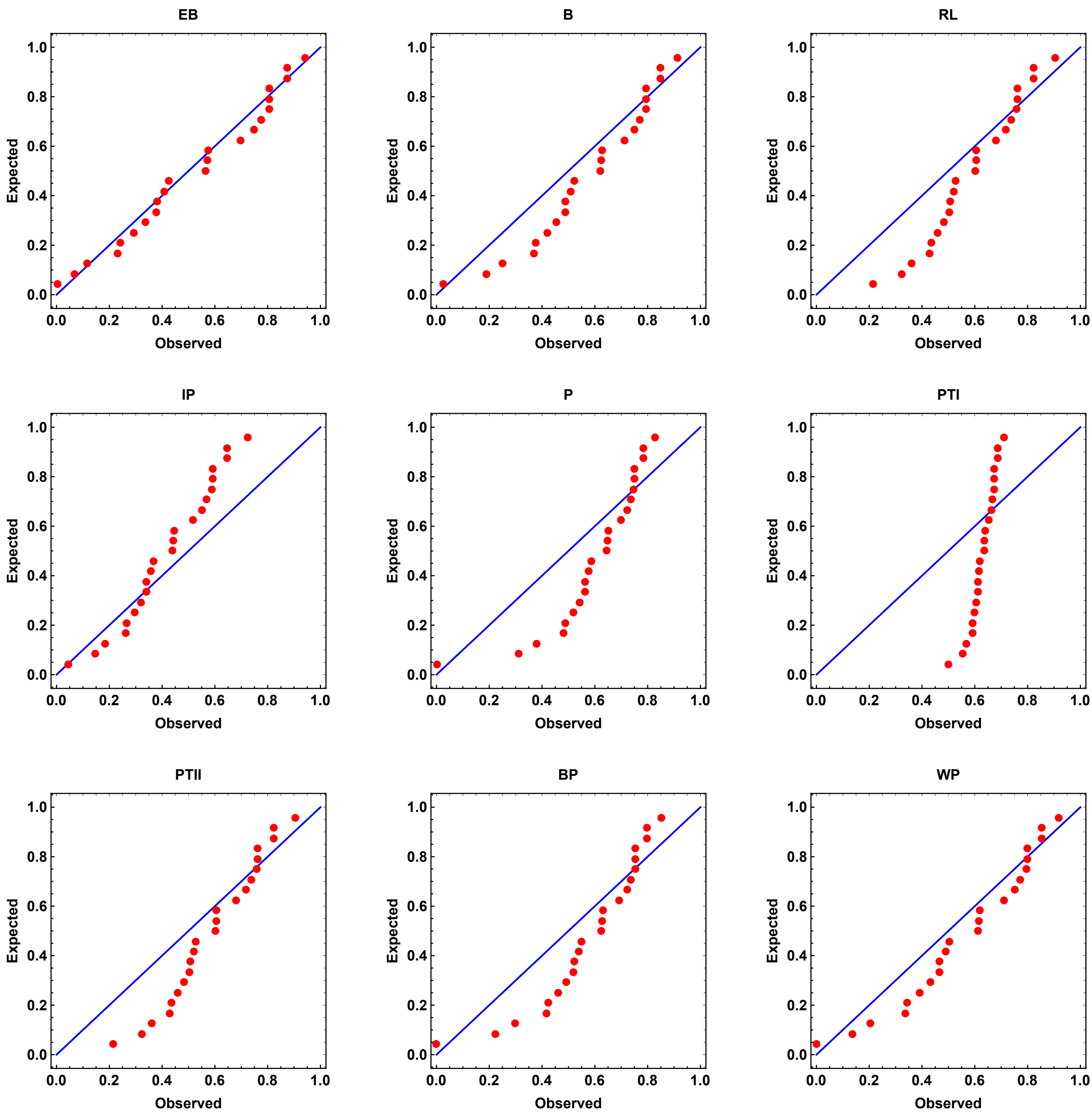

Fig. 7. The P-P plots of the fitted models for the first data set.

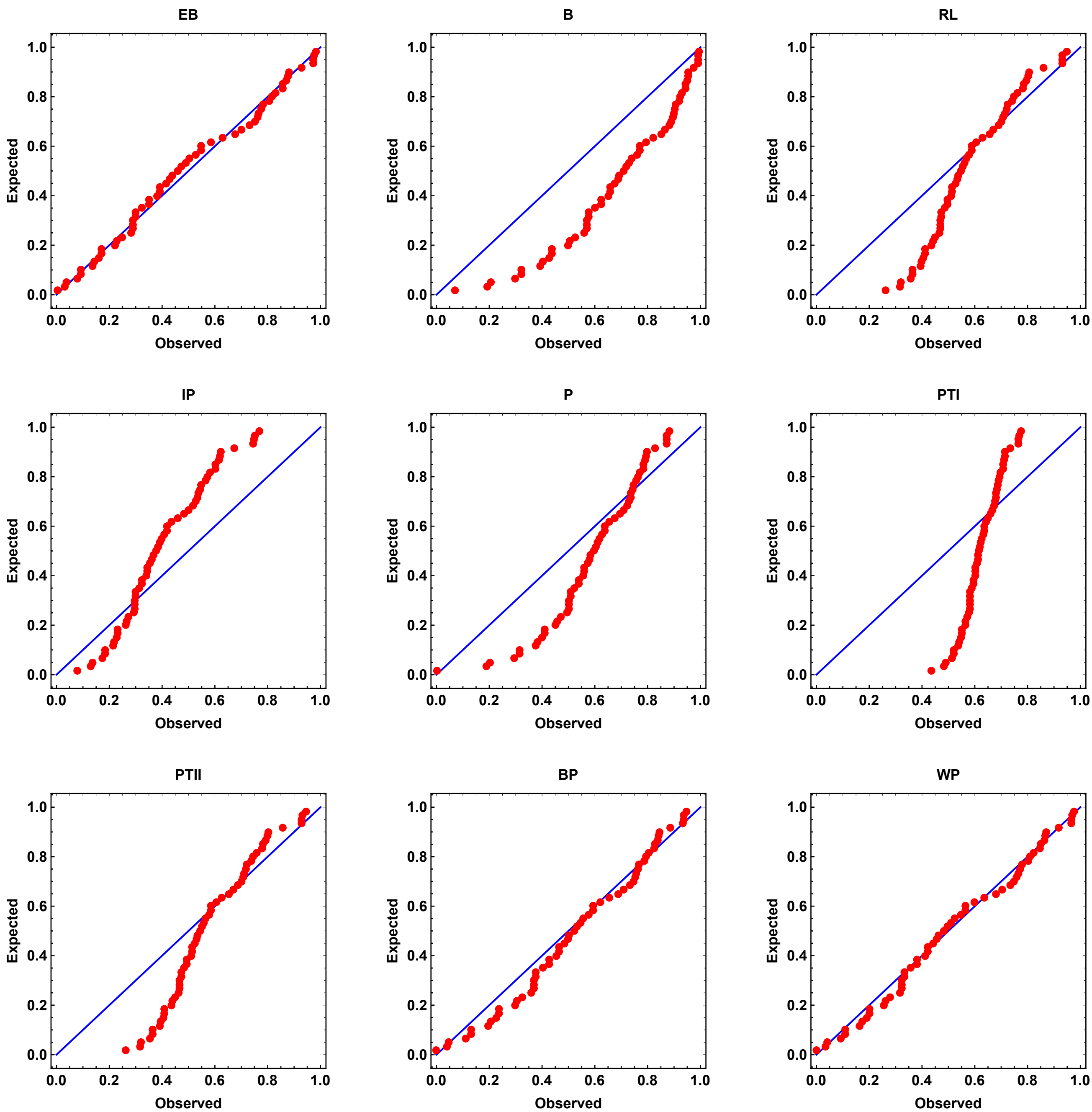

Fig. 8. The P-P plots of the fitted models for the second data set.

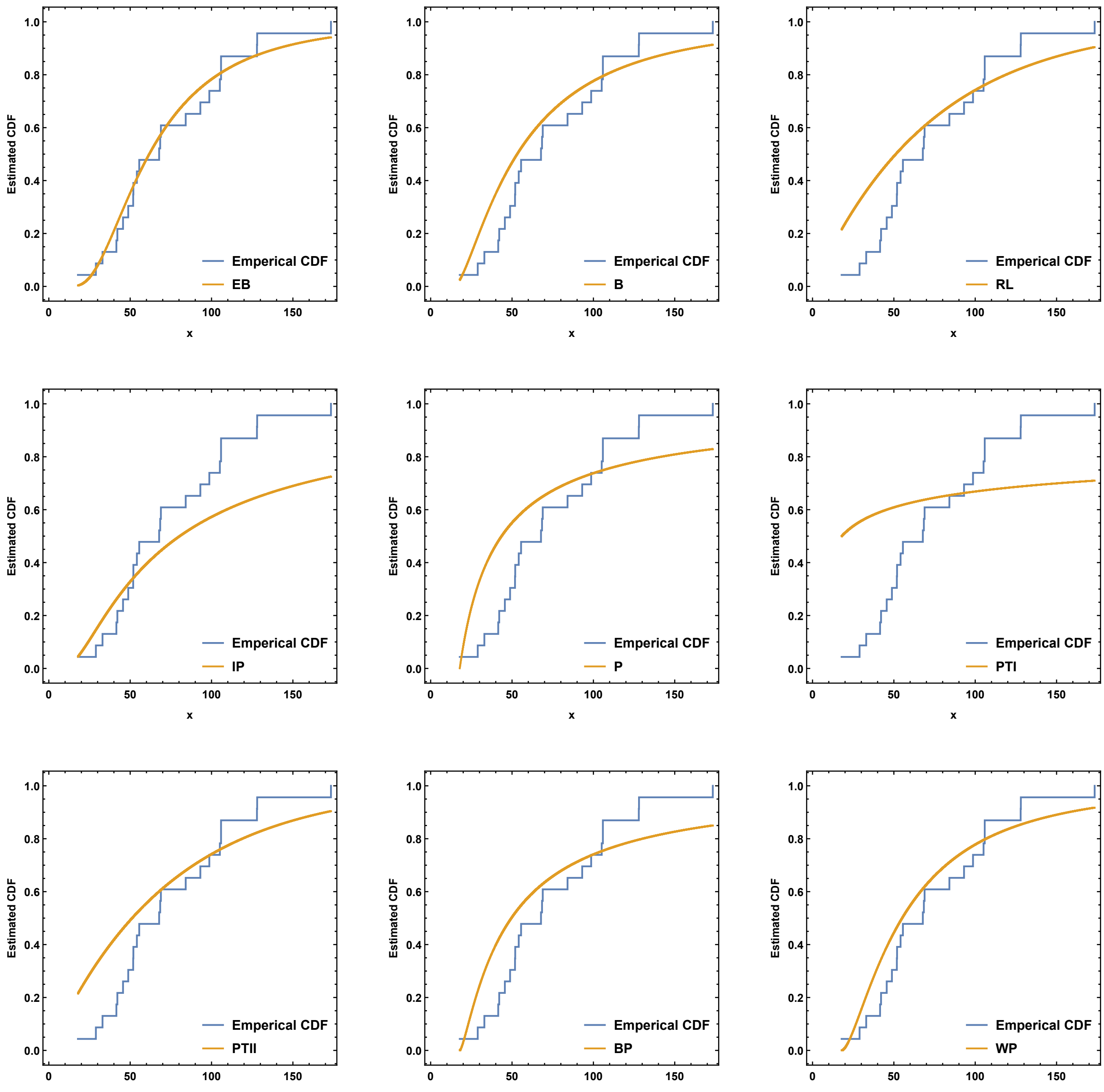

Fig. 9. Plots of the empirical cdfs with the fitted cdfs of all models for the first data set.

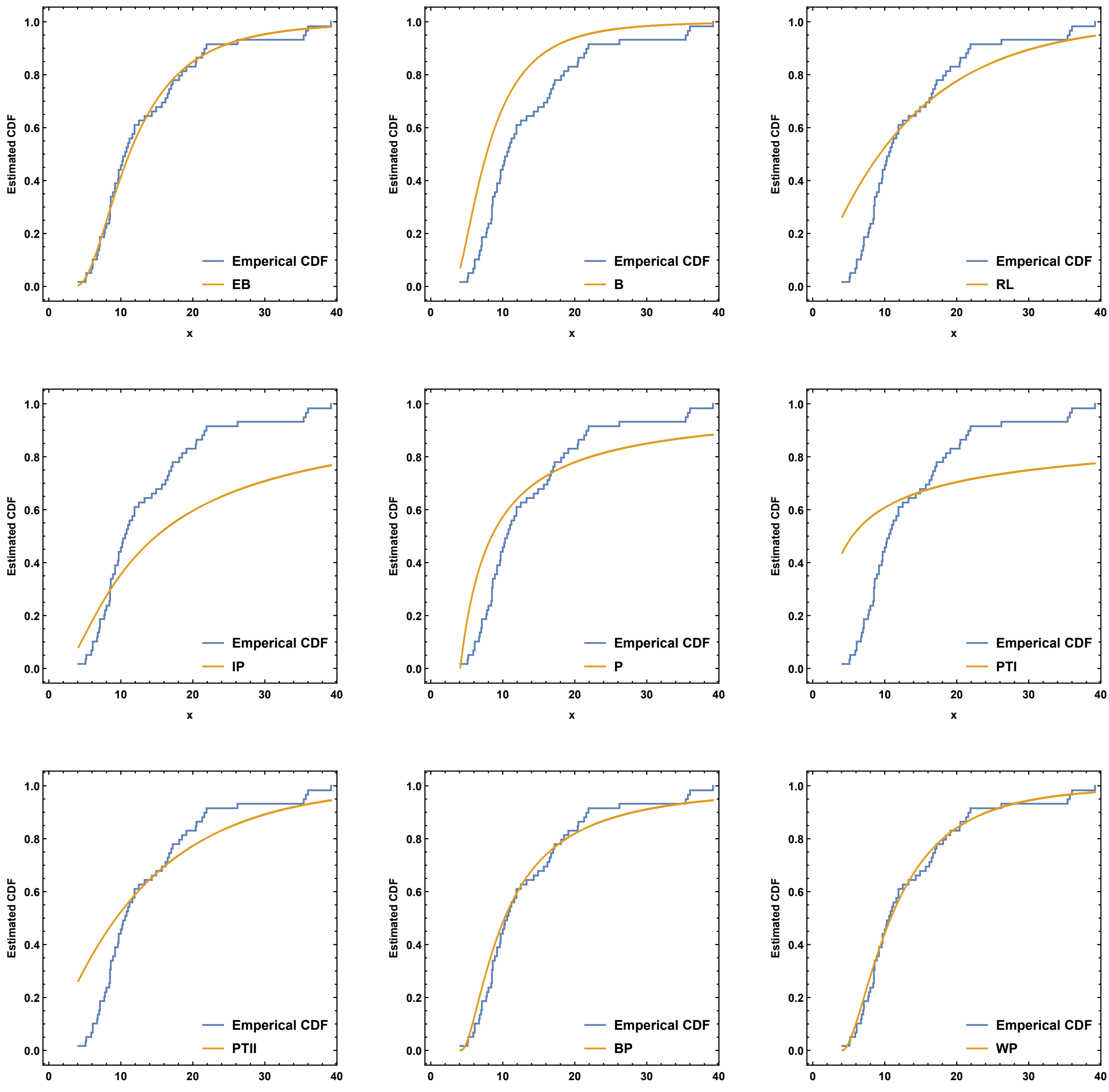

Fig. 10. Plots of the empirical cdfs with the fitted cdfs of all models for the second data set.

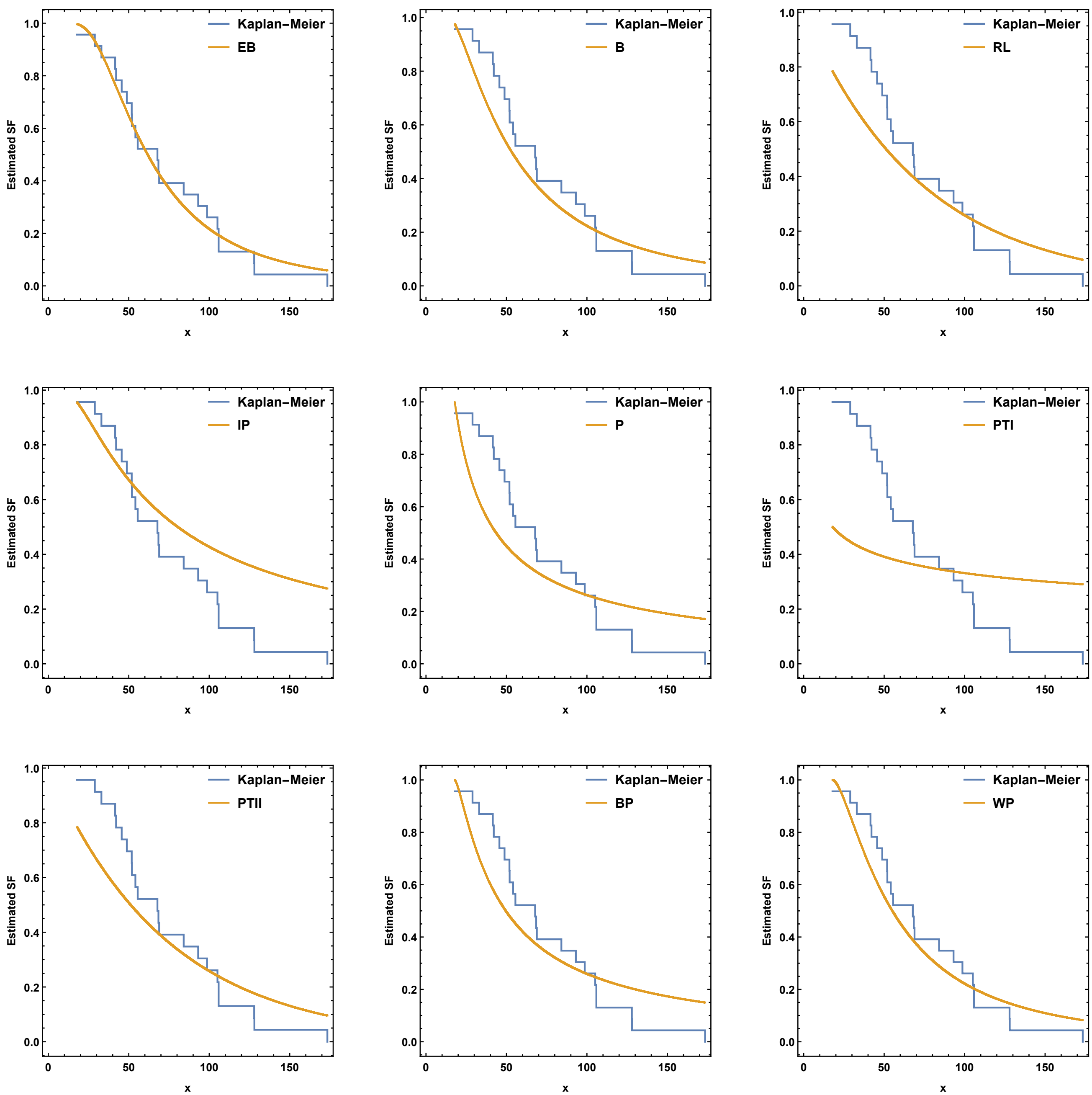

Fig. 11. Plots of the Kaplan-Meier survival functions with the fitted sfs of all models for the first data set.

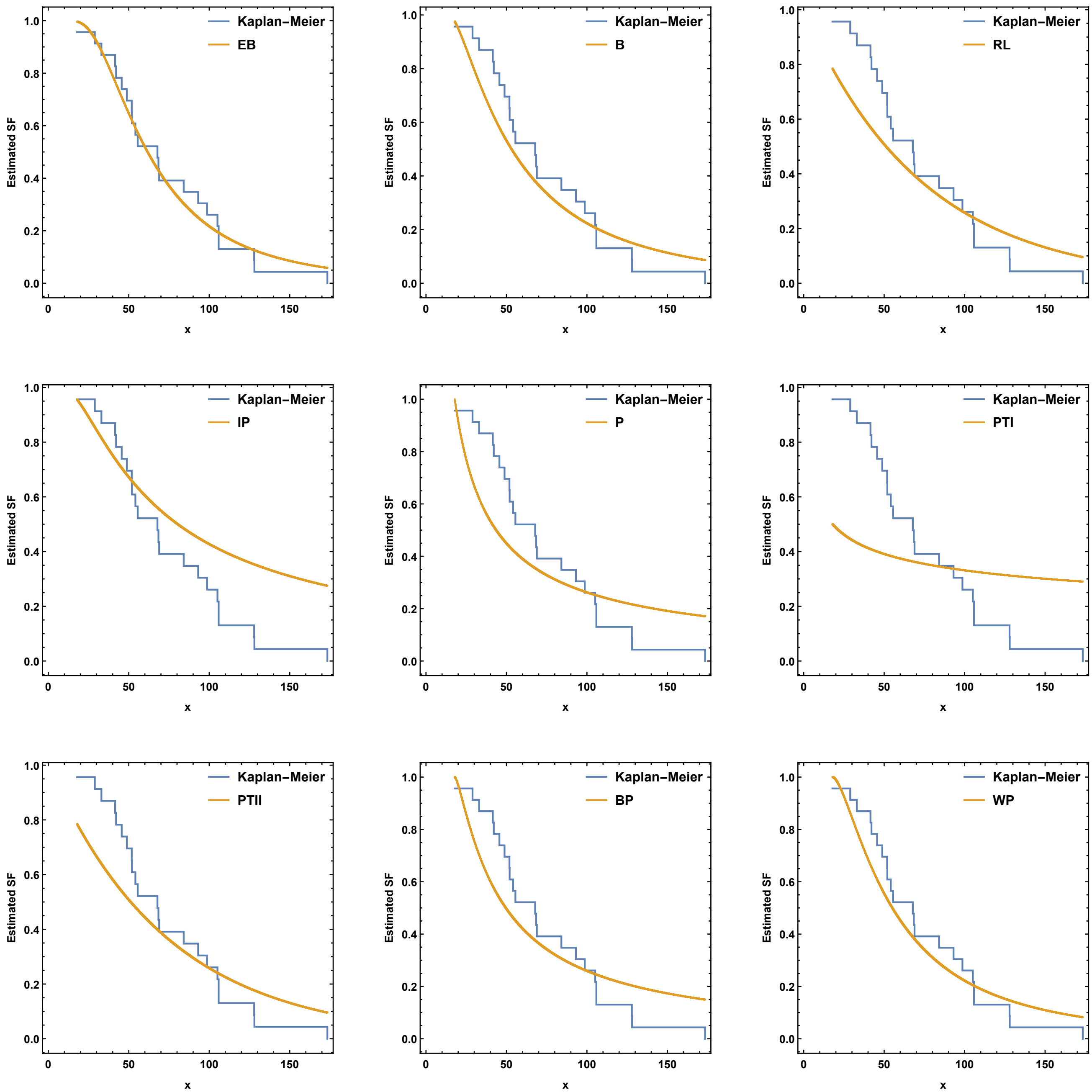

Fig. 12. Plots of the Kaplan-Meier survival functions with the fitted sfs of all models for the second data set.

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Conflicts of Interest

None.

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